South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 301-318

DOI: 10.56827/SEAJMMS.2022.1802.26 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON THE IMAGES OF LM-G-FILTERS AND LM-G-FILTERBASES

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(**Received:** *Jul. 24, 2021* **Accepted:** *Jun. 18, 2022* **Published:** *Aug. 30, 2022)*

Abstract: This paper studies LM-G-filters as a generalization of LM-filters. Images of LM-G-filter spaces and LM-G-filterbases induced by functions are investigated and some of their properties are derived. It is shown that the property of being weakly inspired, catalyzed, s-stratified and stratification of LM-G-filter spaces are preserved by images. Moreover the categorical connections of LM-G-filter spaces with neighborhood systems are also identified.

Keywords and Phrases: LM-G-filters, Images, Quantale, Neighborhood systems.

2020 Mathematics Subject Classification: 54A20, 18A40, 06D22.

1. Introduction

In 1977, Lowen [12] developed the idea of filters in I^X, called prefilters to discuss convergence in fuzzy topological spaces. In 1999 Burton et al. [3] introduced the concept of generalized filters as a map from 2^X to I. Subsequently Höhle and Šostak [4] developed the notion of L-filters and stratified L-filters on a complete quasi-monoidal lattice and discussed their role in the development of fuzzy convergence spaces. Later in 2006, Kim et al. [10] introduced the notion of L-filter base on a strictly two-sided, commutative quantale lattice L and defined two types of images and preimages of L-filter bases. In 2013, Jäger [5] developed a theory of stratified LM-filters which generalizes the theory of stratified L-filters.

In [1], Abbas et al. investigated stratified L-filters and its stratification. In [6], it is found that the stratification of L-filters in [1] need not preserve the L-filter structure. This motivated the authors to introduce the concept of LM-G-filter spaces [6] as a generalization of LM-filters. The study introduced the concept of stratified LM-G-filters and stratification of LM-G-filter spaces by a stratification mapping. Further, some subcategories of LM-**G**, the category of LM-G-filter spaces have been identified by introducing the concepts of inspired, weakly inspired LM-G-filter spaces in [7] and catalyzed LM-G-filter spaces in [8] and their application in mathematical modeling is explored. In [9], the authors studied the categorical connections of L-G-filters with L-filters and L-interior operators and a galois correspondence between the categories of stratified L-G-filter spaces and L-fuzzy pre-proximity spaces is identified.

This paper defines images of LM-G-filters and LM-G-filterbases induced by functions and identifies their properties. It is proved that the properties of being weakly inspired, catalyzed, s-stratified as well as the stratification of LM-G-filters are preserved by images. Moreover, categorical connections of LM-G-filters with LM-fuzzy neighborhood systems and LM-fuzzy quasi-coincident neighborhood system are also obtained.

2. Preliminaries

Throughout this paper X stands for a non-empty ordinary set. For the notions of category theory, the readers can refer to [2].

Definition 2.1. [4] *A triple* $(L, \leq, \odot)$ *is called a strictly two-sided, commutative quantale (stsc-quantale, for short) iff it satisfies the following properties:*

(L1) $L = (L, \leq, 1, 0)$ *is a complete lattice where* 1 *is the universal upper bound and* 0 *denotes the universal lower bound;*

(L2) $(L, \odot)$ *is a commutative semigroup;*

(L3) $a = a \odot 1$, *for each* $a \in L$*;*

(L4) $\odot$ *is distributive over arbitrary joins, i.e.* $(\bigvee_{i \in I} a_i) \odot b = \bigvee_{i \in I}(a_i \odot b)$.

Unless otherwise specified, in this paper, L and M stand for strictly two-sided, commutative quantales.

Remark 2.2. [4] *Every completely distributive lattice, GL-monoid and complete Heyting algebra (i.e. frame) are stsc-quantales.*

All algebraic operations on L can be extended pointwise to L^X as $A \leq B$ if and only if $A(x) \leq B(x)$, $(A \odot B)(x) = A(x) \odot B(x)$ and $(\alpha \odot A)(x) = \alpha \odot A(x)$ for all $x \in X$. The constant function α_X is defined by $\alpha_X(x) = \alpha$ for all $x \in X$.

Definition 2.3. [5] *A mapping $s : L \to M$ with the properties $(M1)$ $s(0_L) = 0_M$; $(M2)$ $s(1_L) = 1_M$ and $(M3)$ $s(\alpha \wedge \beta) = s(\alpha) \wedge s(\beta)$ for all $\alpha, \beta \in L$ where L and M are frames is called a stratification mapping.*

Definition 2.4. [6] *An LM-G-filter on a set X is defined to be a mapping $G : L^X \to M$ satisfying:*

*(**G**1) $G(1_X) = 1$;*

*(**G**2) For every $A, B \in L^X$ such that $A \leq B$, $G(A) \leq G(B)$;*

*(**G**3) For every $A, B \in L^X, G(A \odot B) \geq G(A) \odot G(B)$.*

*The pair (X, G) is called an LM-G-filter space. In addition to the above axioms, if (**G**4) : $G(0_X) = 0$ is also satisfied, then (X, G) becomes an LM-filter space* [4]*.*
If G_1 and G_2 are two LM-G-filters on X such that $G_2(A) \geq G_1(A)$ for all $A \in L^X$, then we say (X, G_1) is weaker (coarser) than (X, G_2) and (X, G_2) is stronger (finer) than (X, G_1).
An LM-G-filter space (X, G) is called s-stratified if $G(\alpha_X \odot A) \geq s(\alpha) \odot G(A)$ for all $\alpha \in L$ and $A \in L^X$ where $s : L \to M$ is a stratification mapping. s-stratified LM-filter is defined analogously [5]*.*

Remark 2.5. *An LM-G-filter space (X, G) is s-stratified if and only if $G(\alpha_X) \geq s(\alpha)$ for all $\alpha \in L$.*

Definition 2.6. [6] *Let (X, G_1) and (Y, G_2) be LM-G-filter spaces. A map $f^{\to} : L^X \to L^Y$ is called an LM-G-filter map if $G_1(f^{\leftarrow}(B)) \geq G_2(B)$, $\forall B \in L^Y$. A map $f^{\to} : L^X \to L^Y$ is called an LM-G-filter preserving map if $G_2(f^{\to}(A)) \geq G_1(A)$, $\forall A \in L^X$.*

The category of LM-G-filter spaces with LM-G-filter maps as morphisms is denoted by LM-**G**.

Definition 2.7. [10] *An L-filterbase on X is a mapping $\mathcal{B} : L^X \to L$ such that for all $A, B \in L^X$:*

($\mathcal{B}$1) $\mathcal{B}(1_X) = 1$ and $\mathcal{B}(0_X) = 0$;

($\mathcal{B}$2) $\langle\mathcal{B}\rangle(A \odot B) \geq \mathcal{B}(A) \odot \mathcal{B}(B)$ *where* $\langle\mathcal{B}\rangle(A) = \bigvee_{B \leq A} \mathcal{B}(B)$.

