South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 419-428

DOI: 10.56827/SEAJMMS.2022.1802.34 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ANALYSIS OF DYNAMIC BEHAVIOUR OF FRACTIONAL ORDER SIR EPIDEMIC MODEL OF CHILDHOOD DISEASES USING RVIM

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(**Received:** *Sep. 01, 2021* **Accepted:** *Aug. 13, 2022* **Published:** *Aug. 30, 2022)*

Abstract: In the given article, the solution of a Mathematical Model of epidemic in childhood diseases is presented. Reconstruction of Variational Iteration Method (RVIM) is used to analyze the model which inculcates the Laplace Transform to reconstruct VIM. This technique is generally used to get solutions of fractional differential equations of linear and non linear form. The result obtained has been simulated and discussed through graphs.

Keywords and Phrases: Epidemics, Caputo fractional derivative, RVIM, Laplace Transform, Convolution.

2020 Mathematics Subject Classification: 92D30, 34A08.

1. Introduction

Study of Epidemics is very important to understand the effect of infectious disease in a community. Generally young children are exposed to various types of bacterial or viral infections. Some of them are Bronchitis (generally effecting children up to 1 year), hand/foot/mouth disease, Diarrhea, chicken pox, measles, meningitis etc. These diseases sometimes cause very serious complications that may lead to death. Whole world is trying to eradicate these deadly diseases by proper Vaccination. WHO has launched Global immunization program to overcome this situation. By framing Mathematical Models, many mathematicians are trying to model these epidemics, estimate parameters and check model sensitivity by changing parameters. By doing Mathematical Modeling, many mathematicians are trying to build models, estimate parameters and check model sensitivity by changing parameters [5]. Singh et al. [6] studied the vaccination process of childhood diseases.

Hesameddini et al. introduced a new approach for the derivation in the method of Variational Iteration using the Laplace transform for solving linear and nonlinear ordinary differential equations named it as the Reconstruction of Variational Iteration Method [2]. RVIM is advantageous in comparison to other techniques as it provides convergent successive approximations and reduces the size of computation. The method is applicable wihout any restrictive assumptions .Also the method improves variational iteration method by not using a Lagrange multiplier because in VIM , fractional Lagrange multiplier can't be found in explicit manner as the fractional derivative term is avoided assuming it to be a restricted variation in correctional function. RVIM also has an upper hand to ADM as no heavy calculations for finding Adomian polynomials is required. Fractional Differential equations show the realistic behavior of infection of disease but at a very slower rate. Here we will find out the numerical solution of Childhood disease using the Reconstruction of Variational Iteration Method [3].

2. Preliminaries and Definitions

Definition 2.1. *Let the function $f \in C[a,b]$, where $C[a,b]$ is the class of real valued continuous functions defined on the interval $[a,b]$ then for $\alpha \geq 0$ the Riemann –Liouville integral of f of order α is expressed as*

$$J^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2.2. *The Caputo's fractional derivative of $f(x)$ is defined as*

$$D^{\alpha_*} f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} f^m(\tau) d\tau, & m-1 < \alpha < m, \quad m \in N^*, \\ \dfrac{d^m}{dt^m} f(t), \quad \alpha = m. \end{cases}$$

Here it is to be noted that

$$J^{\alpha} J^{\beta} f(t) = J^{\alpha+\beta} f(t)$$

$$J^{\alpha}t^{\lambda} = \frac{\Gamma(\lambda+1)}{\Gamma(\alpha+\lambda+1)}t^{\alpha+\lambda}, \quad \alpha > 0, \quad \lambda > -1, \quad t > 0.$$

Definition 2.3. *The Laplace transform of a function* $f(t)$, $t \geq 0$ *is the function* $\mathbb{F}(s)$, s *being the parameter of transform is given by the integral*

$$\mathbb{F}(s) = \mathcal{L}\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)dt,$$

for all values of s *for which the integral converges.*

Definition 2.4. *A function* $f(t)$ *is said to be of exponential order as* $t \to \infty$ *if for nonnegative constants* M, T *and* α

$$|f(t)| \leq Me^{\alpha T} \text{ for } t \geq T.$$

Definition 2.5. *A function* $f(t)$ *is said to be piecewise continuous on an interval* $\alpha \leq t \leq \beta$ *if the interval* $[\alpha, \beta]$ *can be subdivided into finite number of subintervals in such a manner that*
(i) $f(t)$ *is continuous in the interior of each of these subintervals.*
(ii) $f(t)$ *has a finite limit at the endpoints of each of these subintervals.*

