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CERTAIN COEFFICIENT INEQUALITIES FOR THE CLASSES OF q-STARLIKE AND q-CONVEX FUNCTIONS

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Abstract: In this paper we determine certain coefficient inequalities for the classes of q-starlike and q-convex function and find the sufficient conditions for generalized Bessel function to belonging in these classes.

Keywords and Phrases: Univalent functions, q-convex functions, q-starlike functions, q-derivative operator, and generalized Bessel function.

2020 Mathematics Subject Classification: 30C45, 30C50, 30C80.

1. Introduction and Preliminaries

The study of the q-calculus has captivated the ardent attention of researchers and in Science and Engineering the q-calculus introduces an important role.

Recently, Rehman et al. [17] investigated some subclasses of q-starlike functions including numerous coefficient inequalities and a sufficient condition. Furthermore, Srivastava et al. [18, 20, 21, 23] published a series of studies concentrating on the

Janowski function-related classes of q-starlike functions from various angles. Moreover, in this field Srivastava [19] recently published survey-cum-expository. For fractional q-derivative operators in fractional q-calculus, the mathematical description and applications in geometrical function theory are carefully treated.

Srivastava [19] discussed the traditional q-analysis, which we used in this article, as well as the (p, q)-analysis. Specifically, the results in this article for $(0 < q < 1)$ the q-analogues, can easily converted into the results for $0 < q < p \leq 1$ the (p, q)-analogues by returning to Srivastava [18] who applied argument variations and some obvious parametric, the additional parameter p being redundant. As Significantly by Srivastava et al. [21] the majority of searchers employed (p, q)-analysis by inserting an apparently redundant parameter p in the already known findings dealing with the conventional q analysis. Other recent explorations into the q-calculus can be found at [2, 10]

Assume $\mathcal{H}$ is the class that contains all analytic functions in the open unit disk. $\mathbb{D} := \{\xi \in \mathbb{C} : |\xi| < 1\}$, has the form

$$\mathfrak{T}(\xi) = \xi + \sum_{n=2}^{\infty} a_n \, \xi^n, \ (\xi \in \mathbb{D}, \mathfrak{T}(0) = 0, \mathfrak{T}'(0) = 1). \tag{1.1}$$

Let $\mathcal{S}$ be the class of univalent functions as the subclass of $\mathcal{H}$. Further $\mathcal{S}^*$ is the class of starlike functions which is subclass of the univalent functions, where any $\mathfrak{T} \in \mathcal{S}^*$ meet the subsequent conditions:

$$\Re\left\{\frac{\xi \, \mathfrak{T}'(\xi)}{\mathfrak{T}(\xi)}\right\} > 0, \quad \forall \xi \in \mathbb{D}. \tag{1.2}$$

Assume the class of convex functions denoted by $\mathcal{C}^*$ which is subclass of $\mathcal{S}$ such that any function $\mathfrak{T} \in \mathcal{C}^*$ fulfil the subsequent conditions:

$$\Re\left\{\frac{(\xi \, \mathfrak{T}'(\xi))'}{\mathfrak{T}'(\xi)}\right\} = \Re\left\{1 + \frac{\xi \, \mathfrak{T}''(\xi)}{\mathfrak{T}'(\xi)}\right\} > 0, \quad \forall \xi \in \mathbb{D}. \tag{1.3}$$

Definition 1.1. *Allow p be an analytic function through $p(0) = 1$ fits to the class $\mathcal{M}[\tau, \mu]$ with $-1 \leq \mu < \tau \leq 1$ if and only if*

$$p(\xi) \prec \frac{1 + \tau \, \xi}{1 + \mu \, \xi}.$$

This class of analytic functions was introduced and studied by Janowski [8], according to this $\exists$ a function $h \in \mathcal{M}$ if and only if $p \in \mathcal{M}[\tau, \mu]$, such that

$$p(\xi) = \frac{(\tau + 1)h(\xi) - (\tau - 1)}{(\mu + 1)h(\xi) - (\mu - 1)}, \ \xi \in \mathbb{D}.$$