An L-filterbase $\mathcal{B}$ *is said to be stratified if* $\langle\mathcal{B}\rangle(\alpha \wedge A) = \alpha \odot \mathcal{B}(A)$, *for each* $A \in L^X$ *and* $\alpha \in L$.[1]

Definition 2.8. [6] *A function* $\widetilde{\mathcal{B}} : L^X \to M$ *is called an LM-G-filterbase on X if it satisfies the following conditions:*

(B1) $\widetilde{\mathcal{B}}(1_X) = 1$;

(B2) $\langle\widetilde{\mathcal{B}}\rangle(A \odot B) \geq \widetilde{\mathcal{B}}(A) \odot \widetilde{\mathcal{B}}(B)$, *for each* $A, B \in L^X$, *where* $\langle\widetilde{\mathcal{B}}\rangle(A) = \bigvee_{B \leq A} \widetilde{\mathcal{B}}(B)$.

In addition to the above axioms, if $(B3) : \widetilde{\mathcal{B}}(0_X) = 0$ *is also satisfied, then* $(X, \widetilde{\mathcal{B}})$ *becomes an LM-filterbase.*

Definition 2.9. [6] *An LM-G-filterbase* $\widetilde{\mathcal{B}}$ *is said to be s-stratified if* $\widetilde{\mathcal{B}}$ *satisfies* $\langle\widetilde{\mathcal{B}}\rangle(\alpha \odot A) \geq s(\alpha) \odot \widetilde{\mathcal{B}}(A)$, *for each* $A \in L^X$ *and* $\alpha \in L$ *where s is a stratification mapping from L to M. s-stratified LM-filterbase is defined analogously.*

Definition 2.10. [11] *Let L be a complete lattice. The set of all prime elements and co-prime elements in L are denoted by* $pr(L)$ *and* $J(L)$ *respectively. Define a relation* $\preceq$ *in L as follows:* $\forall a, b \in L, a \preceq b$ *if and only if* $\forall S \subset L, \bigvee S \geq b \Rightarrow \exists s \in S$ *such that* $s \geq a$. *Define a relation* $\overline{\succ}$ *in L as follows:* $\forall a, b \in L, a \overline{\succ} b$ *if and only if* $\forall S \subset L, \bigwedge S \leq b \Rightarrow \exists s \in S$ *such that* $s \leq a$.

Definition 2.11. [7, 8] *For* $A \in L^X$, *denote the p-set of A by* $\delta_p(A) = \{x \in X; p \overline{\succ} A(x)\}^c$ *where p is prime in L and* $()^c$ *denotes set complement and 1-set of A by* $\delta_1(A) = \{x \in X; A(x) = 1\}$. *Let* (X, G) *be an LM-G-filter space. If* $G(A) = \bigwedge_{p \in pr(L)} G(1_{\delta_p(A)})$ *for all* $A \in L^X$, *then* (X, G) *is called an inspired LM-G-filter space. If* $G(A) \leq \bigwedge_{p \in pr(L)} G(1_{\delta_p(A)})$ *for all* $A \in L^X$, *then* (X, G) *is called weakly inspired LM-G-filter space. If* $G(A) = G(1_{\delta_1(A)})$ *for all* $A \in L^X$, *then* (X, G) *is called catalyzed LM-G-filter space.*

3. Images of LM-G-filterbases

This section defines two types of images of LM-G-filterbases induced by functions and identifies certain properties of these images. It is proved that the property of being s-stratified and stratification of LM-G-filterbases are preserved by both kind of images. Images induced by functions in the case of LM-filterbases are also analyzed.

Lemma 3.1. *Let $(L, \leq, \odot)$ be a stsc-quantale and $f : X \to Y$ a function. Then $f^{\to}(A \odot \alpha_X) = f^{\to}(A) \odot \alpha_Y$ for all $A \in L^X$ and $\alpha \in L$.*
Proof.

$$\begin{aligned} f^{\to}(A \odot \alpha_X)(y) &= \bigvee\{(A \odot \alpha_X)(x); f(x) = y\} \\ &= \bigvee\{A(x) \odot \alpha; f(x) = y\} \\ &= \bigvee\{A(x); f(x) = y\} \odot \alpha \quad \text{(by property } (L4) \text{ of stsc-quantale)} \\ &= (f^{\to}(A) \odot \alpha_Y)(y) \end{aligned}$$

Theorem 3.2. *Let $\widetilde{\mathcal{B}}$ be an LM-G-filterbase on X and $f : X \to Y$ be a function. Then $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}) : L^Y \to M$ defined by $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(B) = \bigvee\{\widetilde{\mathcal{B}}(A) | f^{\leftarrow}(B) \geq A\}$ for all $B \in L^Y$ is an LM-G-filterbase on Y. If $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ are LM-G-filterbases on X such that $\widetilde{\mathcal{B}}_1 \leq \widetilde{\mathcal{B}}_2$, then $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}_1) \leq f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}_2)$.*

Proof. Clearly $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(1_Y) = 1$. Suppose there exists $B_1, B_2 \in L^Y$ such that $\langle f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})\rangle(B_1 \odot B_2) \ngeq f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(B_1) \odot f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(B_2)$. This implies there exists $C, D \in L^X$ where $f^{\leftarrow}(B_1) \geq C, f^{\leftarrow}(B_2) \geq D$ such that $\langle f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})\rangle(B_1 \odot B_2) \ngeq \widetilde{\mathcal{B}}(C) \odot \widetilde{\mathcal{B}}(D)$. Therefore $\langle f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})\rangle(B_1 \odot B_2) \ngeq \langle\widetilde{\mathcal{B}}\rangle(C \odot D)$. This implies there exists $P \leq C \odot D$ such that $\langle f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})\rangle(B_1 \odot B_2) \ngeq \widetilde{\mathcal{B}}(P)$.
Since $f^{\leftarrow}(B_1 \odot B_2) \geq C \odot D \geq P$, $\langle f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})\rangle(B_1 \odot B_2) \geq \widetilde{\mathcal{B}}(P)$ which is a contradiction. Therefore $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})$ is an LM-G-filterbase on Y. Rest of the proof is trivial.

In [6], we have the following theorem.

Theorem 3.3. [6] *Let $\widetilde{\mathcal{B}}$ be an LM-G-filterbase on X and $s : L \to M$ be a stratification mapping. Then $\widetilde{\mathcal{B}}^s : L^X \to M$ defined by $\widetilde{\mathcal{B}}^s(A) = \bigvee\{\widetilde{\mathcal{B}}(B) \odot s(\alpha) | A \geq B \odot \alpha\}$, where $B \in L^X$, $\alpha \in L$ is the coarsest s-stratified LM-G-filterbase on X which is finer than $\widetilde{\mathcal{B}}$. $\widetilde{\mathcal{B}}^s$ is called the s-stratification of the LM-G-filterbase $\widetilde{\mathcal{B}}$.*

Theorem 3.4. *Let $\widetilde{\mathcal{B}}$ be an LM-G-filterbase on X, $f : X \to Y$ be a function and $s : L \to M$ be a stratification mapping. Then*

(i.) If $\widetilde{\mathcal{B}}$ is s-stratified, then $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})$ is s-stratified.

