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VARIATIONAL ITERATION METHOD FOR FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS - A UNIVERSAL APPROACH BY SUMUDU TRANSFORM

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Abstract: The important feature of this research paper is an extension to solve linear and nonlinear applications of fractional partial differential equations suggested by D. Ziane and M. H. Cherif for various values of α $(1 < \alpha \leq 2)$. The contemplated graphs show that the behavior of exact and approximate solution for different values of fractional order α. The effectiveness and convenience of the method is tested with the help of two illustrative examples. The fractional derivative is described in the Liouville-Caputo sense.

Keywords and Phrases: Fractional Calculus, Sumudu transform, Variational iteration method, Convergence of Variational iteration method, Linear and Nonlinear partial differential equations.

2020 Mathematics Subject Classification: 26A33, 65R10, 65M12, 35R11.

1. Introduction

In 2006, A. A. Kilbas et al. [14] published new book which is completely dedicated on theory and applications of fractional differential equations. In their innovation, they complemented on the concept of fractional differential equations through various books and journals. They suggested many new results on the theory of ordinary and partial differential equations. In systematic manner, they presented various results including the existence and uniqueness of solutions for the

Cauchy type and Cauchy problems involving nonlinear ordinary fractional differential equations, explicit solutions of linear differential equations and its corresponding initial value problems by their reduction to Volterra integral equations with the help of operational and compositional methods. Also, authors focused on different applications of the one and multi-dimensional Laplace, Mellin and Fourier integral transforms in finding closed-form solutions of ordinary and partial differential equations. Finally, they discussed sequential linear fractional differential equations including a generalization of the classical Frobenius method.

The theorems on existence and uniqueness for ordinary differential equations elaborated with special reference of Cauchy Type problems. They considered both linear and nonlinear fractional differential equations in one dimensional and vectorial cases. Explicit and numerical solutions obtained for fractional differential equations and boundary value problems with the reduction to Volterra integral equations upon compositional relations and operational calculus. Further they applied Laplace, Mellin and Fourier integral transforms on distinct applications for explicit solutions of linear differential equations involving Liouville, Caputo and Riesz fractional derivatives with constant coefficients. In the fields of partial differential equations, they used Laplace and Fourier integral transforms for obtaining closed form solutions of the Cauchy type and Cauchy problems for the fractional diffusion wave and evolution equations. Moreover, the authors investigated sequential and non-sequential fractional order linear differential equations as well as linear fractional order differential equations associated with the Riemann-Liouville and Caputo derivatives. Further they developed interesting generalizations of the classical Frobenius method for solving fractional differential equations with variable coefficients and explicit solutions of fractional differential equations. At the end, they systematically presented some important applications involving fractional models.

H. M. Srivastava [19] introduced some parametric and argument variations of the operators of fractional calculus and related special functions and integral transformation. In his work, author investigated and closely examined k-gamma function and the corresponding k-Pochhammer symbol and k-Laplace transform, the pathway integral version and the conformable or non-conformable version. The author concentrated on (k, s) extension of the operators of the traditional Riemann-Liouville fractional calculus and such other familiar operators like Liouville-Caputo fractional derivative operator, the Sumudu transform and the $\mathcal{P}_\delta$-version of the classical Laplace transform, post-quantum or the $(\mathcal{P}, q)$-version of the familiar basic quantum (or q) analysis, the parametric variation of the Bessel and related functions. Also, the author focused on repeated use of classical Laplace transform operator $\mathcal{L}$ in order to effectively solve initial value problems including ordinary and partial differential equations. Furthermore, H. M. Srivastava [20] presented a brief elementary and introductory survey of the theory of the derivative and integral operators of fractional calculus. The author used different applications especially in developing solutions of certain interesting families like Fractional (Relaxation-Oscillation) Ordinary Differential Equation, The Fractional (Diffusion-Wave) Partial Differential Equation, Generalized Fractional Kinetic equations by using the Laplace Transform and Sumudu Transform, Fractional Differential integral Operators based upon the Cauchy-Goursat Integral Formula. H. M. Srivastava [21] analyzed the widespread uses of the operators of fractional calculus that is both fractional integrals and fractional derivatives in the modeling and analysis. They commented on the large variety of applied scientific disciplines and real-world applications in physical, mathematical, biological, engineering and statistical sciences. The author presented the theory and applications of the fractional-calculus operators which depend upon the general Fox-Wright function and its extensively related and potentially useful Mittag-Leffler type functions. Also, they discussed the solutions of fractional Relaxation-Oscillation ordinary differential equation and fractional Diffusion-Wave partial differential equation.