Definition 1.2. [8] *(i) A function* $\mathfrak{T} \in \mathcal{H}$ *be in the class* $\mathcal{S}^*[\tau, \mu]$ *with* $-1 \le \mu < \tau \le 1$ *if and only if*

$$\frac{\xi\, \mathfrak{T}'(\xi)}{\mathfrak{T}(\xi)} \prec \frac{1 + \tau\, \xi}{1 + \mu\, \xi}. \tag{1.4}$$

(ii) A function $\mathfrak{T} \in \mathcal{H}$ *be in the class* $\mathcal{C}^*[\tau, \mu]$ *with* $-1 \le \mu < \tau \le 1$ *if and only if*

$$1 + \frac{\xi \mathfrak{T}''(\xi)}{\mathfrak{T}'(\xi)} \prec \frac{1 + \tau\, \xi}{1 + \mu\, \xi}.$$

Definition 1.3. [23] *For* $q \in (0, 1)$ *the q-number* $[\sigma]_q$ *defined by*

$$[\sigma]_q := \begin{cases} \dfrac{1 - q^\sigma}{1 - q}, & \text{if} \quad \sigma \in \mathbb{C}, \\ \sum\limits_{n=0}^{\sigma - 1} q^n, & \text{if} \quad \sigma \in \mathbb{N} := \{1, 2, \dots\}. \end{cases}$$

Definition 1.4. [23] *The q-derivative of a function* $\mathfrak{T}$ *is labelled as*

$$D_q \mathfrak{T}(\xi) := \begin{cases} \dfrac{\mathfrak{T}(\xi) - \mathfrak{T}(q\, \xi)}{(1 - q)\, \xi}, & \textit{if} \quad \xi \in \mathbb{C} \setminus \{0\}, \\ \mathfrak{T}'(0), & \textit{if} \quad \xi = 0, \end{cases}$$

Presuming that $\mathfrak{T}'(0)$ *exists, and* $0 < q < 1$.
If $q \to 1$ then from above definition, we have

$$\lim_{q \to 1^-} D_q \mathfrak{T}(\xi) = \lim_{q \to 1^-} \frac{\mathfrak{T}(\xi) - \mathfrak{T}(q\, \xi)}{(1 - q)\, \xi} = \mathfrak{T}'(\xi).$$

Srivastava and Bansal [22] introduced

$$D_q\, \mathfrak{T}(\xi) = 1 + \sum_{n=2}^{\infty} [n]_q\, a_n\, \xi^{n-1}$$

for $\xi \in \mathbb{U}$ and $\mathfrak{T} \in \mathcal{H}$, In 1990, Ismail et al. [7] employed the q-derivative operator D_q to investigate a q-extension of the class $\mathcal{S}^*$ of starlike functions in $\mathbb{D}$.

Definition 1.5. *A function* $\mathfrak{T} \in \mathcal{H}$ *belongs to the class* $\mathcal{S}_q^*$ *iff*

$$\left| \frac{\xi}{\mathfrak{T}(\xi)} D_q \mathfrak{T}(\xi) - \frac{1}{1 - q} \right| < \frac{1}{1 - q}, \; \xi \in \mathbb{D}. \tag{1.5}$$

The class $\mathcal{S}_q^*$ of q-starlike functions and the right-half plane diminishes to the acquainted class $\mathcal{S}^*$ by the closed disk $\left|w - \frac{1}{1-q}\right| < \frac{1}{1-q}$ for $q \to 1^-$. Consistently, from definition of analytic functions and subordination, the inequality (1.5) wrote as below

$$\frac{\xi}{\mathfrak{T}(\xi)} D_q \mathfrak{T}(\xi) \prec \frac{1+\xi}{1-q\,\xi}. \tag{1.6}$$

One method to generalize the class $\mathcal{S}^*[\tau,\mu]$ of definition 1.2 is to substitute the function $(1+\tau\,\xi)/(1+\mu\,\xi)$ in (1.4) with the function $(1+\xi)/(1-q\,\xi)$ which is involved in (1.6). The appropriate definition of the corresponding q-extension $\mathcal{S}_q^*[\tau,\mu]$ is specified below.