(ii.) $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s) = (f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}))^s$.

Proof. Proof of (i.) is trivial. Since $\widetilde{\mathcal{B}} \leq \widetilde{\mathcal{B}}^s$, $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}) \leq f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s)$. Also since $\widetilde{\mathcal{B}}^s$ is s-stratified, $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s)$ is s-stratified. Therefore $(f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}))^s \leq f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s)$.

For $B \in L^Y$,

$$\begin{aligned} f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s)(B) &= \bigvee_{f^{\leftarrow}(B)\geq A} \widetilde{\mathcal{B}}^s(A) \\ &= \bigvee_{f^{\leftarrow}(B)\geq A} \bigvee\{\widetilde{\mathcal{B}}(P)\odot s(\alpha); A \geq P\odot \alpha_X\} \\ (f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}))^s(B) &= \bigvee\{f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(R)\odot s(\beta); B\geq R\odot \beta_Y\} \\ &= \bigvee\{[\bigvee_{f^{\leftarrow}(R)\geq Q} \widetilde{\mathcal{B}}(Q)]\odot s(\beta); B\geq R\odot\beta_Y\} \end{aligned}$$

Since $f^{\leftarrow}(B) \geq A \geq P \odot \alpha_X$, we have $B \geq f^{\rightarrow}(f^{\leftarrow}(B)) \geq f^{\rightarrow}(P) \odot \alpha_Y$. Therefore, by comparing the expressions of $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s)$ and $(f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}))^s$, it is clear that $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s)(B) \leq (f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}))^s(B)$. Therefore $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}^s) = (f^{\twoheadrightarrow}(\widetilde{\mathcal{B}}))^s$.

Example 3.5. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\}$ and $L = M = [0, 1]$. $s : L \to M$ defined by $s(1) = 1, s(0) = 0$ and $s(\alpha) = 0.2$ for all $\alpha \in (0, 1)$ is a stratification mapping. Let $A_1 \in L^X$ be defined by $A_1(x_1) = 0.8, A_1(x_2) = 0.9$. Then $\widetilde{\mathcal{B}} : L^X \to M$ defined by

$$\widetilde{\mathcal{B}}(A) = \begin{cases} 1 & \text{if } A = 1_X\,, \\ 0.3 & \text{if } A \geq A_1 \text{ and } A \neq 1_X \\ 0.2 & \text{otherwise} \end{cases}$$

is an s-stratified LM-G-filterbase on X. Let $f : X \to Y$ be defined by $f(x_1) = f(x_2) = y_1$. Then $f^{\rightarrow}(1_X) = B_1$ where $B_1(y_1) = 1, B_1(y_2) = 0, B_1(y_3) = 0$ and $f^{\rightarrow}(A_1) = B_2$ where $B_2(y_1) = 0.9, B_2(y_2) = 0, B_2(y_3) = 0$. Then $f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(B) : L^Y \to M$ defined by

$$f^{\twoheadrightarrow}(\widetilde{\mathcal{B}})(B) = \begin{cases} 1 & \text{if } B \geq B_1\,, \\ 0.3 & \text{if } B \geq B_2 \text{ and } B \ngeq B_1 \\ 0.2 & \text{otherwise} \end{cases}$$

is an s-stratified LM-G-filterbase on Y.

It is easy to prove that

Theorem 3.6. *Let $\mathcal{B}$ be an LM-filterbase on X and $f : X \to Y$ be a function. Then $f^{\twoheadrightarrow}(\mathcal{B}) : L^Y \to M$ defined by $f^{\twoheadrightarrow}(\mathcal{B})(B) = \bigvee\{\mathcal{B}(A)|f^{\leftarrow}(B) \geq A\}$ for all $B \in L^Y$ is an LM-filterbase on Y. If $\mathcal{B}$ is s-stratified then $f^{\twoheadrightarrow}(\mathcal{B})$ is also s-stratified.*

Theorem 3.7. *Let $\widetilde{\mathcal{B}}$ be an LM-G-filterbase on Y and $f : X \to Y$ be a function. Then $f^{\leftarrow}(\widetilde{\mathcal{B}}) : L^X \to M$ defined by $f^{\leftarrow}(\widetilde{\mathcal{B}})(A) = \bigvee\{\widetilde{\mathcal{B}}(B)|A \geq f^{\leftarrow}(B)\}$ for all $A \in L^X$ is an LM-G-filterbase on X. If $\widetilde{\mathcal{B}}_1$ and $\widetilde{\mathcal{B}}_2$ are LM-G-filterbases on Y such* *that $\widetilde{\mathcal{B}}_1 \leq \widetilde{\mathcal{B}}_2$, then $f^{\Leftarrow}(\widetilde{\mathcal{B}}_1) \leq f^{\Leftarrow}(\widetilde{\mathcal{B}}_2)$.*

Proof. $(\widetilde{\mathcal{B}}1)$ is obvious. Suppose there exists $A_1, A_2 \in L^X$ such that $\langle f^{\Leftarrow}(\widetilde{\mathcal{B}})\rangle(A_1 \odot A_2) \ngeq f^{\Leftarrow}(\widetilde{\mathcal{B}})(A_1) \odot f^{\Leftarrow}(\widetilde{\mathcal{B}})(A_2)$. This implies there exists $U, V \in L^Y$ where $A_1 \geq f^{\leftarrow}(U), A_2 \geq f^{\leftarrow}(V)$ such that $\langle f^{\Leftarrow}(\widetilde{\mathcal{B}})\rangle(A_1 \odot A_2) \ngeq \widetilde{\mathcal{B}}(U) \odot \widetilde{\mathcal{B}}(V)$. Therefore $\langle f^{\Leftarrow}(\widetilde{\mathcal{B}})\rangle(A_1 \odot A_2) \ngeq \langle\widetilde{\mathcal{B}}\rangle(U \odot V)$. This implies there exists $W \leq U \odot V$ such that $\langle f^{\Leftarrow}(\widetilde{\mathcal{B}})\rangle(A_1 \odot A_2) \ngeq \widetilde{\mathcal{B}}(W)$.
Since $A_1 \odot A_2 \geq f^{\leftarrow}(U) \odot f^{\leftarrow}(V) \geq f^{\leftarrow}(U \odot V) \geq f^{\leftarrow}(W)$, $f^{\Leftarrow}(\widetilde{\mathcal{B}})(A_1 \odot A_2) \geq \widetilde{\mathcal{B}}(W)$ and hence $\langle f^{\Leftarrow}(\widetilde{\mathcal{B}})\rangle(A_1 \odot A_2) \geq \widetilde{\mathcal{B}}(W)$ which is a contradiction. Therefore $f^{\Leftarrow}(\widetilde{\mathcal{B}})$ is an LM-G-filterbase on X. Rest of the proof is trivial.