In recent decades, in many areas of science and engineering, a large number of phenomena has been successfully modeled with the help of fractional derivatives and integrals. Those fields are fluid mechanics, viscoelasticity, biology and physics. The several researchers have applied various analytical and numerical methods to solve fractional ordinary differential equations, integral equations and fractional partial differential equations as per physical interest. Mostly further methods are employed on fractional differential and integral equations and those are: Adomian Decomposition Method (ADM), Variational Iteration Method (VIM), Fractional Difference Method (FDM), Differential Transform Method (DTM), Homotopy Perturbation Method (HPM) and Sumudu decomposition method (SDM). Also, a few classical solution methods include Laplace transform method, Sumudu transform method, Fractional Green's function method, Mellin transform method, method of orthogonal polynomials, Galerkin Method and least square polynomial method. Among these solution methods, the variational iteration method and the Adomian decomposition method are the most reliable methods for solution of fractional differential and integral equations, because they provide immediate and visible symbolic terms of analytic solutions and numerical approximate solutions to both linear and nonlinear differential equations without linearization or discretization [28].

In 1998, J. H. He was the first to introduce a very powerful analytical technique named as Variational iteration method (VIM) [8-11] for solving non-linear equations. The iterations found by this method are valid both for small parameter and also for very large parameter. Moreover, their first order approximations reach extreme accuracy. This method provides convergent successive approximate series solution and also may give the exact solution if such a solution exists. In some of the applications, it is not possible to obtain the exact solution. For these types of number cases, the approximations can be used for numerical purposes. This technique needs to construct a correctional functional for further identification by the variational theory along with involvement of Lagrange multiplier and the multiplier λ can be optimally recognized by variational theory. The application of restricted variations in correction functional prepares it much easier to calculate the multiplier. The initial approximation can be freely chosen with unknown constants which can be obtained through different methods.

As compared to other methods such as ADM, Galerkin method and HPM, VIM which have not specific demands for nonlinear terms. J. H. He [12] recently mentioned some effective modifications to find the value of Lagrange's multiplier. Now he suggested an alternative technology to construct the iteration formulation that is the well-known Laplace transform technology. Recently, many mathematicians have applied Variational iteration method and its modifications to solve fractional order partial differential equations [4, 5, 16, 18, 23, 26]. Nowadays, Sumudu transform has been successfully employed on various fractional differential equations and fractional partial differential equations for linear and nonlinear terms which produces approximate analytical series solution [6, 13, 17]. S. Vilu et al. [22] presented a new approach for solution of delay differential equations (DDEs) which is the combination of Sumudu transform and Variational iteration method (VIM). They discussed a new idea in finding the unknown Lagrange multiplier with the help of uncommon Sumudu transform alongside variational theory. Moreover, they reduced the complexity of computational work in comparison with the conventional approaches. M. Goyal et al. [7] applied mixture of Homotopy perturbation technique and Sumudu transform for solving time-fractional vibration equation. The authors also commented on their technique that reduces time as well as size of the computation when exact solution of a nonlinear differential equation is unknown.

In the early 1990s, K. Li et al. [15] introduced the Sumudu transform along with interesting advantages over other integral transforms (Fourier, Laplace, etc.). Especially the novelty of Sumudu transform provides convenience when solving differential equations and which is 'unity' feature. The Sumudu transform is similar to Laplace transform in terms of several examples shows its unique character. The integral transform method originated in calculating a functions frequency component. Today, most of them lost its original physics background, turning out to pure mathematical tools appearing in all kind of engineering problems. Among them, the Fourier transform is concerned with the primitive Fourier series. It has great power in the field of frequency analysis. The Fourier's transform with complex domain only considers the positive area which provides Laplace transform. It solves various differential equations. Among different integral transform, the Sumudu transform makes up the defects of the Fourier and Laplace transform for making expressions form more intuitive. Meanwhile, we can realize directly the units of results without complete the whole solutions when using the Sumudu transform. Thus, we can transform from one domain into another much more conveniently for the engineering problem itself.

S. A. Zahedi [27] determined analytical solution of time dependent nonlinear partial differential equation using HAM, HPM and VIM. Among various analytical methods like HPM, ADM, HAM, DTM, etc., VIM has been successfully employed to get the analytical solution of time-fractional (linear or nonlinear) partial differential equations because this method gives successive rapidly convergent approximations of the exact solution without any restrictive assumptions or transformations causing changes in the physical properties of the problem. Increase in numbers of iterations leads to the explicit solution for the problem. Furthermore, the VIM does not need small parameters in the equation so that it overcomes the limitations that have risen in traditional perturbation methods.

The fundamental theme of this paper is to use coupling of correction functional of Variational Iteration Method (VIM) and Sumudu transform. The authors D. Ziane and M. H. Cherif [28] motivate us to implement Variational Iteration Method coupled with Sumudu transform to solve linear and nonlinear partial differential equations with time-fractional derivative of order α. In this paper, the Sumudu transform is accepted to construct the variational iteration algorithm instead of the traditional VIM. This novel combination may also be defined as He-Sumudu method in literature. The new integral transform-Sumudu transform was applied by G. K. Watugala [24] for solving differential equations in control engineering problems.