Definition 1.6. *A function $\mathfrak{T} \in \mathcal{H}$ belongs to the class $\mathcal{S}_q^*[\tau,\mu]$ iff*

$$\frac{\xi D_q \mathfrak{T}(\xi)}{\mathfrak{T}(\xi)} = \frac{(\tau+1)\mathfrak{T}(\xi) - (\tau-1)}{(\mu+1)\mathfrak{T}(\xi) - (\mu-1)}, \ \xi \in \mathbb{D}, \tag{1.7}$$

where

$$T(\xi) = \frac{1+\xi}{1-q\xi},$$

which can be written as follows by using the definition of the subordination:

$$\frac{\xi D_q \mathfrak{T}(\xi)}{\mathfrak{T}(\xi)} \prec \phi(\xi),$$

where

$$\phi(\xi) := \frac{(\tau+1)\xi + 2 + (\tau-1)q\,\xi}{(\mu+1)\xi + 2 + (\mu-1)q\,\xi}, \ -1 \le \mu < \tau \le 1, \ q \in (0,1).$$

Remark 1.1. *(i) We can observed that*

$$\lim_{q \to 1^-} \mathcal{S}_q^*[\tau,\mu] = \mathcal{S}^*[\tau,\mu].$$

Also, $\mathcal{S}_q^*[1,-1] = \mathcal{S}_q^*$, see ([7]).

(ii) From Duren [4], we have

$$\mathfrak{T} \in \mathcal{C}_q^*[\tau,\mu] \Leftrightarrow \xi\, D_q \mathfrak{T}(\xi) \in \mathcal{S}_q^*[\tau,\mu].$$

The generalized Bessel function w_l is defined as (See [3]),

$$w_l(\xi) = \sum_{n=0}^{\infty} \frac{(-1)^n\, \mathfrak{b}^n}{n!\, \left(l + n + \frac{\mathfrak{b}-1}{2}\right)!} \left(\frac{\xi}{2}\right)^{2n+l}; \ \xi \in \mathbb{C}$$

which is the specific solution of the following linear differential equation

$$\xi\, w^{''}(\xi) + \mathfrak{a}\, \xi\, w^{'}(\xi) + \left\{\mathfrak{b}\, \xi^2 - l^2 + (1-\mathfrak{a})l\right\} w(\xi) = 0$$

where $\mathfrak{a}$, $\mathfrak{b}$, $l \in \mathbb{C}$.
Now, the generalized Bessel function (normalized function) is defined as

$$u_l\,(\xi) = 2^l \left(l + n\frac{(\mathfrak{a}-1)}{2}\right)!\ \xi^{\frac{-l}{2}}\, w_l\,(\xi^{\frac{1}{2}}) = \sum_{n=0}^{\infty} \frac{(\frac{-\mathfrak{b}}{4})^n}{(\sigma)_n\, n!}\, \xi^n,$$

where

$$\sigma_n = \begin{cases} 1, & \text{for } n = 0 \\ \prod_{i=0}^{n-1}(\sigma + i), & \text{for } n \in \mathbb{N}. \end{cases} \quad \left(\sigma = l + \frac{(a+1)}{2},\ \sigma \neq 0, -1, -2 \ldots\right),$$

Normalized Bessel function defined by using the convolution between two functions, in below

$$\xi\, u_l(\xi) = \xi + \sum_{n=2}^{\infty} \frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\, \xi^n.$$

In this paper, we investigate and obtain the sufficient conditions of q-starlikeness and q-convexity for functions which are linked with generalized Bessel function by using following sufficient conditions obtained by Srivastava [23]. A similar work be also investigated by Gour et al. in [5] and Seoudy in [17] and Goyal et al. obtained sufficient conditions of starlikeness for the multivalent functions in [6].

Lemma 1.1. [23] *Suppose $\mathfrak{T} \in \mathcal{S}_q^*[\tau, \mu]$, if it is achieving below condition*

$$\sum_{n=2}^{\infty} \Big(2q[n-1]_q + |(\mu+1)[n]_q - (\tau+1)\,|\,\Big)|a_n| < |\mu - \tau\,| \tag{1.8}$$

Lemma 1.2. [23] *Suppose $\mathfrak{T} \in \mathcal{C}_q^*[\tau, \mu]$, if it is achieving below condition*

$$\sum_{n=2}^{\infty} [n]_q \Big(2q[n-1]_q + |(\mu+1)[n]_q - (\tau+1)\,|\,\Big)|a_n| < |\mu - \tau\,| \tag{1.9}$$