Theorem 3.8. *Let $\widetilde{\mathcal{B}}$ be an LM-G-filterbase on Y, $f : X \to Y$ be a function and $s : L \to M$ be a stratification mapping. Then*

(i.) If $\widetilde{\mathcal{B}}$ is s-stratified, then $f^{\Leftarrow}(\widetilde{\mathcal{B}})$ is s-stratified.

(ii.) $f^{\Leftarrow}(\widetilde{\mathcal{B}}^s) = (f^{\Leftarrow}(\widetilde{\mathcal{B}}))^s$.

Proof. We prove only (ii.). Since $\widetilde{\mathcal{B}} \leq \widetilde{\mathcal{B}}^s$, $f^{\Leftarrow}(\widetilde{\mathcal{B}}) \leq f^{\Leftarrow}(\widetilde{\mathcal{B}}^s)$. Also since $\widetilde{\mathcal{B}}^s$ is s-stratified, $f^{\Leftarrow}(\widetilde{\mathcal{B}}^s)$ is s-stratified. Therefore $(f^{\Leftarrow}(\widetilde{\mathcal{B}}))^s \leq f^{\Leftarrow}(\widetilde{\mathcal{B}}^s)$.
For $A \in L^X$,

$$
\begin{aligned}
f^{\Leftarrow}(\widetilde{\mathcal{B}}^s)(A) &= \bigvee_{A \geq f^{\leftarrow}(B)} \widetilde{\mathcal{B}}^s(B) \\
&= \bigvee_{A \geq f^{\leftarrow}(B)} \bigvee\{\widetilde{\mathcal{B}}(R) \odot s(\beta); B \geq R \odot \beta_Y\} \\
(f^{\Leftarrow}(\widetilde{\mathcal{B}}))^s(A) &= \bigvee\{f^{\Leftarrow}(\widetilde{\mathcal{B}})(P) \odot s(\alpha); A \geq P \odot \alpha_X\} \\
&= \bigvee\{[\bigvee_{P \geq f^{\leftarrow}(S)} \widetilde{\mathcal{B}}(S)] \odot s(\alpha); A \geq P \odot \alpha_X\}
\end{aligned}
$$

Since $A \geq f^{\leftarrow}(B) \geq f^{\leftarrow}(R) \odot \beta_X$, comparing the expressions of $f^{\Leftarrow}(\widetilde{\mathcal{B}}^s)$ and $(f^{\Leftarrow}(\widetilde{\mathcal{B}}))^s$, it is clear that $f^{\Leftarrow}(\widetilde{\mathcal{B}}^s)(A) \leq (f^{\Leftarrow}(\widetilde{\mathcal{B}}))^s(A)$. Therefore $f^{\Leftarrow}(\widetilde{\mathcal{B}}^s) = (f^{\Leftarrow}(\widetilde{\mathcal{B}}))^s$.

Example 3.9. Let $X = \{x_1, x_2\}, Y = \{y_1, y_2, y_3\}$ and $L = M = [0, 1]$. $s : L \to M$ defined by $s(1) = 1, s(0) = 0$ and $s(\alpha) = 0.5$ for all $\alpha \in (0, 1)$ is a stratification mapping. Let $B_1 \in L^Y$ be defined by $B_1(y_1) = 0.7, B_1(y_2) = 0.8$ and $B_1(y_3) = 0.4$. Then $\widetilde{\mathcal{B}} : L^Y \to M$ defined by

$$
\widetilde{\mathcal{B}}(B) = \begin{cases} 1 & \text{if } B = 1_Y\,, \\ 0.8 & \text{if } B \geq B_1 \text{ and } B \neq 1_Y \\ 0.5 & \text{otherwise} \end{cases}
$$

is an s-stratified LM-G-filterbase on Y. Let $f : X \to Y$ be defined by $f(x_1) = y_1$ and $f(x_2) = y_3$. Since $f^{\leftarrow}(B_1) = A_1$ where $A_1(x_1) = 0.7$ and $A_1(x_2) = 0.4$, $f^{\twoheadleftarrow}(\widetilde{\mathcal{B}}) : L^X \to M$ defined by

$$f^{\twoheadleftarrow}(\widetilde{\mathcal{B}})(A) = \begin{cases} 1 & \text{if } A = 1_X\,, \\ 0.8 & \text{if } A \geq A_1 \text{ and } A \neq 1_X \\ 0.5 & \text{otherwise} \end{cases}$$

is an s-stratified LM-G-filterbase on X.

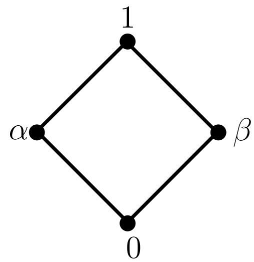

Figure 1: The diamond type lattice

Remark 3.10. *Let $\mathcal{B}$ be an LM-filterbase on Y and $f : X \to Y$ be a function. Then $f^{\twoheadleftarrow}(\mathcal{B}) : L^X \to M$ defined by $f^{\twoheadleftarrow}(\mathcal{B})(A) = \bigvee\{\mathcal{B}(B) | A \geq f^{\leftarrow}(B)\}$ for all $A \in L^X$ need not be an LM-filterbase on X.*

For example, let $X = \{x_1, x_2, x_3\}$, $Y = \{y_1, y_2\}$ and $L = M$ be the lattice shown in Figure 1. Let $f : X \to Y$ be defined by $f(x_1) = f(x_2) = f(x_3) = y_1$ and the LM-filterbase on Y, $\mathcal{B}$ be defined by

	B_1	B_2	B_3	B_4	B_5	B_6	B_7	B_8	B_9	B_{10}	B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}
y_1	0	0	0	0	α	α	α	α	β	β	β	β	1	1	1	1
y_2	0	α	β	1	0	α	β	1	0	α	β	1	0	α	β	1
$\mathcal{B}(B_i)$	0	β	α	1	0	β	α	1	0	β	α	1	0	β	α	1

Then $f^{\leftarrow}(B_4) = 0_X$. Therefore by definition, $f^{\twoheadleftarrow}(\mathcal{B})(0_X) = 1$ and hence $f^{\twoheadleftarrow}(\mathcal{B})$ is not an LM-filterbase on X.

It is easy to observe that

Theorem 3.11. *Let $\mathcal{B}$ be an LM-filterbase on Y and $f : X \to Y$ be a surjective function. Then $f^{\twoheadleftarrow}(\mathcal{B}) : L^X \to M$ defined by $f^{\twoheadleftarrow}(\mathcal{B})(A) = \bigvee\{\mathcal{B}(B) | A \geq f^{\leftarrow}(B)\}$ is an LM-filterbase on X.*

4. Images of LM-G-filter Spaces

This section defines images of LM-G-filters induced by functions. Images of s-stratified, inspired, weakly inspired and catalyzed LM-G-filter spaces are discussed.

Images induced by functions in the case of LM-filters are also analyzed.