In present paper, we have taken the literature review in the first section. We have defined basic definitions of fractional calculus, Sumudu transform and properties of Sumudu transform in the second section. We have elaborated the scheme of traditional Variational Iteration Method (VIM) in the third section. There is a description of fractional Variational Iteration transform method in the fourth section. We have explained convergence analysis of the VIM in the fifth section. We have described comparative study of approximate and exact solution of the FPDEs in the sixth section and finally we briefly concluded summary of the paper in the seventh section.

2. Preliminaries

The basic definitions of the fractional calculus, Sumudu transform and a few properties of Sumudu transform are discussed in this section and these definitions are useful for defining the proposed methodology of Sumudu Transform-Variational iteration method.

Definition 2.1. [28] *Let* $\Omega = [a, b](-\infty < a < b < \infty)$ *be a finite interval on the real axis* $\mathbb{R}$ *: The Riemann-Liouville fractional integrals* $I^{\alpha}_{a+}f$ *of order* $\alpha \in \mathbb{C}$ *(*$Re(\alpha) > 0$*) is defined by*

$$I^{\alpha}_{a+}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-\xi)^{\alpha-1}f(\xi)d\xi, t > 0, Re(\alpha) > 0, \tag{1}$$

$$I^{0}_{a+}f(t) = f(t). \tag{2}$$

Definition 2.2. [28] *Let* $Re(\alpha) > 0$ *and let n=[Re(*α*)]+1. If* $f(t) \in AC^n[a, b]$ *then the Liouville-Caputo fractional derivatives* $D^{\alpha}_t f(t)$ *of order* α *exist almost everywhere on [a, b]. If* $\alpha \notin \mathbb{N}$, $D^{\alpha}_t f(t)$ *is defined by*

$$D^{\alpha}_t f(t) = J^{m-\alpha}D^{\alpha}_t f(t) = \frac{1}{\Gamma(m-\alpha)}\int_a^t (t-\xi)^{m-\alpha-1}f^{(m)}(\xi)d\xi, \tag{3}$$

for $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $t > 0$.
If $\alpha \notin \mathbb{N}$, *we obtain* $D^{\alpha}_t f(t) = f^n(t)$.
Moreover, the operator D^{α}_t satisfies the following basic properties.

Lemma 2.1. [18] *Let* $m - 1 < \alpha \leq m$, $m \in \mathbb{N}$, $f \in \mathbb{C}^m_{\mu}$, $\mu \geq -1$ *and* $\gamma > \alpha - 1$ *then*

1. $D^{\alpha}_t D^{\beta}_t f(t) = D^{\alpha+\beta}_t f(t)$
2. $D^{\alpha}_t x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}x^{\gamma-\alpha}$
3. $D^{\alpha}_t I^{\alpha}_t f(t) = f(t)$
4. $I^{\alpha}_t D^{\alpha}_t f(t) = f(t) - \sum_{k=0}^{m} f^{(k)}(0^+)\left(\frac{t^k}{k!}\right), t > 0$

In the present work, we applied Liouville-Caputo fractional derivative on traditional initial and boundary conditions in the formulation of the physical problems.

Definition 2.3. [2] *The Sumudu transform is defined over the set of functions*

$$A = \{f(t) \colon \exists M, \tau_1, \tau_2 > 0, f(t) < Me^{\frac{t}{\tau_j}}, if t \in (-1)^j \times [0, \infty)\}, \tag{4}$$

which is defined through definite integral by using the following formula:

$$F(u) = S[f(t)] = \frac{1}{u}\int_0^{\infty} e^{\frac{-t}{u}} f(t)dt, u \in (-\tau_1, \tau_2). \tag{5}$$

There are various properties of the Sumudu transform are included in [2]. Some selected properties of the Sumudu transform are as follows.
1.$S\{1\} = 1,$
2.$S\{t^n\} = u^n\Gamma(n+1), n > 0,$
3.$S\{f(t) \pm g(t)\} = S\{f(t)\} \pm S\{g(t)\}.$

Definition 2.4. [1] *The Sumudu transform of Liouville-Caputo fractional derivative is defined as follows:*

$$S\{D_t^{\alpha} f(x,t)\} = u^{-\alpha} S\{f(x,t)\} - \sum_{k=0}^{m-1} u^{-\alpha+k} f^{(k)}(x,0), \ldots m-1 < \alpha \leq m. \tag{6}$$

Definition 2.5. [3] *The Mittag-Leffler function in one parameter which is a generalization of exponential function is defined as*

$$E_{\alpha}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+1)} (\alpha > 0). \tag{7}$$

Definition 2.6. [14] *A two-parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)} (\alpha > 0, \beta > 0). \tag{8}$$