2. Main Results

Theorem 2.1. *Let $E(\sigma, q)$ be defined as follows:*

$$E(\sigma,q) := \left(\frac{2q+(\mu+1)}{1-q}+(\tau+1)\right) {}_0F_1\left(-;\sigma;\left|\frac{\mathfrak{b}}{4}\right|\right) - \frac{(\mu+3)q}{1-q}\, {}_0F_1\left(-;\sigma;\left|\frac{\mathfrak{b}\,q}{4}\right|\right) + (\tau-\mu).$$

If the inequality

$$E(\sigma,q) < |\mu-\tau|$$

holds, then function $\xi\, u_l\,(\xi)$ fits to the class $\mathcal{S}_q^*[\tau,\mu]$.

Proof. Since

$$\xi\, u_l(\xi) = \xi + \sum_{n=2}^{\infty} \frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\,\xi^n,\ \xi\in\mathbb{D},$$

we know that from Lemma 1.1, any function $\mathfrak{T}\in\mathcal{S}_q^*[\tau,\mu]$ fulfils (1.8). Then, for $f(\xi) := \xi\, u_l(\xi)$ it is sufficient to show that (1.8) holds, for

$$a_n = \frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}, \quad \text{and} \quad [n]_q = \frac{1-q^n}{1-q}.$$

Using the triangle's inequality we get

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Big(2q[n-1]_q + |\,(\mu+1)[n]_q-(\tau+1)\,|\Big)\,|\,a_n\,| \\
&\le \sum_{n=2}^{\infty} 2q\frac{1-q^{n-1}}{1-q}\,|\,a_n\,| + \sum_{n=2}^{\infty}(\mu+1)\frac{1-q^n}{1-q}\,|\,a_n\,| + \sum_{n=2}^{\infty}(\tau+1)\,|\,a_n\,| \\
&= \sum_{n=2}^{\infty}\left(\frac{2q+(\mu+1)}{1-q}+(\tau+1)\right)|\,a_n\,| - \sum_{n=2}^{\infty}\frac{(\mu+3)q^n}{1-q}\,|\,a_n\,| \\
&= \left(\frac{2q+(\mu+1)}{1-q}+(\tau+1)\right)\sum_{n=2}^{\infty}\left|\frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\right| - \frac{(\mu+3)}{1-q}\sum_{n=2}^{\infty}\left|\frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\right|q^n \\
&= \left(\frac{2q+(\mu+1)}{1-q}+(\tau+1)\right)\sum_{n=1}^{\infty}\frac{\left|\frac{-\mathfrak{b}}{4}\right|^n}{(\sigma)_n(n)!} - \frac{(\mu+3)}{1-q}\sum_{n=1}^{\infty}\frac{\left|\frac{-\mathfrak{b}}{4}\right|^n}{(\sigma)_n(n)!}q^{n+1}
\end{aligned}$$

$$= \left(\frac{2q+(\mu+1)}{1-q}+(\tau+1)\right)\left({}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}}{4}|\right)-1\right)-\frac{(\mu+3)q}{1-q}\left({}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|\right)-1\right)$$
$$= \left(\frac{2q+(\mu+1)}{1-q}+(\tau+1)\right){}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}}{4}|\right)-\frac{(\mu+3)q}{1-q}{}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|\right)+(\tau-\mu)$$
$$=: E(\sigma,q)$$

and the theorem's assumption implies (1.8).
Hence $\xi\, u_l(\xi) \in \mathcal{S}_q^*[\tau,\mu]$.
Here ${}_0F_1(-;\gamma;x)$ is the special case of Gauss hypergeometric function ${}_2F_1(\xi,\tau;\gamma;x)$.