Theorem 4.1. *Let (Y, G) be an LM-G-filter space and $f : X \to Y$ be a function. Define $f^{\twoheadleftarrow}(G) : L^X \to M$ by $f^{\twoheadleftarrow}(G)(A) = \bigvee\{G(B)|A \geq f^{\leftarrow}(B)\}$ for all $A \in L^X$. Then*

(i.) $f^{\twoheadleftarrow}(G)$ *is an LM-G-filter on X.*

(ii.) If G_1 and G_2 are LM-G-filters on Y such that $G_1 \leq G_2$, then $f^{\twoheadleftarrow}(G_1) \leq f^{\twoheadleftarrow}(G_2)$.

(iii.) $f^{\twoheadleftarrow}(G)$ is the coarsest LM-G-filter on X for which $f^{\to} : (X, f^{\twoheadleftarrow}(G)) \to (Y, G)$ is an LM-G-filter map.

Proof. We prove only (iii.). It is clear that $f^{\to} : (X, f^{\twoheadleftarrow}(G)) \to (Y, G)$ is an LM-G-filter map. Let H be an LM-G-filter on X such that $f^{\to} : (X, H) \to (Y, G)$ is an LM-G-filter map. Therefore, for $B \in L^Y$, $H(f^{\leftarrow}(B)) \geq G(B)$.

$$\begin{aligned} f^{\twoheadleftarrow}(G)(A) &= \bigvee\{G(B)|A \geq f^{\leftarrow}(B)\} \\ &\leq \bigvee\{H(f^{\leftarrow}(B))|A \geq f^{\leftarrow}(B)\} \leq H(A). \end{aligned}$$

In [6], we have the following theorem.

Theorem 4.2. [6] *Let (X, G) be an LM-G-filter space and $s : L \to M$ be a stratification mapping. Then $G^s : L^X \to M$ defined by $G^s(A) = \bigvee\{G(B) \odot s(\alpha)|A \geq B \odot \alpha\}$, where $B \in L^X$, $\alpha \in L$ is the coarsest s-stratified LM-G-filter on X which is finer than G. G^s is called the s-stratification of the LM-G-filter G.*

Theorem 4.3. *Let (Y, G) be an LM-G-filter space, $f : X \to Y$ be a function and $s : L \to M$ be a stratification mapping. Then*

(i.) If G is s-stratified, then $f^{\twoheadleftarrow}(G)$ is s-stratified.

(ii.) $f^{\twoheadleftarrow}(G^s) = (f^{\twoheadleftarrow}(G))^s$.

Proof. We prove only (ii.). Since $G \leq G^s$, $f^{\twoheadleftarrow}(G) \leq f^{\twoheadleftarrow}(G^s)$. Also since G^s is s-stratified, $f^{\twoheadleftarrow}(G^s)$ is s-stratified. Therefore $(f^{\twoheadleftarrow}(G))^s \leq f^{\twoheadleftarrow}(G^s)$. For $A \in L^X$,

$$\begin{aligned} f^{\twoheadleftarrow}(G^s)(A) &= \bigvee_{A \geq f^{\leftarrow}(B)} G^s(B) \\ &= \bigvee_{A \geq f^{\leftarrow}(B)} \bigvee\{G(R) \odot s(\beta); B \geq R \odot \beta_Y\} \end{aligned}$$

$$\begin{aligned}(f^{\leftarrow}(G))^s(A) &= \bigvee\{f^{\leftarrow}(G)G(P)\odot s(\alpha); A\geq P\odot\alpha_X\}\\ &= \bigvee\{[\bigvee_{P\geq f^{\leftarrow}(S)} G(S)]\odot s(\alpha); A\geq P\odot\alpha_X\}\end{aligned}$$

Since $A \geq f^{\leftarrow}(B) \geq f^{\leftarrow}(R)\odot\beta_X$, comparing the expressions of $f^{\leftarrow}(G^s)$ and $(f^{\leftarrow}(G))^s$, it is clear that $f^{\leftarrow}(G^s)(A)\leq(f^{\leftarrow}(G))^s(A)$. Therefore $f^{\leftarrow}(G^s) = (f^{\leftarrow}(G))^s$.

Example 4.4. Let $X=\{x_1,x_2\}, Y=\{y_1,y_2\}$ and $L=M=[0,1]$. $s: L\to M$ defined by $s(1)=1, s(0)=0$ and $s(\alpha)=0.3$ for all $\alpha\in(0,1)$ is a stratification mapping. Let $B_1\in L^Y$ be defined by $B_1(y_1)=0.3, B_1(y_2)=0.4$. Then $G: L^Y\to M$ defined by

$$G(B)=\begin{cases}1 & \text{if } B=1_Y\text{ },\\ 0.7 & \text{if } B\geq B_1 \text{ and } B\neq 1_Y\\ 0.3 & \text{ otherwise}\end{cases}$$

is an s-stratified LM-G-filter on Y. Let $f: X\to Y$ be defined by $f(x_1)=f(x_2)=y_2$. Since $f^{\leftarrow}(B_1)=A_1$ where $A_1(x_1)=A_1(x_2)=0.4$, $f^{\leftarrow}(G): L^X\to M$ defined by

$$f^{\leftarrow}(G)(A)=\begin{cases}1 & \text{if } A=1_X\text{ },\\ 0.7 & \text{if } A\geq A_1 \text{ and } A\neq 1_X\\ 0.3 & \text{ otherwise}\end{cases}$$

is an s-stratified LM-G-filter on X.

Theorem 4.5. *Let L and M be completely distributive lattices, (Y,G) be an LM-G-filter space and $f: X\to Y$ be a function. Then*

(i.) If (Y,G) is weakly inspired LM-G-filter space, then $(X,f^{\leftarrow}(G))$ is weakly inspired.

(ii.) If (Y,G) is catalyzed LM-G-filter space, then $(X,f^{\leftarrow}(G))$ is catalyzed.

Proof.

(i.) Let (Y,G) be weakly inspired LM-G-filter space. For $A\in L^X$,

$$\begin{aligned}f^{\leftarrow}(G)(A) &= \bigvee\{G(B)|A\geq f^{\leftarrow}(B)\}\\ &\leq \bigvee\{G(1_{\delta_p(B)})|1_{\delta_p(A)}\geq f^{\leftarrow}(1_{\delta_p(B)})\}\ \text{ for all } p\in pr(L)\\ &\quad [\text{since } \delta_p(A)\leq\delta_p(B) \text{ when } A\leq B \text{ and } \delta_p(f^{\leftarrow}(B))=f^{-1}(\delta_p(B))\ ]\\ &\leq f^{\leftarrow}(G)(1_{\delta_p(A)})\ \text{ for all } p\in pr(L)\end{aligned}$$

Therefore, $f^{\twoheadleftarrow}(G)(A) \leq \bigwedge_{p \in pr(L)} f^{\twoheadleftarrow}(G)(1_{\delta_p(A)})$.