3. Basics of Variational Iteration Method (VIM)

This section deals with the idea of the variational iteration method [25] that is based on constructing a correction functional by a general Lagrange multiplier λ. The improvement of correction solution with respect to the initial approximation or to the trial function is done by using the selection of Lagrange multiplier λ. The basic idea of the variational method is described as follows. Consider the following nonlinear equation:

$$Lu(t) + Nu(t) = g(t) \tag{9}$$

Where L is a linear operator, N is a nonlinear operator and *g(t)* is a known analytic function. According to the variational iteration method mentioned by J. H. He, we can construct the following correction functional:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)[Lu_n(\tau) + N\tilde{u_n}(\tau) - g(t)]d\tau \tag{10}$$

Where λ is a general Lagrange multiplier which can be identified optimally via variational theory and $\tilde{u}_n(\xi)$ is considered as a restricted variation which means $\delta\tilde{u}_n(\xi) = 0$. Using this method, we first find the Lagrangian multiplier that will be identified optimally via integration by parts. With available value of λ, then several approximations $u_n(x,t)$, $n \geq 0$ follows immediately. The exact solution may be constructed by using limit and which is the limiting value of approximate solution. Ultimately, the exact solution is obtained by using series approximations as $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

4. Fractional Variational Iteration Transform Method (FVITM)

In this section, the extension of available method in the literature is suggested by Ziane and Cherif [28] and it has been used for the utilization of another integral transform which is Sumudu transform. Many mathematicians have successfully applied such an integral transform in their work. We consider the general form of linear and nonlinear partial differential equations with time-fractional derivative of order α.

$$D_t^\alpha u(x,t) + Ru(x,t) + Nu(x,t) = f(x,t), 1 < \alpha \leq 2 \tag{11}$$

where $D_t^\alpha = \frac{\partial^\alpha}{\partial t^\alpha}$ =fractional order partial derivative,
subject to the initial conditions,

$$u(x,0) = h(x), u_t(x,0) = g(x). \tag{12}$$

Applying Sumudu transform on both sides of equation (11)
$S\{D_t^\alpha u(x,t)\} + S\{Ru(x,t)\} + S\{Nu(x,t)\} = S\{f(x,t)\}$
Using the definition 2.4, we have
$\frac{S\{u(x,t)\}}{u^\alpha} - \frac{u(x,0)}{u^\alpha} - \frac{u_t(x,0)}{u^{\alpha-1}} = S\{f(x,t)\} - S\{Ru(x,t)\} - S\{Nu(x,t)\}$
Also, by using initial conditions (12), we obtain

$$S\{u(x,t)\} = h(x) + ug(x) + u^\alpha S\{f(x,t)\} - u^\alpha S\{Ru(x,t)\} - u^\alpha S\{Nu(x,t)\} \tag{13}$$

Operating with the inverse Sumudu transform on both sides of equation (13), we get

$$\begin{aligned} u(x,t) = h(x) + g(x)t + S^{-1}\{u^\alpha S\{f(x,t)\}\} - S^{-1}\{u^\alpha S\{Ru(x,t)\}\} \\ - S^{-1}\{u^\alpha S\{Nu(x,t)\}\} \end{aligned} \tag{14}$$

Applying $\frac{\partial}{\partial t}$ on both sides of equation (14), we have
$\frac{\partial u}{\partial t} + \frac{\partial}{\partial t} S^{-1}\{u^\alpha S\{Ru(x,t)\}\} + \frac{\partial}{\partial t} S^{-1}\{u^\alpha S\{Nu(x,t)\}\} - \frac{\partial}{\partial t} S^{-1}\{u^\alpha S\{f(x,t)\}\} - g(x) =$

0

The correction functional of the variational iteration method is defined by

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \tau} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Ru_n(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Nu_n(x,t)\}\} - \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{f(x,t)\}\} - g(x) \right] d\tau \tag{15}$$

By Considering the given initial conditions (12), we can select the 0^{th} approximation $u_0 = h(x) + g(x)t$. Further we use this selection into (15) we obtain the following successive approximations

$u_0 = h(x) + g(x)t$

$u_1 = u_0 - \int_0^t \left[\frac{\partial u_0}{\partial \tau} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Ru_0(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Nu_0(x,t)\}\} - \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{f(x,t)\}\} - g(x) \right] d\tau$

$u_2 = u_1 - \int_0^t \left[\frac{\partial u_1}{\partial \tau} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Ru_1(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Nu_1(x,t)\}\} - \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{f(x,t)\}\} - g(x) \right] d\tau$

$u_3 = u_2 - \int_0^t \left[\frac{\partial u_2}{\partial \tau} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Ru_2(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Nu_2(x,t)\}\} - \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{f(x,t)\}\} - g(x) \right] d\tau$

$\vdots$

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \tau} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Ru_n(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{Nu_n(x,t)\}\} - \frac{\partial}{\partial \tau} S^{-1}\{u^\alpha S\{f(x,t)\}\} - g(x) \right] d\tau \tag{16}$$

The $(n+1)^{th}$ order successive approximations produce the exact solution in the closed form $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. Where the approximate series solution is written as $u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$.

