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AN EXTENSION OF BILATERAL GENERATING FUNCTIONS OF GEGENBAUER POLYNOMIALS

K.P. Samanta, B.C. Chandra* & C.S. Bera**
Department of Mathematics, Bengal Engineering and Science University, Shibpur,
P.O.- Botanic Garden, Howrah, Pin-711103, W.B., India.
E-mail Address : kalipadasamanta2010@gmail.com
* Sree Vidyaniketan; 9A, J.M. Avenue, Kolkata - 700006, W.B, India.
E-mail Address : chandrasekharbera75@gmail.com
** Department of Mathematics, Bagnan College, Bagnan, Howrah-711303, W.B., India.
E-mail Address : chandrasekharbera75@gmail.com

Abstract : In this note, we have obtained a novel extension of a bilateral generating relations involving modified Gegenbauer polynomials, $C_n^{\lambda+n}(x)$ from the existence of quasi-bilinear generating relation by group-theoretic method.

1. Introduction:

The Gegenbauer polynomials, $C_n^\lambda(x)$ is defined by [2]:

$$C_n^\lambda(x) = \sum_{p=0}^{[\frac{n}{2}]} (-1)^p \; \frac{(\lambda)_{n-p}(2x)^{n-2p}}{p! \; (n-2p)!}.$$

In[1], the quasi bilateral generating function is defined by

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; p_n^{(\alpha)}(x) \; q_m^{(n)}(u) \; w^n,$$

where a_n,the coefficients are quite arbitrary and $p_n^{(\alpha)}(x), q_m^{(n)}(u)$are two special functions of orders n, m and of parameters α and n respectively.If $q_m^{(n)}(u) \equiv p_m^{(n)}(u)$,the generating relation is known as quasi bilinear.

The aim at presenting this note is to prove the existence of a more general generating relation from the existence of a quasi-bilinear generating relation involving modified Gegenbauer polynomials by group theoretic method. In [3], authors have proved the following theorem on bilateral generating functions involving $C_n^{\lambda+n}(x)$, a modified form of $C_n^\lambda(x)$ by group-theoretic method.

Theorem 1: If there exists a unilateral generating relation of the form

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$$G(x,w) = \sum_{n=0}^{\infty} a_n \, C_n^{\lambda+n}(x) \, w^n \tag{1.1}$$

then

$$\frac{(1-w)^{\lambda+\frac{1}{2}}}{(1-w+wx^2)^{\lambda}} \, G\left(\frac{x}{(1-w+wx^2)^{\frac{1}{2}}}, \frac{wy(1-w)}{(1-w+wx^2)^{\frac{3}{2}}}\right) = \sum_{n=0}^{\infty} w^n \, \sigma_n(x,y), \tag{1.2}$$

where

$$\sigma_n(x,y) = \sum_{k=0}^{n} a_k \, \frac{(\frac{k+1}{2})_{n-k}(\frac{k+2}{2})_{n-k}}{(n-k)!(1-\lambda-k)_{n-k}} \, C_{2n-k}^{\lambda-n+2k}(x) \, y^k.$$

The importance of the above theorem lies in the fact that whenever one knows a generating relation of the form (1.1) then the corresponding bilateral generating relation can at once be written down from (1.2). So one can get a large number of bilateral generating relations by attributing different suitable values to a_n in (1.1).

In the present paper, we have obtained the following extension (Theorem 2) of the Theorem 1 from the existence of quasi bilinear generating relation.