(ii.) Let (Y,G) be catalyzed LM-G-filter space. Since $A \geq 1_{\delta_1(A)}$ for $A \in L^X$, $f^{\twoheadleftarrow}(G)(1_{\delta_1(A)}) \leq f^{\twoheadleftarrow}(G)(A)$. The reverse inequality is obtained by

$$\begin{aligned} f^{\twoheadleftarrow}(G)(A) &= \bigvee\{G(B) | A \geq f^{\leftarrow}(B)\} \\ &= \bigvee\{G(1_{\delta_1(B)}) | 1_{\delta_1(A)} \geq f^{\leftarrow}(1_{\delta_1(B)})\} \\ &\quad [\text{since } \delta_1(A) \leq \delta_1(B) \text{ when } A \leq B \text{ and } \delta_1(f^{\leftarrow}(B)) = f^{-1}(\delta_1(B))] \\ &\leq f^{\twoheadleftarrow}(G)(1_{\delta_1(A)}) \end{aligned}$$

Therefore, $f^{\twoheadleftarrow}(G)(A) = f^{\twoheadleftarrow}(G)(1_{\delta_1(A)})$.

We leave the following question open.

Question 4.6. *Is $(X, f^{\twoheadleftarrow}(G))$ an inspired LM-G-filter space if (Y,G) is inspired ?*

Remark 4.7. *Let (Y,F) be an LM-filter space and $f : X \to Y$ be a function. Then $f^{\twoheadleftarrow}(F) : L^X \to M$ defined by $f^{\twoheadleftarrow}(F)(A) = \bigvee\{F(B) | A \geq f^{\leftarrow}(B)\}$ for all $A \in L^X$ need not be an LM-filter on X. For example, let $X = \{x\}$, $Y = \{y_1, y_2\}$ and $L = M$ be the lattice shown in Figure 1. Let $f : X \to Y$ be defined by $f(x) = y_2$ and the LM-filter on Y, F be defined by*

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	A_{13}	A_{14}	A_{15}	A_{16}
y_1	0	0	0	0	α	α	α	α	β	β	β	β	1	1	1	1
y_2	0	α	β	1	0	α	β	1	0	α	β	1	0	α	β	1
$F(A_i)$	0	0	0	0	α	α	α	α	β	β	β	β	1	1	1	1

Then $f^{\leftarrow}(A_{13}) = 0_X$. Therefore by definition, $f^{\twoheadleftarrow}(F)(0_X) = 1$ and hence $f^{\twoheadleftarrow}(F)$ is not an LM-filter on X.

It is easy to observe that

Theorem 4.8. *Let F be an LM-filter on Y and $f : X \to Y$ be a surjective function. Then $f^{\twoheadleftarrow}(F) : L^X \to M$ defined by $f^{\twoheadleftarrow}(F)(A) = \bigvee\{F(B) | A \geq f^{\leftarrow}(B)\}$ for all $A \in L^X$ is an LM-filter on X.*

Theorem 4.9. *Let (X,G) be an LM-G-filter space and $f : X \to Y$ be a function. Define $f^{\twoheadrightarrow}(G) : L^Y \to M$ by $f^{\twoheadrightarrow}(G)(B) = G(f^{\leftarrow}(B))$ for all $B \in L^Y$. Then*

(i.) $f^{\twoheadrightarrow}(G)$ is an LM-G-filter on Y.

(ii.) If G_1 and G_2 are LM-G-filters on X such that $G_1 \leq G_2$, then $f^{\twoheadrightarrow}(G_1) \leq f^{\twoheadrightarrow}(G_2)$.

(iii.) $f^{\twoheadrightarrow}(G)$ *is the coarsest* LM*-G-filter on* Y *for which* $f^{\rightarrow} : (X, G) \rightarrow (Y, f^{\twoheadrightarrow}(G))$ *is an* LM*-G-filter preserving map.*

Proof. We prove only (iii.). For all $A \in L^X$, $f^{\twoheadrightarrow}(G)(f^{\rightarrow}(A)) = G(f^{\leftarrow}(f^{\rightarrow}(A))) \geq G(A)$. Therefore, $f^{\rightarrow} : (X, G) \rightarrow (Y, G^{f^{\rightarrow}})$ is an LM-G-filter preserving map.
Let H be an LM-G-filter on Y such that $f^{\rightarrow} : (X, G) \rightarrow (Y, H)$ is an LM-G-filter preserving map. Therefore, for $A \in L^X$, $G(A) \leq H(f^{\rightarrow}(A))$.

$$\begin{aligned} f^{\twoheadrightarrow}(G)(B) &= G(f^{\leftarrow}(B)) \\ &\leq H(f^{\rightarrow}(f^{\leftarrow}(B)) \leq H(B). \end{aligned}$$

Theorem 4.10. *Let* (X, G) *be an* LM*-G-filter space,* $f : X \rightarrow Y$ *be a function and* $s : L \rightarrow M$ *be a stratification mapping. Then*

(i.) If G *is* s*-stratified, then* $f^{\twoheadrightarrow}(G)$ *is* s*-stratified.*

(ii.) $f^{\twoheadrightarrow}(G^s) = (f^{\twoheadrightarrow}(G))^s$.

Proof. Proof of (i.) is obvious. Since $G \leq G^s$, $f^{\twoheadrightarrow}(G) \leq f^{\twoheadrightarrow}(G^s)$. Also since G^s is s-stratified, $f^{\twoheadrightarrow}(G^s)$ is s-stratified. Therefore $(f^{\twoheadrightarrow}(G))^s \leq f^{\twoheadrightarrow}(G^s)$. For $B \in L^Y$,

$$\begin{aligned} f^{\twoheadrightarrow}(G^s)(B) &= G^s(f^{\leftarrow}(B)) \\ &= \bigvee \{G(A) \odot s(\alpha); f^{\leftarrow}(B) \geq A \odot \alpha_X\} \\ (f^{\twoheadrightarrow}(G))^s(B) &= \bigvee \{f^{\twoheadrightarrow}(G)(R) \odot s(\beta); B \geq R \odot \beta_Y\} \\ &= \bigvee \{G(f^{\leftarrow}(R)) \odot s(\beta); B \geq R \odot \beta_Y\} \end{aligned}$$

Since $f^{\leftarrow}(B) \geq A \odot \alpha_X$, we have $B \geq f^{\rightarrow}(f^{\leftarrow}(B)) \geq f^{\rightarrow}(A \odot \alpha_X) = f^{\rightarrow}(A) \odot \alpha_Y$. Therefore, by comparing the expressions of $f^{\twoheadrightarrow}(G^s)$ and $(f^{\twoheadrightarrow}(G))^s$, it is clear that $f^{\twoheadrightarrow}(G^s)(A) \leq (f^{\twoheadrightarrow}(G))^s(A)$. Therefore $f^{\twoheadrightarrow}(G^s) = (f^{\twoheadrightarrow}(G))^s$.