Definition 2.6. *If the function* $f(t)$ *is piecewise continuous for* $t \geq 0$ *and is of exponential order as* $t \to \infty$, *then its Laplace Transform* $\mathbb{F}(s)$ *exists for all* $s > \alpha$.

Definition 2.7. *Convolution of two functions* $f(t)$ *and* $g(t)$, $t \geq 0$ *is denoted by* $(f * g)(t)$ *and is given by the integral*

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau = \mathcal{L}^{-1}\{\mathbb{F}(s)\mathbb{G}(s)\},$$

where $\mathcal{L}\{f(t); s\} = \mathbb{F}(s)$ *and* $\mathcal{L}\{g(t); s\} = \mathbb{G}(s)$ *and* $\mathcal{L}^{-1}$ *is the inverse Laplace transform operator.*

Definition 2.8. *The Laplace transform of Caputo fractional derivative* $D_*^{\alpha} f(t)$ *is defined as*

$$\mathcal{L}\{D_*^{\alpha} f(t), s\} = s^{\alpha}\mathbb{F}(s) - \sum_{r=0}^{m-1} s^{(\alpha-r-1)} f^{r}(0^{+}),$$

where $\mathbb{F}(s) = \mathcal{L}\{f(t); s\}, \quad m - 1 \leq \alpha < m.$

3. Methodology of RVIM

To understand the idea of RVIM, let us consider a system of fractional differential equations as

$$D_*^{\mu} x_i(t) = \mathbb{N}_i(x_1, x_2, ..., x_n, t), \; i = 1, 2, ..., n. \tag{3.1}$$

where $D_*^{\mu_i}s$ is the μ_i^{th} order Caputo derivative of x_i, $N_i's$ are linear/nonlinear functions of $t, m-1 \leq \mu_i < m$ with $m \geq 1$, subject to the initial conditions:

$$x_i^{(k)} = c_k^i < \ 0 \leq k \leq [\mu_i],\ 1 \leq i < n. \tag{3.2}$$

On Taking Laplace transform of both sides for (3.1) and using the homogeneous initial condition or the zero artificial initial condition, the result obtained is as follows

$$\mathcal{L}\{x_i(t); s\} = \frac{1}{s^{\mu}} \mathcal{L}\{\mathbb{N}_i(x_1, x_2, ..., x_n, t)\} \tag{3.3}$$

Now using the convolution theorem to find the inverse Laplace transform of (3.3), we get

$$x_i(t) = \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \mathbb{N}_i(x_1(\tau), x_2(\tau), ..., x_n(\tau), \tau) d\tau \tag{3.4}$$

In order to obtain the solution, we use the actual initial condition. Thus the following iteration formula is obtain :

$$x_i^{n+1}(t) = x_i^0(t) + \frac{1}{\Gamma(\mu)} \int_0^t (t-\tau)^{\mu-1} \mathbb{N}_i(x_1(\tau), x_2(\tau), ..., x_n(\tau), \tau) d\tau, \quad i = 1, 2, ..., n. \tag{3.4}$$

Here $x_i^0(t)$ is obtained by actual initial condition and $x_i^n(t)$ denotes the n^{th} approximation of x_i. Identifying the initial approximation of x_i^0 for each i, the successive approximations $x_i^n(t), n > 0$ can be calculated with the help of previous ones. According to RVIM, the series solution for $x_i(t)$ may be written as a sum of components in the form

$$x_i(t) = \lim_{n \to \infty} x_i^n(t), \quad i = 1, 2, ..., n.$$

4. Mathematical Model for Epidemic Childhood Diseases

The mathematical model presented here is to identify pattern and common variations in disease function and consequently enabling us to find some of the factors affecting the disease. In the given set of equations depicting the model, the emphasis is on to show the efficacy of vaccination on infectious diseases so some common parameters such as vaccination and incidence are taken in to account irrespective of focusing on any particular disease. The model predicts about who may get infected and to what extent vaccination efforts will be helpful in curbing the spread of disease. Hence, in general model is representative pertaining to the aforesaid parameters.