On taking $\tau = 1-2\xi$, $0 \le \xi < 1$, and $\mu = -1$, we have $\mathcal{S}_q^*[1-2\xi,-1] =: \mathcal{S}_q^*(\xi)$ and Theorem 2.1 reduced in result given follows:

Corollary 2.1. *Let $E^*(\sigma,q)$ be defined as follows:*

$$E^*(\sigma,q) := \left(\frac{2q}{1-q}+2(1-\tau)\right){}_0F_1\left(-;\sigma;\left|\frac{\mathfrak{b}}{4}\right|\right)-\frac{2q}{1-q}\,{}_0F_1\left(-;\sigma;\left|\frac{\mathfrak{b}\,q}{4}\right|\right)+2(1-\tau)$$

If the inequality

$$E^*(\sigma,q) < 2|\tau-1|$$

holds, then function $\xi\, u_l(\xi) \in \mathcal{S}_q^*(\xi)$.
For $\xi = 0$, we have the following example:

Example 2.1. Let $\widetilde{E}(\sigma,q)$ be labelled as follows:

$$\widetilde{E}(\sigma,q) := \frac{2}{1-q}\,{}_0F_1\left(-;\sigma;\left|\frac{\mathfrak{b}}{4}\right|\right)-\frac{2q}{1-q}\,{}_0F_1\left(-;\sigma;\left|\frac{\mathfrak{b}\,q}{4}\right|\right)+2$$

If the inequality

$$\widetilde{E}(\sigma,q) < 2$$

holds, then function $\xi\, u_l(\xi) \in \mathcal{S}_q^*(0)$.

Theorem 2.2. *Let $G(\sigma,q)$, be demarcated as follows:*

$$\begin{aligned} G(\sigma,q) :=& \frac{1}{(1-q)^2}\Big\{(q+\mu+2+\tau(1-q))\;{}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}}{4}|\right) \\ &- q(\tau(1-q)+2\mu+q+5)\;{}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|\right)+q^2(\mu+3) \\ &{}_0F_1\left(-;\sigma;|\frac{\mathfrak{b}\,q^2}{4}|\right)+(\tau+\mu+2)(1-q)^2\Big\}\end{aligned}$$

If the inequality

$$G(\sigma,q) < |\mu-\tau|$$

holds, then function $\xi\, u_l(\xi) \in \mathcal{C}_q^*[\tau,\mu]$.

Proof. We Know that from Lemma 1.2, any function $\mathfrak{T} \in \mathcal{C}_q^*[\tau,\mu]$ if it fulfils (1.9). Then for $\mathfrak{T}(\xi) := \xi\, u_l(\xi)$, it is sufficient show that (1.9) holds for

$$a_n = \frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}, \quad \text{and} \quad [n]_q = \frac{1-q^n}{1-q}.$$

Using first triangle's inequality, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}[n]_q\big(2q[n-1]_q + \mid (\mu+1)[n]_q - (\tau+1) \mid\big) \mid a_n \mid \\
&\leq \sum_{n=2}^{\infty} 2q[n]_q[n-1]_q \mid a_n \mid + \sum_{n=2}^{\infty}(\mu+1)[n]_q[n]_q \mid a_n \mid + \sum_{n=2}^{\infty}(\tau+1)[n]_q \mid a_n \mid \\
&= \sum_{n=2}^{\infty} 2q\frac{1-q^n}{1-q}\frac{1-q^{n-1}}{1-q} \mid a_n \mid + \sum_{n=2}^{\infty}(\mu+1)\frac{1-q^n}{1-q}\frac{1-q^n}{1-q} \mid a_n \mid \\
&\quad + \sum_{n=2}^{\infty}(\tau+1)\frac{1-q^n}{1-q} \mid a_n \mid \\
&= \sum_{n=2}^{\infty} \frac{2q+(\mu+1)+(\tau+1)(1-q)}{(1-q)^2} \mid a_n \mid + \sum_{n=2}^{\infty}\frac{2+(\mu+1)}{(1-q)^2}q^{2n} \mid a_n \mid \\
&\quad - \sum_{n=2}^{\infty}\frac{(\tau+1)(1-q)+2(\mu+1)+2q+2}{(1-q)^2}q^n \mid a_n \mid \\
&= \frac{q+\mu+2+\tau(1-q)}{(1-q)^2}\sum_{n=2}^{\infty} \mid a_n \mid - \frac{\tau(1-q)+2\mu+q+5}{(1-q)^2}\sum_{n=2}^{\infty} q^n \mid a_n \mid \\
&\quad + \frac{\mu+3}{(1-q)^2}\sum_{n=2}^{\infty} q^{2n} \mid a_n \mid \\
&= \frac{q+\mu+2+\tau(1-q)}{(1-q)^2}\sum_{n=2}^{\infty}\left|\frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\right| - \frac{\tau(1-q)+2\mu+q+5}{(1-q)^2} \\
&\quad \sum_{n=2}^{\infty}\left|\frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\right| q^n + \frac{\mu+3}{(1-q)^2}\sum_{n=2}^{\infty}\left|\frac{(\frac{-\mathfrak{b}}{4})^{n-1}}{(\sigma)_{n-1}(n-1)!}\right| q^{2n} \\
&= \frac{q+\mu+2+\tau(1-q)}{(1-q)^2}\sum_{n=1}^{\infty}\frac{\left|\frac{-\mathfrak{b}}{4}\right|^n}{(\sigma)_n(n)!} - \frac{\tau(1-q)+2\mu+q+5}{(1-q)^2}\sum_{n=1}^{\infty}\frac{\left|\frac{-\mathfrak{b}}{4}\right|^n}{(\sigma)_n(n)!}q^{n+1}
\end{aligned}$$