Example 4.11. Let $X = \{x_1, x_2, x_3\}, Y = \{y_1, y_2, y_3\}$ and $L = M = [0, 1]$. $s : L \rightarrow M$ defined by $s(1) = 1, s(0) = 0$ and $s(\alpha) = 0.4$ for all $\alpha \in (0, 1)$ is a stratification mapping. Let $A_1 \in L^X$ be defined by $A_1(x_1) = 0.5, A_1(x_2) = 0.9, A_1(x_3) = 0.2$. Then $G : L^X \rightarrow M$ defined by

$$G(A) = \begin{cases} 1 & \text{if } A = 1_X, \\ 0.7 & \text{if } A \geq A_1 \text{ and } A \neq 1_X \\ 0.5 & \text{otherwise} \end{cases}$$

is an s-stratified LM-G-filter on X. Let $f : X \to Y$ be defined by $f(x_1) = f(x_2) = y_1, f(x_3) = y_2$. Let $B_1, B_2 \in L^Y$ be defined by $B_1(y_1) = 1, B_1(y_2) = 1, B_1(y_3) = 0$ and $B_2(y_1) = 0.9, B_2(y_2) = 0.2, B_2(y_3) = 0$ Then $f^{\twoheadrightarrow}(G)(B) : L^Y \to M$ defined by

$$f^{\twoheadrightarrow}(G)(B) = \begin{cases} 1 & \text{if } B \geq B_1\,, \\ 0.7 & \text{if } B \geq B_2 \text{ and } B \ngeq B_1 \\ 0.5 & \text{otherwise} \end{cases}$$

is an s-stratified LM-G-filter on Y.

Proceeding as in Theorem 4.5, it is easy to prove the following theorem:

Theorem 4.12. *Let L and M be completely distributive lattices, (X, G) be an LM-G-filter space and $f : X \to Y$ be a function. Then*

(i.) If (X, G) is inspired LM-G-filter space, then $(Y, f^{\twoheadrightarrow}(G))$ is inspired.

(ii.) If (X, G) is weakly inspired LM-G-filter space, then $(Y, f^{\twoheadrightarrow}(G))$ is weakly inspired.

(iii.) If (X, G) is catalyzed LM-G-filter space, then $(Y, f^{\twoheadrightarrow}(G))$ is catalyzed.

It is easy to observe that

Theorem 4.13. *Let F be an LM-filter on X and $f : X \to Y$ be a function. Then $f^{\twoheadrightarrow}(F) : L^Y \to M$ defined by $f^{\twoheadrightarrow}(F)(B) = F(f^{\leftarrow}(B))$ for all $B \in L^Y$ is an LM-filter on Y.*

5. LM-G-filter Spaces and Neighborhood Systems

This section reveals the categorical connection between LM-G-filter spaces and neighborhood systems. In this section L and M are assumed to be completely distributive lattices with an order reversing involution " $'$ ".

Notation 5.1. [11] *The set of all fuzzy points x_λ $(\lambda \in J(L))$ is denoted by $J(L^X)$. A fuzzy point $x_\lambda \in J(L^X)$ quasi-coincides with $A \in L^X$ if $\lambda \nleq A'(x)$ and is denoted by $x_\lambda \hat{q} A$. The relation " does not quasi-coincides " is denoted by $\neg\hat{q}$.*

Definition 5.2. (See [14] for L-fuzzifying neighborhood system) *An LM-fuzzy neighborhood system on X is defined to be a set $\mathcal{N} = \{\mathcal{N}_{x_\lambda} : x_\lambda \in J(L^X)\}$ of maps $\{\mathcal{N}_{x_\lambda} : L^X \to M\}$ satisfying the following conditions:*

(FN1) *$\mathcal{N}_{x_\lambda}(1_X) = 1$ and $\mathcal{N}_{x_\lambda}(0_X) = 0$;*

(FN2) *$\mathcal{N}_{x_\lambda}(A) = 0$ if $x_\lambda \nleq A$;*

(FN3) *$\mathcal{N}_{x_\lambda}(A \wedge B) = \mathcal{N}_{x_\lambda}(A) \wedge \mathcal{N}_{x_\lambda}(B)$.*

*The pair $(X, \mathcal{N})$ is called an LM-fuzzy neighborhood space and it will be called topological if it satisfies moreover, **(FN4)** $\mathcal{N}_{x_\lambda}(A) = \bigvee_{x_\lambda \leq B \leq A} \bigwedge_{y_\mu \preceq B} \mathcal{N}_{y_\mu}(B)$. A mapping $f : X \to Y$ between two LM-fuzzy neighborhood spaces $(X, \mathcal{N}_1)$ and $(Y, \mathcal{N}_2)$ is called continuous if $\forall x_\lambda \in J(L^X), \forall B \in L^Y$, $(\mathcal{N}_1)_{x_\lambda}(f^{\leftarrow}(B)) \geq (\mathcal{N}_2)_{f(x)_\lambda}(B)$. The category of topological LM-fuzzy neighborhood spaces with continuous mappings as morphisms is denoted by (LM)-**FN**.*

Definition 5.3. [13] *An LM-fuzzy quasi-coincident neighborhood system on X is defined to be a set $\mathcal{Q} = \{\mathcal{Q}_{x_\lambda} : x_\lambda \in J(L^X)\}$ of maps $\{\mathcal{Q}_{x_\lambda} : L^X \to M\}$ satisfying the following conditions:*

(FQN1) $\mathcal{Q}_{x_\lambda}(1_X) = 1$ *and* $\mathcal{Q}_{x_\lambda}(0_X) = 0$;

(FQN2) $\mathcal{Q}_{x_\lambda}(A) = 0$ *if* $x_\lambda \neg \hat{q} A$;

(FQN3) $\mathcal{Q}_{x_\lambda}(A \wedge B) = \mathcal{Q}_{x_\lambda}(A) \wedge \mathcal{Q}_{x_\lambda}(B)$.

*The pair $(X, \mathcal{Q})$ is called an LM-fuzzy quasi-coincident neighborhood space and it will be called topological if it satisfies moreover, **(FQN4)**$\mathcal{Q}_{x_\lambda}(A) = \bigvee_{x_\lambda \hat{q} B \leq A} \bigwedge_{y_\mu \hat{q} B} \mathcal{Q}_{y_\mu}(B)$. A mapping $f : X \to Y$ between two LM-fuzzy quasi-coincident neighborhood spaces $(X, \mathcal{Q}_1)$ and $(Y, \mathcal{Q}_2)$ is called continuous if $\forall x_\lambda \in J(L^X), \forall B \in L^Y$, $(\mathcal{Q}_1)_{x_\lambda}(f^{\leftarrow}(B)) \geq (\mathcal{Q}_2)_{f(x)_\lambda}(B)$. The category of topological LM-fuzzy quasi-coincident neighborhood spaces with continuous mappings as morphisms is denoted by (LM)-**FQN**.*

Theorem 5.4. *Let (X, G) be an LM-G-filter space. Define $G_{x_\lambda} : L^X \to M$ by*

$$G_{x_\lambda}(A) = \begin{cases} G(A) & \text{if } x_\lambda \leq A\,, \\ 0 & \text{otherwise} \end{cases}$$

Then

(i.) $\mathcal{N}_G = \{G_{x_\lambda}; x_\lambda \in J(L^X)\}$ is a topological LM-fuzzy neighborhood system.