A Susceptible -Infected - Recovered model was given by Makinde [4] as follows:

$$
\begin{aligned}
\frac{ds}{dt} &= (1-p)\sigma - \eta\frac{si}{N} - \sigma s \\
\frac{di}{dt} &= \eta\frac{si}{N} - (\gamma+\sigma)i \\
\frac{dr}{dt} &= p\sigma + \gamma i - \sigma r
\end{aligned} \tag{4.1}
$$

Arafa et al. [1] reframed this model using the terms $s/N = S$, $i/N = I$, $r/N = R$ and $N = s + i + r$ to get a new model as

$$
\begin{aligned}
\frac{dS}{dt} &= (1-p)\sigma - \eta\frac{SI}{-}\sigma S \\
\frac{dI}{dt} &= \eta SI - (\gamma+\sigma)I \\
\frac{dR}{dt} &= p\sigma + \gamma I - \sigma R
\end{aligned} \tag{4.2}
$$

Here the population is divided in to three classes namely a susceptible class (S), an infected class (I) and the third class being (R) including vaccinated and recovered ones with permanent immunity. We can easily see from the above model that vaccination is efficient in all manners and the natural death rate is not same. Therefore the total population size N is never constant. The rate of birth is represented by σ. The parameter, p, represents that fraction of the population who are vaccinated at the time of birth, where $0 < p < 1$, considering that the rest of population, i.e. non vaccinated, is susceptible to the disease. A susceptible person can get the disease through a contact with infected person at the rate of η. Infected persons recover at the rate of γ.

5. Application of Reconstruction of Variational Iteration Method (RVIM) to solve Epidemic Model of Childhood Disease

Consider the following balanced form of system of fractional ordinary differential equations of order μ (fractional derivatives are taken in Caputo sense) representing the epidemic model of childhood disease:

$$
\begin{aligned}
D^{\mu}S(t) &= (1-p)\sigma^{\mu} - \eta^{\mu}SI - \sigma^{\mu}S \\
D^{\mu}I(t) &= \eta^{\mu}SI - (\gamma^{\mu} + \sigma^{\mu})I \\
D^{\mu}R(t) &= p\sigma^{\mu} + \gamma^{\mu}I - \sigma^{\mu}R, \quad 0 < \mu \leq 1,
\end{aligned} \tag{5.1}
$$

where

$$
S(0) = 1;\ I(0) = 0:5;\ R(0) = 0. \tag{5.2}
$$

Applying Reconstructed Variational Iterative Method to (4.1), we get:

$$S(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\{(1-p)\sigma^\mu - \eta^\mu SI - \sigma^\mu S\}d\tau$$
$$I(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\{\eta^\mu SI - (\gamma^\mu + \sigma^\mu)I\}d\tau$$
$$R(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\{p\sigma^\mu + \gamma^\mu I - \sigma^\mu R\}d\tau \quad (5.3)$$

Therefore according to RVIM, the iterative formulae for approximate solution can be written as

$$S_{n+1}(t) = S_0(t) + \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\{(1-p)\sigma^\mu - \eta^\mu S_n(\tau)I_n(\tau) - \sigma^\mu S_n(\tau)\}d\tau$$
$$I_{n+1}(t) = I_0(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\{\eta^\mu S_n(\tau)I_n(\tau) - (\gamma^\mu + \sigma^\mu)I_n(\tau)\}d\tau$$
$$R_{n+1}(t) = R_0(t) = \frac{1}{\Gamma(\mu)}\int_0^t (t-\tau)^{\mu-1}\{p\sigma^\mu + \gamma^\mu I_n(\tau) - \sigma^\mu R_n(\tau)\}d\tau \quad (5.4)$$