5. Convergence Analysis

Theorem 5.1. [25] *Assume that $y(t), y_i(t) \in C[0,T], i = 1, 2 \ldots,$ Then from equation (16), we get the true solution $y(t)$ of solution sequences converge at the problem (11) to (12).*

6. Illustrative Applications

This section illustrates two numerical examples by applying Variational Iteration Method (VIM) coupled with Sumudu transform for the Liouville-Caputo fractional derivative to solve linear and nonlinear time-fractional partial differential equations.

Example 6.1. We consider the linear time-fractional partial differential equation [28]

$$D_t^{\alpha} u = u_{xx} - 3u, 1 < \alpha \leq 2, 0 < x < \pi, t > 0 \tag{17}$$

subject to the initial conditions,

$$u(x,0) = 0, u_t(x,0) = 2cosx. \tag{18}$$

After implementation of Sumudu transform and Inverse Sumudu transform along with definition 2.4, we can construct the iteration formula by using equation (17) to (18), we have

$$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \tau} - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{u_{nxx}(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{3u_n(x,t)\}\} - 2cosx\right] d\tau$$

Also, we obtain the following approximations

$$u_0 = cosx(2t)$$

$$u_1 = u_0 - \int_0^t \left[\frac{\partial u_0}{\partial \tau} - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{u_{0xx}(x,t)\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{3u_0(x,t)\}\} - 2cosx\right] d\tau$$

$$u_1 = cosx(2t) - \int_0^t \left[\frac{\partial}{\partial \tau}(cosx(2t)) - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{u_{xx}(cosx(2t))\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{3cosx(2t)\}\} - 2cosx\right] d\tau$$

$$u_1 = cosx(2t) - \int_0^t \left[2cosx - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{(-cosx(2t))\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{3cosx(2t)\}\} - 2cosx\right] d\tau$$

$$u_1 = cosx(2t) - \int_0^t \left[-\frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{(-cosx(2t))\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} S\{3cosx(2t)\}\}\right] d\tau$$

$$u_1 = cosx(2t) - \int_0^t \left[\frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} 2ucosx\}\} + \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} 6ucosx\}\}\right] d\tau$$

$$u_1 = cosx(2t) - \int_0^t \left[\frac{\partial}{\partial \tau} 2cosx \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right) + \frac{\partial}{\partial \tau} 6cosx \left(\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)\right] d\tau$$

$$u_1 = cosx(2t) - 8cosx \frac{1}{\Gamma(\alpha+2)} \int_0^t \left[\frac{\partial}{\partial \tau} t^{\alpha+1}\right] d\tau$$

$$u_1 = cosx(2t) - 8cosx \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$$

$$u_1 = cosx \left(2t - 8\frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right)$$

$$u_2 = cosx \left(2t - 8\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 32\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right)$$

$$u_3 = cosx \left(2t - 8\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 32\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 128\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right)$$

$$\vdots$$

$u_n = cosx\left(2t - 8\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + 32\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - 128\frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots + (-1)^n 2^{2n+1}\frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)}\right)$

The exact solution can be written as

$u(x,t) = \lim_{n\to\infty} u_n(x,t)$

whereas the approximate series solution is defined as

$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = cosx\sum_{k=0}^{\infty}(-1)^k 2^{2k+1}\frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)}$

In particular for $\alpha = 2$, the exact solution is given by

$u(x,t) = \lim_{n\to\infty} u_n(x,t) = cosx\sum_{k=0}^{\infty}(-1)^k 2^{2k+1}\frac{t^{2k+1}}{\Gamma(2k+2)}$

$u(x,t) = cosx\sum_{k=0}^{\infty}(-1)^k \frac{(2t)^{2k+1}}{(2k+1)!}$

$u(x,t) = cosxsin2t$

The behaviour of solution of $u(x,t)$ for different values of (x,t) is included in Table 1. The approximate solution of FVITM is recorded for distinct values of $\alpha = 1.5, 1.75, 2$ by taking first four terms. The smaller number of absolute errors mentions that the results agreed well with the exact solutions.