(ii.) If $f^{\rightarrow} : (X, G_1) \to (Y, G_2)$ is an LM-G-filter map, then $f^{\rightarrow} : (X, \mathcal{N}_{G_1}) \to (Y, \mathcal{N}_{G_2})$ is continuous.

Proof.

(i.) (**FN1**) and (**FN2**) are obvious.

(FN3) When $A \leq B$, $G_{x_\lambda}(A) \leq G_{x_\lambda}(B)$. For $A, B \in L^X$ such that $x_\lambda \leq A$ and $x_\lambda \leq B$, we have $x_\lambda \leq A \wedge B$. Hence $G_{x_\lambda}(A) \wedge G_{x_\lambda}(B) = G(A) \wedge G(B) \leq G(A \wedge B) = G_{x_\lambda}(A \wedge B)$. Therefore, $G_{x_\lambda}(A \wedge B) = G_{x_\lambda}(A) \wedge G_{x_\lambda}(B)$.

(FN4) If $x_\lambda \nleq A$, then $G_{x_\lambda}(A) = \bigvee_{x_\lambda \leq B \leq A} \bigwedge_{y_\mu \preceq B} G_{y_\mu}(B) = 0$.

If $x_\lambda \leq A$, then $\bigvee_{x_\lambda \leq B \leq A} \bigwedge_{y_\mu \preceq B} G_{y_\mu}(B) = \bigvee_{x_\lambda \leq B \leq A} G(B) = G(A) = G_{x_\lambda}(A)$.

Therefore, $G_{x_\lambda}(A) = \bigvee_{x_\lambda \leq B \leq A} \bigwedge_{y_\mu \preceq B} G_{y_\mu}(B)$.

(ii.) If $(G_2)_{f(x)_\lambda}(B) \neq 0$ for any $B \in L^Y$, then $f(x)_\lambda \leq B$, which implies $x_\lambda \leq f^{\leftarrow}(B)$. Therefore $(G_1)_{x_\lambda}(f^{\leftarrow}(B)) = G_1(f^{\leftarrow}(B)) \geq G_2(B) = (G_2)_{f(x)_\lambda}(B)$ for each $x_\lambda \in J(L^X)$. Hence $f^{\rightarrow} : (X, \mathcal{N}_{G_1}) \rightarrow (Y, \mathcal{N}_{G_2})$ is continuous.

Corollary 5.5. *Let (X, G) be an LM-G-filter space and $\omega(G) = \mathcal{N}_G$. Then ω is a functor from LM-**G** to (LM)-**FN***

Theorem 5.6. *Let (X, G) be an LM-G-filter space. Define $\widehat{G}_{x_\lambda} : L^X \rightarrow M$ by*

$$\widehat{G}_{x_\lambda}(A) = \begin{cases} G(A) & \text{if } x_\lambda \hat{q} A\,, \\ 0 & \text{if } x_\lambda \neg \hat{q} A \end{cases}$$

Then

(i.) $\mathcal{Q}_G = \{\widehat{G}_{x_\lambda}; x_\lambda \in J(L^X)\}$ is a topological LM-fuzzy quasi-coincident neighborhood system.

(ii.) If $f^{\rightarrow} : (X, G_1) \rightarrow (Y, G_2)$ is an LM-G-filter map, then $f^{\rightarrow} : (X, \mathcal{Q}_{G_1}) \rightarrow (Y, \mathcal{Q}_{G_2})$ is continuous.

Proof.

(i.) (**FQN1**) and (**FQN2**) are obvious.

(FQN3) When $A \leq B$, $\widehat{G}_{x_\lambda}(A) \leq \widehat{G}_{x_\lambda}(B)$. For $A, B \in L^X$ such that $x_\lambda \hat{q} A$ and $x_\lambda \hat{q} B$, we have $x_\lambda \hat{q} A \wedge B$. Hence $\widehat{G}_{x_\lambda}(A) \wedge \widehat{G}_{x_\lambda}(B) = G(A) \wedge G(B) \leq G(A \wedge B) = \widehat{G}_{x_\lambda}(A \wedge B)$. Therefore, $\widehat{G}_{x_\lambda}(A \wedge B) = \widehat{G}_{x_\lambda}(A) \wedge \widehat{G}_{x_\lambda}(B)$.

(FQN4) If $x_\lambda \neg\hat{q} A$, then $\widehat{G}_{x_\lambda}(A) = \bigvee_{x_\lambda \hat{q} B \leq A} \bigwedge_{y_\mu \hat{q} B} \widehat{G}_{y_\mu}(B) = 0$.

If $x_\lambda \hat{q} A$, then $\bigvee_{x_\lambda \hat{q} B \leq A} \bigwedge_{y_\mu \hat{q} B} \widehat{G}_{y_\mu}(B) = \bigvee_{x_\lambda \hat{q} B \leq A} G(B) = G(A) = \widehat{G}_{x_\lambda}(A)$.

Therefore, $\widehat{G}_{x_\lambda}(A) = \bigvee_{x_\lambda \hat{q} B \leq A} \bigwedge_{y_\mu \hat{q} B} \widehat{G}_{y_\mu}(B)$.

(ii.) If $(\widehat{G}_2)_{f(x)_\lambda}(B) \neq 0$ for any $B \in L^Y$, then $f(x)_\lambda \hat{q} B$, which implies $x_\lambda \hat{q} f^{\leftarrow}(B)$. Therefore $(\widehat{G}_1)_{x_\lambda}(f^{\leftarrow}(B)) = G_1(f^{\leftarrow}(B)) \geq G_2(B) = (\widehat{G}_2)_{f(x)_\lambda}(B)$ for each $x_\lambda \in J(L^X)$. Hence $f^{\rightarrow} : (X, \mathcal{Q}_{G_1}) \rightarrow (Y, \mathcal{Q}_{G_2})$ is continuous.

Corollary 5.7. *Let (X, G) be an LM-G-filter space and $\Omega(G) = \mathcal{Q}_G$. Then Ω is a functor from LM-**G** to (LM)-**FQN***

6. Conclusion

The study has identified images of LM-G-filters and LM-G-filterbases induced by functions and investigated their properties. It is proved that the image of s-stratified LM-G-filterbase is again an s-stratified LM-G-filterbase. The properties of being weakly inspired, catalyzed, s-stratified as well as the stratification of LM-G-filters are preserved by images.

In addition the study has obtained topological neighborhood systems from LM-G-filter spaces and categorical connections of LM-**G** with (LM)-**FN** and (LM)-**FQN** are identified. Identifying more relations of LM-G-filters with neighborhood systems and investigating the role of stratified, inspired, weakly inspired and catalyzed LM-G-filters in convergence theory are part of our future research.

Acknowledgements

The authors are thankful to a referee for his valuable suggestions and comments. The first author wishes to thank CSIR, India for giving financial assistance under the Senior Research Fellowship awarded by order No. 08/528(0004)/2019-EMR-1 dated 08/04/2021.

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