From above we get the successive terms of approximation solutions as :

$$S_0(t) = 1,\ I_0(t) = 0.5,\ R_0(t) = 0.$$

$$S_1(t) = 1 - \frac{[p\sigma^\mu + .5\eta^\mu]t^\mu}{\Gamma(\mu+1)}$$
$$I_1(t) = 0.5 + \frac{0.5\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\}t^\mu}{\Gamma(\mu+1)}$$
$$R_1(t) = \frac{[p\sigma^\mu + 0.5\gamma^\mu]t^\mu}{\Gamma(\mu+1)}$$
$$S_2(t) = 1 - \frac{[p\sigma^\mu + .5\eta^\mu]t^\mu}{\Gamma(\mu+1)} - \frac{[(p\sigma^\mu + .5\eta^\mu)(.5\eta^\mu + \sigma^\mu) - .5\eta^\mu\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\}]t^{2\mu}}{\Gamma(2\mu+1)}$$
$$- \frac{.5\eta^\mu\{(p\sigma^\mu + .5\eta^\mu)\}\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\}\Gamma(2\mu+1)t^{3\mu}}{(\Gamma(\mu+1))^2\Gamma(3\mu+1)}$$
$$I_2(t) = 0.5 + \frac{0.5\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\}t^\mu}{\Gamma(\mu+1)} + \frac{0.5[\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\}^2 - \{p\sigma^\mu + .5\eta^\mu\}\eta^\mu]t^{2\mu}}{\Gamma(2\mu+1)}$$
$$- \frac{.5\eta^\mu\{(p\sigma^\mu + .5\eta^\mu)\}\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\}\Gamma(2\mu+1)t^{3\mu}}{(\Gamma(\mu+1))^2\Gamma(3\mu+1)}$$
$$R_2(t) = \frac{[p\sigma^\mu + 0.5\gamma^\mu]t^\mu}{\Gamma(\mu+1)} + \frac{[.5\gamma^\mu\{\eta^\mu - (\gamma^\mu + \sigma^\mu)\} - \sigma^\mu\{p\sigma^\mu + 0.5\gamma^\mu\}]t^{2\mu}}{\Gamma(2\mu+1)} \quad (5.5)$$

and so on.

6. Graphical Representation of Results

In this section the solution obtained is presented graphically. Figure 1 depict the variation of susceptible, infected and recovered populations for varying values of the fractional parameter μ. Effect of vaccination parameter p on the susceptible and recovered populations is shown through figure 2, which show the decrease in susceptible and increase in recovered population with the increase in vaccination. Figure 3 analyses the effect of rate of contact η on Susceptible and Infected population showing the decrease in $S(t)$ and increase in $I(t)$ with respect to the corresponding increase in η. Figure 4 show that increment in the recovery parameter γ results in decrease in infected population $I(t)$ and increase in recovered population $R(t)$.