Theorem 2: If there exists a quasi-bilinear generating relation of the following form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \, C_n^{\lambda+n}(x) C_m^n(u) \, w^n \tag{1.3}$$

then

$$\begin{aligned} &(1-2w)^{\lambda+\frac{1}{2}}\{1-2w(1-x^2)\}^{-\lambda}(1-2w)^{-\frac{m}{2}} \\ &\times \, G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{u}{(1-2w)^{\frac{1}{2}}}, \frac{wy}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right) \qquad (1.4) \\ &= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n \frac{w^{n+p+q}}{p!q!} \, 2^{q-p} \frac{(n)_q(n+1)_{2p}}{(1-\lambda-n)_p} C_{n+2p}^{\lambda+n-p}(x) \, C_m^{n+q}(u) y^n. \end{aligned}$$

2. Proof of the theorem

For the Gegenbauer polynomials, we consider the following operators[3,4]:

$$R_1 = x(1-x^2)\frac{y^2}{z^3}\frac{\partial}{\partial x} + (1-3x^2)\frac{y^3}{z^3}\frac{\partial}{\partial y} - \frac{2x^2\, y^2}{z^2}\frac{\partial}{\partial z} + \frac{y^2}{z^3}$$

and

$$R_2 = ut\frac{\partial}{\partial u} + 2t^2\frac{\partial}{\partial t} + mt$$

such that

$$R_1\left(C_n^{\lambda+n}(x)\ y^n z^\lambda\right) = \frac{(n+1)(n+2)}{2(1-\lambda-n)}\ C_{n+2}^{\lambda+n-1}(x)\ y^{n+2} z^{\lambda-3}, \tag{2.1}$$

$$R_2\left(C_m^n(u)\ t^n\right) = 2nC_m^{n+1}(u)t^{n+1} \tag{2.2}$$

and

$$e^{wR_1}f(x,y,z) = \{1-2w\frac{y^2}{z^3}\}^{\frac{1}{2}}$$
$$\times f\left(\frac{x}{\{1-2w(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}}, \frac{y(1-2w\frac{y^2}{z^3})}{\{1-2w(1-x^2)\frac{y^2}{z^3}\}^{\frac{3}{2}}}, \frac{z(1-2w\frac{y^2}{z^3})}{\{1-2w(1-x^2)\frac{y^2}{z^3}\}}\right) \tag{2.3}$$

$$e^{wR_2}f(u,v) = (1-2wv)^{-\frac{m}{2}}\ f\left(\frac{u}{(1-2wv)^{\frac{1}{2}}}, \frac{v}{(1-2wv)}\right). \tag{2.4}$$

Let us now consider the generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ C_n^{\lambda+n}(x) C_m^n(u)\ w^n. \tag{2.5}$$

Replacing w by wvy and multiplying both sides of (2.5) by z^λ, we get

$$z^\lambda G(x,u,wvy) = \sum_{n=0}^{\infty} a_n\ \left(C_n^{\lambda+n}(x) y^n\ z^\lambda\right)\left(C_m^n(u)\ v^n\right) w^n. \tag{2.6}$$

Now operating $e^{wR_1}e^{wR_2}$ on both sides of (2.6), we get

$$e^{wR_1}\ e^{wR_2}\left[z^\lambda G(x,u,wvy)\right] = e^{wR_1}\ e^{wR_2}\left[\sum_{n=0}^{\infty} a_n\ \left(C_n^{\lambda+n}(x)y^n\ z^\lambda\right)\left(C_m^n(u)\ v^n\right)w^n\right]. \tag{2.7}$$

Now the left member of (2.7), with the help of (2.3) and (2.4), becomes

$$\left(1-2w\frac{y^2}{z^3}\right)^{\lambda+\frac{1}{2}}\left\{1-2w(1-x^2)\frac{y^2}{z^3}\right\}^{-\lambda}(1-2wv)^{-\frac{m}{2}}\ z^\lambda$$
$$\times\ G\left(\frac{x}{\{1-2w(1-x^2)\frac{y^2}{z^3}\}^{\frac{1}{2}}}, \frac{u}{(1-2wv)^{\frac{1}{2}}}, \frac{wvy(1-2w\frac{y^2}{z^3})}{\{1-2w(1-x^2)\frac{y^2}{z^3}\}^{\frac{3}{2}}(1-2wv)}\right). \tag{2.8}$$