$$+\frac{\mu+3}{(1-q)^2}\sum_{n=1}^{\infty}\frac{\left|\frac{-\mathfrak{b}}{4}\right|^n}{(\sigma)_n(n)!}q^{2n+2}$$

$$=\frac{q+\mu+2+\tau(1-q)}{(1-q)^2}\Big({}_0F_1(-;\sigma;|\frac{\mathfrak{b}}{4}|)-1\Big)-\frac{\tau(1-q)+2\mu+q+5}{(1-q)^2}q$$

$$\Big({}_0F_1(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|)-1\Big)+\frac{\mu+3}{(1-q)^2}q^2\Big({}_0F_1(-;\sigma;|\frac{\mathfrak{b}\,q^2}{4}|)-1\Big)$$

$$=\frac{1}{(1-q)^2}\Big\{(q+\mu+2+\tau(1-q))\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}}{4}|\Big)-q(\tau(1-q)+2\mu+q+5)$$

$${}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|\Big)+q^2(\mu+3)\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}\,q^2}{4}|\Big)+(\tau+\mu+2)(1-q)^2\Big\}$$

$$=:G(\sigma,q)$$

Therefore, the Theorem's assumption implies (1.9), hence $\xi\,u_l(\xi)\in\mathcal{C}_q^*[\tau,\mu]$.

On taking $\tau=1-2\xi$, $0\le\xi<1$ and $\mu=-1$, we have $\mathcal{C}_q^*[1-2\tau,-1]=:\mathcal{C}_q^*(\xi)$, and Theorem 2.2 reduced in result given follows:

Corollary 2.2. *Let $G^*(\sigma,q)$ be defined as follows:*

$$G^*(\sigma,q):=\frac{1}{(1-q)^2}\Big\{2(1-\tau(1-q))\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}}{4}|\Big)-2q(2-\tau(1-q))$$
$${}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|\Big)+2q^2\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}\,q^2}{4}|\Big)+2(1-\tau)(1-q)^2\Big\}$$

If inequality

$$G^*(\sigma,q)<2|\tau-1|$$

holds for $0\le\tau<1$, then function $\xi\,u_l(\xi)\in\mathcal{C}_q^*(\xi)$.
For $\xi=0$, we have the following example:

Example 2.2. Let $\widetilde{G}(m,q)$ be defined as follows:

$$\widetilde{G}(\sigma,q):=\frac{1}{(1-q)^2}\Big\{2\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}}{4}|\Big)-4q\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}\,q}{4}|\Big)$$
$$+2q^2\,{}_0F_1\Big(-;\sigma;|\frac{\mathfrak{b}\,q^2}{4}|\Big)+2(1-q)^2\Big\}$$

If for any $j\in\{1,2\}$ the inequality

$$\widetilde{G}(\sigma,q)<2$$

holds, then function $\xi\, u_l(\xi) \in \mathcal{C}_q^*(0)$.

3. Conclusion

In this paper, we obtained sufficient conditions for q-starlikenss and q-convexity for a function associated with normalized Bessel function as following Srivastava ([19]).

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