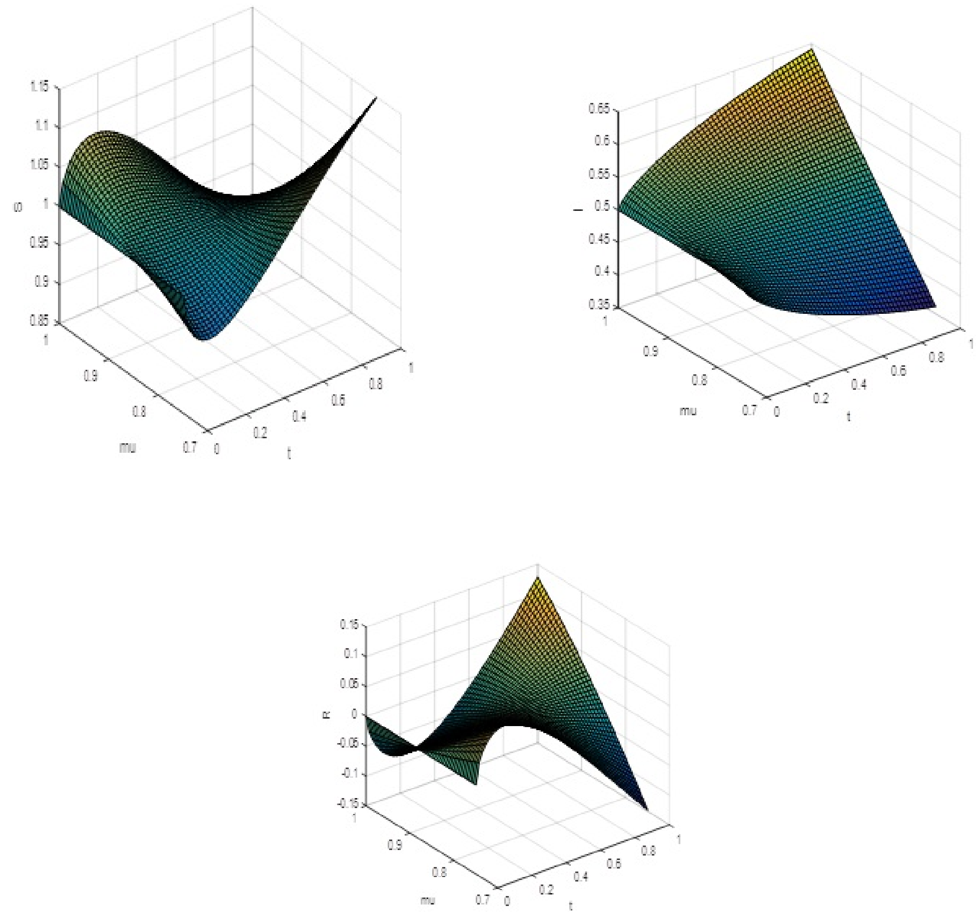

Figure 1: Variation of $S(t)$, $I(t)$, $R(t)$ with time and μ

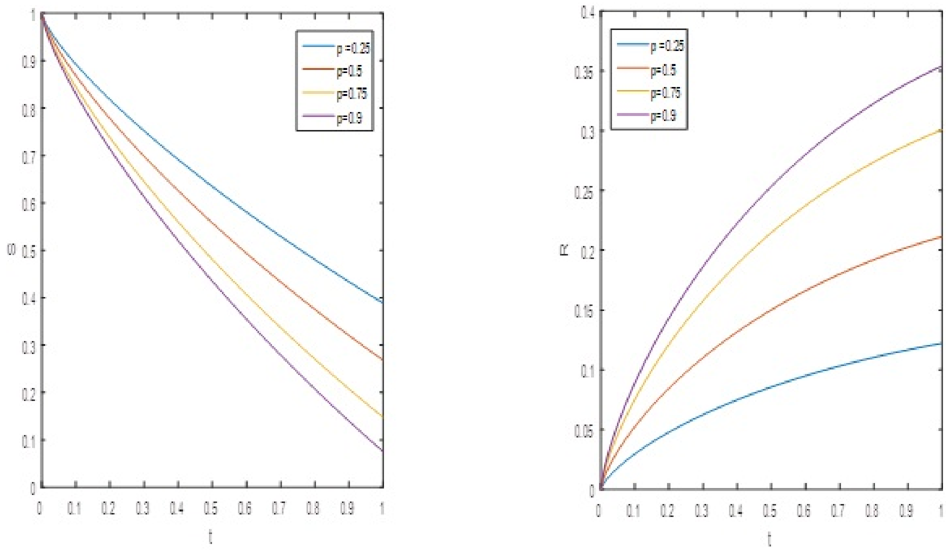

Figure 2: Effect of vaccination on susceptible population $S(t)$ and recovered population $R(t)$