Table 1: Comparison of exact solution with numerical solution of SDM for distinct values of x and t. (for $\alpha = 1.5, 1.75, 2$)

x	t	SDM			Exact solution	Absolute Errors
		$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$		for $\alpha = 2$
$\frac{\pi}{6}$	$\frac{\pi}{6}$	0.569161	0.669680	0.749996	0.75	3.57900×10^{-6}
$\frac{\pi}{4}$	$\frac{\pi}{4}$	0.455681	0.580988	0.706996	0.707107	1.10945×10^{-4}
$\frac{\pi}{3}$	$\frac{\pi}{3}$	0.211732	0.322825	0.431986	0.433013	1.026969×10^{-3}
$\frac{2\pi}{3}$	$\frac{2\pi}{3}$	5.684062	1.924487	0.431986	0.433013	1.026969×10^{-3}
$\frac{3\pi}{4}$	$\frac{3\pi}{4}$	17.160899	6.326415	0.706996	0.707107	1.10945×10^{-4}
$\frac{5\pi}{6}$	$\frac{5\pi}{6}$	40.962588	16.612834	0.749996	0.75	3.57900×10^{-6}

In Figures 1, 2 and 3, we have plotted the surface of $u(x,t)$ corresponding to the values $\alpha = 2, 1.5$ for FVITM; the three figures indicate that the similarity among FVITM by Sumudu transform and FVITM by Laplace transform [28]. We can observe that the approximate solution and exact solution for $\alpha = 2$ are very close to each other. According to convergence analysis theorem of variational iteration method, it is clear that the approximate series solution is excellently convergent with the exact solution. Figure 4 indicates two dimensional plots for comparison of approximate solution for $\alpha = 2, 1.75, 1.5$.

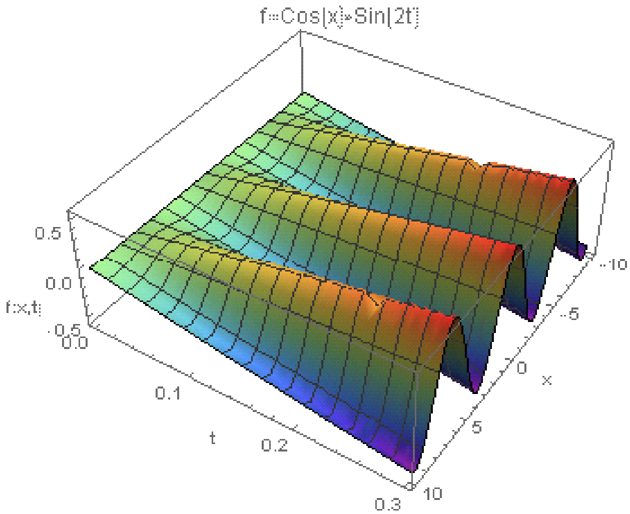

Figure 1: shows that the behaviour of exact solution for $\alpha = 2$.

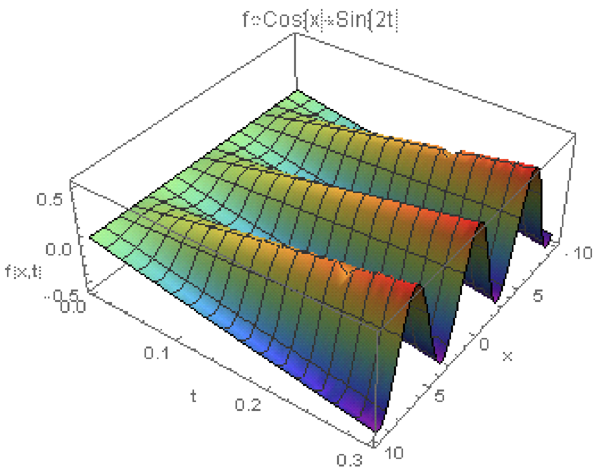

Figure 2: shows that the behaviour of approximate solution for $\alpha = 2$.

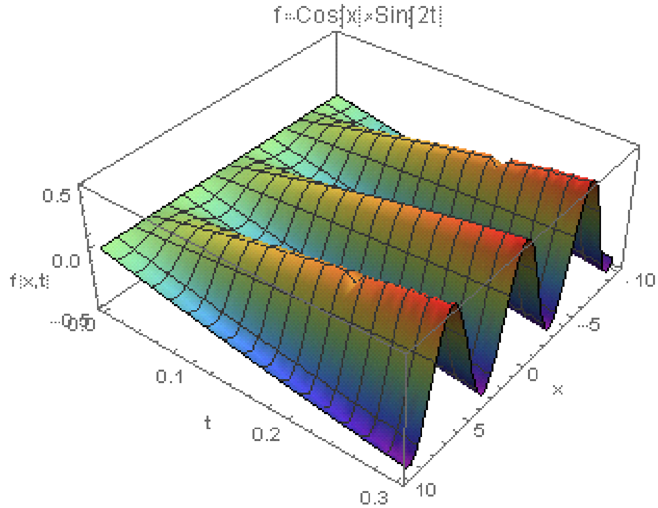

Figure 3: shows that the behaviour of approximate solution for $\alpha = 1.5$.

