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ON THE SOLUTION OF A CLASS OF EXPONENTIAL DIOPHANTINE EQUATIONS

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Abstract: In this note, we show that for $n = 4N + 3, N \in \mathbb{N} \cup \{0\}$, the exponential Diophantine equation $n^x + 24^y = z^2$ has exactly two solutions if $n + 1$ or equivalently $N + 1$ is an square. When $N + 1 = m^2$, the solutions are given by $(0, 1, 5)$ and $(1, 0, 2m)$. Otherwise it has a unique solution $(0, 1, 5)$ in non-negative integers. Finally, we leave an open problem to explore.

Keywords and Phrases: Catalan's Conjecture solutions, Exponential Diophantine equations, Integer solutions.

2020 Mathematics Subject Classification: 11D61, 11D72.

1. Introduction

Many authors have studied the exponential Diophantine equation for a long time [7]. In 1844, the great Mathematician, Eugene Charles Catalan formulated a conjecture that the exponential Diophantine equation $a^x - b^y = 1$ where $a, b, x, y \in \mathbb{Z}$ with $min\{a, b, x, y\} > 1$ has a unique solution $(a, b, x, y) = (3, 2, 2, 3)$ [8]. Since then, numerous mathematicians have attempted to solve it, with varying degrees of success (see for instance [2, 10, 13, 14]). Eventually, Preda Mihăilescu [15] proved the conjecture in 2004. For various values of a and b, the equations $a^x + b^y = z^2$ have been vastly studied in non-negative integers, see for example [1, 3, 4, 5, 6, 12, 16, 19, 21].

W. S. Gayo, and J. B. Bacan [11], studied and solved the exponential Diophantine equation of the form $M_p^x + (M_q + 1)^y = z^2$ for Mersenne primes M_p and M_q and non-negative integers x, y, and z. K. Chakraborty, A. Hoque, and K. Srinivas [9] investigated and explained the answer on the Diophantine equation $cx^2 + p^{2m} = 4y^n$ (see also [17, 18]). We continue to carry out research on similar type of exponential Diophantine equations with different bases, motivated by the strategies utilised in the aforementioned works. The novelty of the paper is that we mainly use elementary methods to solve a particular class of exponential Diophantine equations.

2. Main Results

For our result we need Catalan's conjecture ([15]):

Conjecture 2.1. *The unique solution for the Diophantine equation* $a^x - b^y = 1$ *where* $a, b, x, y \in \mathbb{Z}$ *with* $min\{a, b, x, y\} > 1$ *is* $(3, 2, 2, 3)$.

Theorem 2.2. *For* $n = 4N+3, N \in \mathbb{N}\cup\{0\}$, *the exponential Diophantine equation* $n^x + 24^y = z^2$ *has*
(i)exactly two solutions if $n + 1$ *or equivalently* $N + 1$ *is a perfect square. When* $n + 1 = m^2, m \geq 0$, *the solutions are given by* $(0, 1, 5)$ *and* $(1, 0, m)$;
(ii) otherwise, the equation has a unique solution given by $(0, 1, 5)$.
Proof. The equation is

$$n^x + 24^y = z^2 \tag{1}$$

(i) **Case 1.** $y = 0$
Therefore $z^2 - n^x = 1$ implies that $x = 0$ or 1 by Catalan conjecture. Now, $x = 0$ implies that $z^2 = 2$ which implies that no solution exists. For $x = 1$ giving $n = z^2 - 1$ which implies that $z^2 = n + 1 = m^2$ implies $z = \sqrt{n+1} = m$. Therefore $(x, y, z) = (1, 0, m)$ is a solution.
Case 2. $y \neq 0$
Then z is odd which implies that $z^2 \equiv 1(\text{mod}4)$. Also, $n \equiv -1(\text{mod}4)$. Therefore equation (1) gives $(-1)^x \equiv 1(\text{mod}4)$ implying x is even. Let $x = 2k, k \geq 0$, then equation (1) becomes $24^y = z^2 - n^{2k} = (z - n^k)(z + n^k)$. Now, $z \pm n^k$ are even which implies that the following subcases are possible.
Subcase 1. $z - n^k = 2^y, z + n^k = 12^y$.
Subtracting we get, $2n^k = 2^y(6^y - 1)$ implying $y = 1, n^k = 5$ implies $n = 5, k = 1$. But 5 is not of the form $4N + 3$ and so we discard this solution.
Subcase 2. $z - n^k = 4^y, z + n^k = 6^y$.
Subtracting we get, $2n^k = 2^y(3^y - 2^y)$ implying $y = 1, n^k = 3^1 - 2^1 = 1$. This