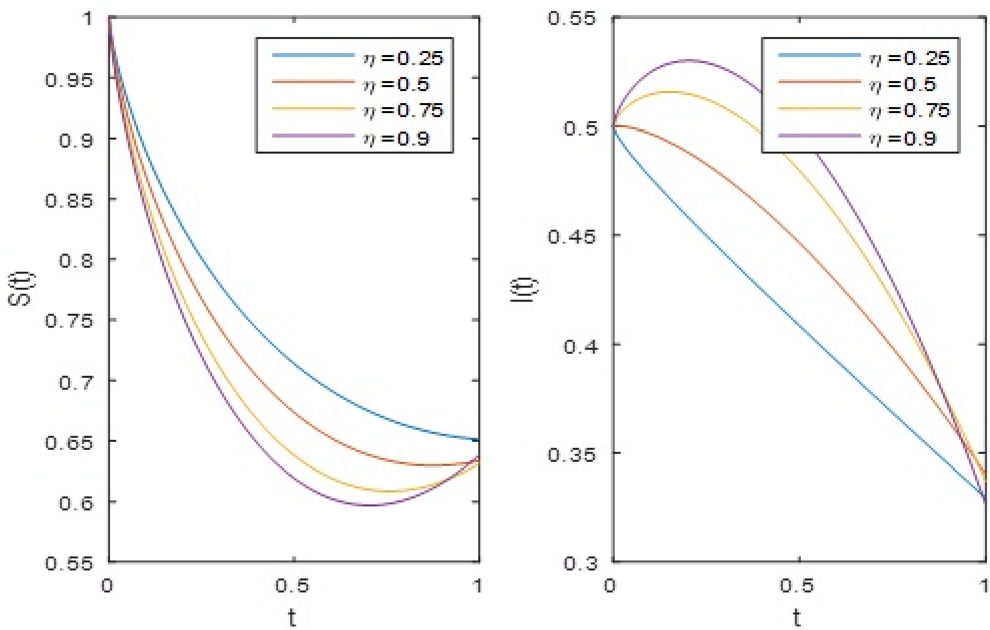

Figure 3: Effect of η on Susceptible and Infected population

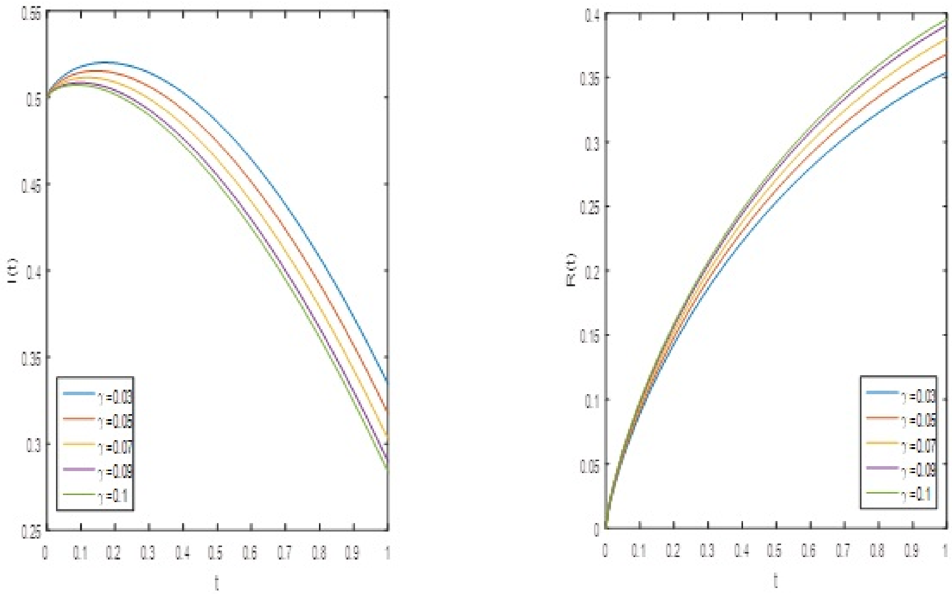

Figure 4: Effect of γ on infected population and Recovered population

7. Conclusion

In this paper, Reconstruction of Variational Iteration Method (RVIM) is used to solve systems of differential equations of fractional order representing the epidemic model of Childhood diseases. During Computation it has been realized that the method is a strong and reliable technique to solve linear as well as nonlinear systems of fractional differential equations. Good results were obtained using this method with less computation. It can be easily deduced from graphs that effect of vaccination on susceptible population is remarkable. Results obtained as an application of aforesaid technique are discussed in the form of graphs by changing various parameters of the given model. Thus RVIM is an efficient method and can be broadly applied to study various problems occurring in the field of Mathematics and Engineering.

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