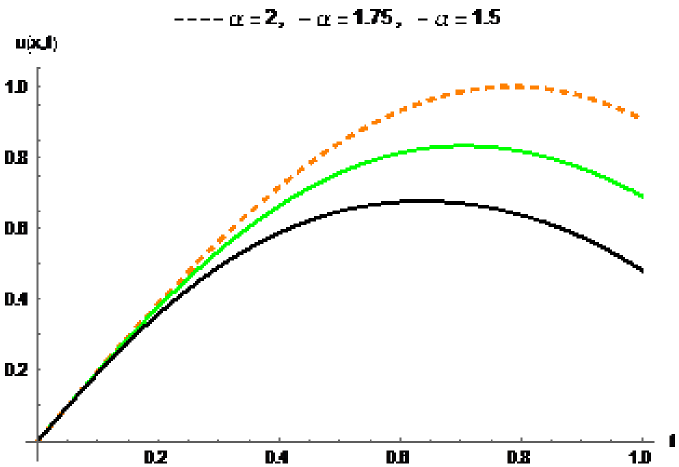

Figure 4: shows that the behaviour of approximate solution for $\alpha = 2, 1.75, 1.5$.

Example 6.2. We consider the linear time-fractional partial differential equation [28]

$$D_t^{\alpha} u = \frac{x^2}{2} u_{xx}, 1 < \alpha \leq 2, 0 < x < 1, t > 0 \tag{19}$$

subject to the initial conditions,

$$u(x,0) = 0, u_t(x,0) = x^2. \tag{20}$$

After implementation of Sumudu transform and Inverse Sumudu transform along with definition 2.4, we can construct the iteration formula by using equation (19) to (20), we have

$u_{n+1} = u_n - \int_0^t \left[\frac{\partial u_n}{\partial \tau} - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} \frac{x^2}{2} S\{u_{nxx}(x,t)\}\} - x^2 \right] d\tau$

Also, we obtain the following approximations

$u_0 = x^2 t$

$u_1 = u_0 - \int_0^t \left[\frac{\partial u_0}{\partial \tau} - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} \frac{x^2}{2} S\{u_{0xx}(x,t)\}\} - x^2 \right] d\tau$

$u_1 = x^2 t - \int_0^t \left[x^2 - \frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} \frac{x^2}{2} S\{2t\}\} - x^2 \right] d\tau$

$u_1 = x^2 t - \int_0^t \left[-\frac{\partial}{\partial \tau} S^{-1}\{u^{\alpha} \frac{x^2}{2} 2u\} \right] d\tau$

$u_1 = x^2 t - \int_0^t \left[-\frac{\partial}{\partial \tau} x^2 S^{-1}\{u^{\alpha+1}\} \right] d\tau$

$u_1 = x^2 t - \int_0^t \left[-\frac{\partial}{\partial \tau} x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right] d\tau$

$u_1 = x^2 t + x^2 \frac{1}{\Gamma(\alpha+2)} \int_0^t \left[\frac{\partial}{\partial \tau} t^{\alpha+1} \right] d\tau$

$u_1 = x^2 t + x^2 \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}$

$u_1 = x^2 \left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} \right)$

$u_2 = x^2 \left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} \right)$

$u_3 = x^2 \left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} \right)$

$\vdots$

$u_n = x^2 \left(t + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots + \frac{t^{n\alpha+1}}{\Gamma(n\alpha+2)} \right)$

The approximate series solution is defined as

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t) = x^2 \sum_{k=0}^{\infty} \frac{t^{k\alpha+1}}{\Gamma(k\alpha+2)} = x^2 E_{\alpha,2}(t),$$

where $E_{\alpha,2}(t)$ is Mittag-leffler function [20] which is defined in the definition 2.6. In particular for $\alpha = 2$, the exact solution is given by

$$u(x,t) = \lim_{n \to \infty} u_n(x,t) = x^2 \sum_{k=0}^{\infty} \frac{t^{2k+1}}{\Gamma(2k+2)}$$

$$u(x,t) = x^2 \sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}$$

$$u(x,t) = x^2 sinht$$

The behaviour of solution of $u(x,t)$ for different values of (x,t) is included in Table 2. The approximate solution of FVITM is recorded for distinct values of $\alpha = 1.5, 1.75, 2$ by taking first four terms. The smaller number of absolute errors mentions that the results agreed well with the exact solutions.

Table 2: Comparison of exact solution with numerical solution of SDM for distinct values of x and t. (for $\alpha = 1.5, 1.75, 2$)

x	t	SDM			Exact solution	Absolute Errors
		$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$		for $\alpha = 2$
0.2	0.2	8.215860×10^{-3}	8.728344×10^{-3}	8.053440×10^{-3}	8.053440×10^{-3}	0.0
0.4	0.4	0.068944	0.066961	0.065720	0.065720	0.0
0.6	0.6	0.247449	0.236679	0.229195	0.229195	0.0
0.8	0.8	0.631627	0.594953	0.568388	0.568388	0.0

In Figures 5, 6 and 7, we have plotted the surface of $u(x,t)$ corresponding to the values $\alpha = 2, 1.5$ for FVITM; the three figures indicate that the similarity among FVITM by Sumudu transform and FVITM by Laplace transform [28]. We can observe that the approximate solution and exact solution for $\alpha = 2$ are very close to each other. According to convergence analysis theorem of variational iteration method, it is clear that the approximate series solution is excellently convergent with the exact solution.