The right member of (2.7), with the help of (2.1) and (2.2), becomes

$$\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\frac{w^{n+p+q}}{p!q!}\ 2^{q-p}\frac{(n)_q(n+1)_{2p}}{(1-\lambda-n)_p}C_{n+2p}^{\lambda+n-p}(x)\ C_m^{n+q}(u)y^{n+2p}z^{\lambda-3p}v^{n+q}. \tag{2.9}$$

Now equating both members, and then substituting $\frac{y^2}{z^3}=1, v=1$, we get

$$\left(1-2w\right)^{\lambda+\frac{1}{2}}\left\{1-2w(1-x^2)\right\}^{-\lambda}(1-2w)^{-\frac{m}{2}}$$
$$\times\ G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{u}{(1-2w)^{\frac{1}{2}}}, \frac{wy}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} a_n\frac{w^{n+p+q}}{p!q!}\ 2^{q-p}\frac{(n)_q(n+1)_{2p}}{(1-\lambda-n)_p}C_{n+2p}^{\lambda+n-p}(x)\ C_m^{n+q}(u)y^n. \tag{2.10}$$

This completes the proof of theorem 2.

Corollary: If we put $m=0$, we notice that $G(x,u,w)$ becomes $G(x,w)$ since $C_0^{(n+q)}(u)=1$. Hence from (2.10), we get

$$
\begin{aligned}
&(1-2w)^{\lambda+\frac{1}{2}}\{1-2w(1-x^2)\}^{-\lambda} \\
&\times G\left(\frac{x}{\{1-2w(1-x^2)\}^{\frac{1}{2}}}, \frac{wy}{\{1-2w(1-x^2)\}^{\frac{3}{2}}}\right) \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!}\frac{(n+1)_{2p}}{2^p(1-\lambda-n)_p} C_{n+2p}^{\lambda+n-p}(x) y^n \left(\sum_{q=0}^{\infty}\frac{(2w)^q (n)_q}{q!}\right). \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!}\, 2^{n-p}\frac{(n+1)_{2p}}{(1-\lambda-n)_p} C_{n+2p}^{\lambda+n-p}(x) \left\{\frac{y}{2(1-2w)}\right\}^n.
\end{aligned}
$$

Replacing $\{\frac{y}{2(1-2w)}\}$ by y and then $2w$ by w on both sides, we get

$$
\begin{aligned}
&(1-w)^{\lambda+\frac{1}{2}}\{1-w(1-x^2)\}^{-\lambda}\, G\left(\frac{x}{\{1-w(1-x^2)\}^{\frac{1}{2}}}, \frac{wy(1-w)}{\{1-w(1-x^2)\}^{\frac{3}{2}}}\right) \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} a_n \frac{w^{n+p}}{p!}\frac{(n+1)_{2p}}{2^{2p}(1-\lambda-n)_p} C_{n+2p}^{\lambda+n-p}(x) y^n. \\
&= \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p} \frac{w^{n}}{p!}\frac{(n-p+1)_{2p}}{2^{2p}(1-\lambda-n)_p} C_{n+p}^{\lambda+n-2p}(x) y^{n-p}.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
&(1-w)^{\lambda+\frac{1}{2}}\{1-w(1-x^2)\}^{-\lambda}\, G\left(\frac{x}{\{1-w(1-x^2)\}^{\frac{1}{2}}}, \frac{wy(1-w)}{\{1-w(1-x^2)\}^{\frac{3}{2}}}\right) \\
&= \sum_{n=0}^{\infty} w^n\, \sigma_n(x,y),
\end{aligned}
$$

where

$$
\sigma_n(x,y) = \sum_{p=0}^{n} a_p \frac{(\frac{p+1}{2})_{n-p}(\frac{p+2}{2})_{n-p}}{(n-p)!(1-\lambda-p)_{n-p}}\; C_{2n-p}^{\lambda-n+2p}(x)\; y^p,
$$

which is Theorem 1.

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