implies $k = 0$ giving $x = 0$. Therefore equation (1) implies $z^2 = n^0 + 24^1 = 5^2$ rendering $z = 5$. Therefore $(x, y, z) = (0, 1, 5)$ is another solution.
The other subcases are not possible. For example,
Subcase 3. $z - n^k = 1, z + n^k = 24^y$ implies $2n^k = 24^y - 1$ This is not possible since lhs is even but rhs is odd.
Subcase 4. $z - n^k = 3^y, z + n^k = 8^y$ implies $2n^k = 8^y - 3^y$ This is not possible since lhs is even but rhs is odd.
(ii) Proceeding as in (i), we get $(0, 1, 5)$ as the unique solution of the equation (1). This completes the theorem.

We now provide some examples below.

Examples.
(i) The only non-negative integral solution of the exponential Diophantine equation $23^x + 24^y = z^2$ is $(0, 1, 5)$ [17], since $23 = 4(5) + 3$, and 24 is not a perfect square.
(ii) The exponential Diophantine equation $15^x + 24^y = z^2$ has exactly two non-negative integral solutions, viz. $(0, 1, 5), (1, 0, 4)$, since $15 = 4(3) + 3$, and 16 is a perfect square.

An immediate result is the following:

Corollary 2.3. *If $n \equiv 3(\mathrm{mod} 4)$ the exponential Diophantine equation $n^x + 24^y = z^2$ has no solution in positive integers.*

Example. None of the equations $11^x + 24^y = z^2$, $19^x + 24^y = z^2$, $47^x + 24^y = z^2$ has a positive integral solution.

Corollary 2.4. *If $n \equiv 3(\mathrm{mod} 4)$ the exponential Diophantine equation $n^x + 24^y = w^{2m}, m \in \mathbb{N}$ has no positive integral solution.*
Proof. Follows from Corollary 2.3 above.

Example. $15^x + 24^y = w^4$, $31^x + 24^y = w^6$, or $19^x + 24^y = w^8$ has no positive integral solution.

Corollary 2.5. *Let $n \equiv 3(\mathrm{mod} 4)$. In non negative integers the exponential Diophantine equation $n^x + 24^y = w^{2m}, 1 < m \in \mathbb{N}$ has*
(i) a unique solution if $n + 1 = l^{2m}$ for some $l \in \mathbb{N}$. In this case the solution is $(1, 0, l)$.
(ii) no solution otherwise.
Proof. Given equation is

$$n^x + 24^y = w^{2m},\ m > 1. \tag{2}$$

Now, (x, y, w) is a solution of equation (2) implies $(x, y, z = w^m)$ is a solution of equation (1). This means $w^m = 5$ or $\sqrt{n+1}$, by Theorem 2.2. As $w^m \neq 5$ for

$m > 1$, we have $w^m = \sqrt{n+1}$. Thus, equation (2) will have a solution if and only if $(n+1)^{1/2m}$ is a natural number. In this case, $(n+1)^{1/2m} = l$ say. Then, $(x, y, z = w^m) = (1, 0, l^m)$ is solution of equation (1). That is, $(x, y, w) = (1, 0, l), l \in \mathbb{N}$ is a solution of equation (2). The Proof is complete.

Example.
(i) $3^x + 24^y = w^4$ has no solution in non-negative integers as $4 \neq l^4$ for any $l \in \mathbb{N}$.
(ii) $15^x + 24^y = w^4$ has a unique solution in non-negative integers as $15+1 = 16 = 2^4$ and the solution is $(1, 0, 2)$
(iii) $103^x + 24^y = w^6$ has no solution in non-negative integers as $104 \neq l^6$, for any $l \in \mathbb{N}$.

3. Conclusion

In this work we studied the equations $n^x + 24^y = z^2$ where $n = 4N + 3, N \in \mathbb{N} \cup \{0\}$. We wish to explore the exponential Diophantine equation $n^x + 24^y = z^2$ where $n = 4N + 1, N \in \mathbb{N}$ in our future work. It is an open problem to explore $n^x + (m^2 - 1)^y = z^2$ for any $m \in \mathbb{N}$. We conclude with the observation that $(0, 1, m)$ is always a solution of this exponential Diophantine equation.

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