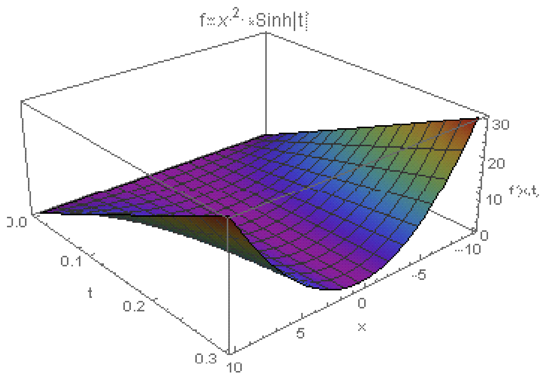

Figure 5: shows that the behaviour of exact solution for $\alpha = 2$.

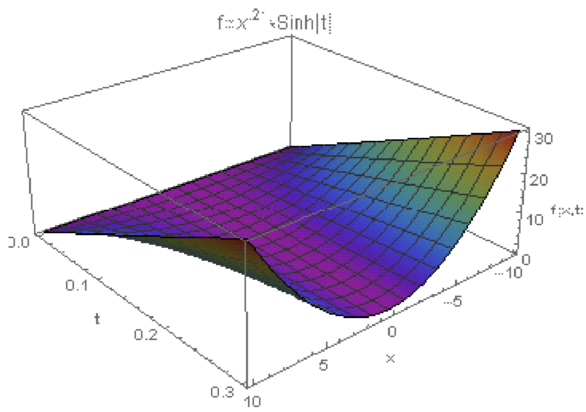

Figure 6: shows that the behaviour of approximate solution for $\alpha = 2$.

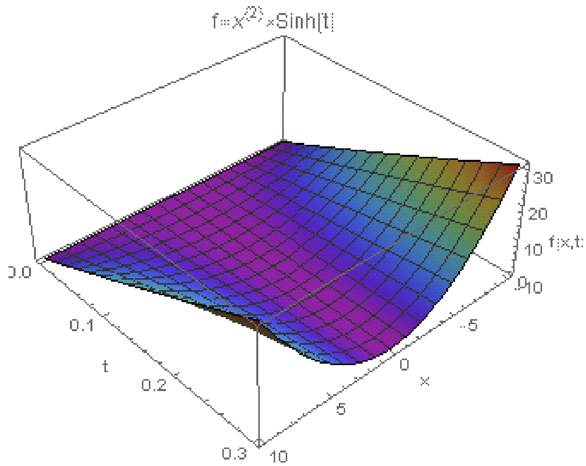

Figure 7: shows that the behaviour of approximate solution for $\alpha = 1.5$.

Figure 8 indicates two dimensional plots for comparison of approximate solution for $\alpha = 2, 1.75, 1.5$.

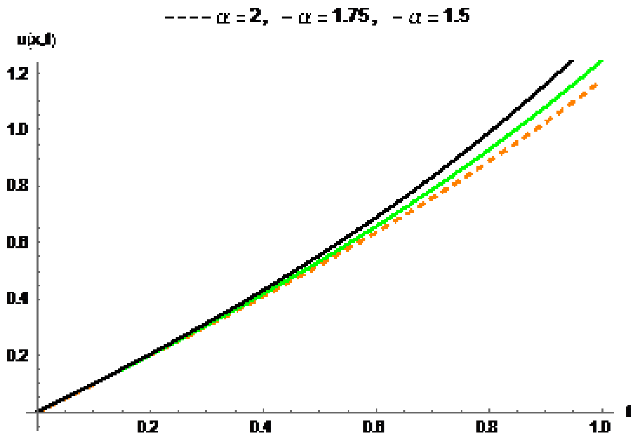

Figure 8: shows that the behaviour of approximate solution for $\alpha = 2, 1.75, 1.5$.

7. Conclusion

In this paper, we successfully applied the technique of Variational iteration method [VIM] coupling with Sumudu transform for solving linear and nonlinear partial differential equations with time-fractional derivative. There is no effective method to identify the Lagrange multipliers especially for the FDEs. The approach applied in this paper is simple and straightforward because it is not necessary to calculate the value of Lagrange multipliers with the help of integration by parts rule. We can observe the numerical results, it is clear that the fractional variational iteration transform method produces very accurate approximate solutions using only a few iterations. The proposed technique has provided more realistic series solutions which converges very rapidly with exact solution. As compared to the other previous version of VIM, this modification is much better. This paper shows that the Sumudu transform technique is extremely effective to construct the VIM like Laplace transform. In future, the extension of this technique can be seen in fractional partial differential equations in various areas of science and engineering.

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