South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 229-244

DOI: 10.56827/SEAJMMS.2022.1802.21 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

COMMON FIXED POINT THEOREM FOR SIX MAPPINGS

Kakali Roy and Kalishankar Tiwary

Department of Mathematics,
Raiganj University,
Raiganj - 733134, West Bengal, INDIA

E-mail : kakaliroy2610@gmail.com, tiwarykalishankar@yahoo.com

(**Received:** *Aug. 27, 2021* **Accepted:** *Jul. 28, 2022* **Published:** *Aug. 30, 2022)*

Abstract: In this paper we shall obtain a common fixed point of six mappings in a metric space which extend the results proved in {[10], [11], [24]}.

Keywords and Phrases: Compatible mappings, weakly compitable mappings, EA property, CLR property, common fixed point.

2020 Mathematics Subject Classification: 47H09, 47H10, 54H25.

1. Introduction and Preliminaries

During the last three decades research workers have devoted their much time to generalise the Banach contraction theorem in various ways. They have obtained common fixed point of two mappings using commutative property. The notion of commutativity has been weakend in terms of weakly commutative, compatibity, weak compatibity etc. . A number of common fixed point theorems have been obtained using compatibity, weak compatibity etc. We shall quote some definitions and theorems from the literature to complete our paper. Let f and g be selfmaps on the metric space (X, d). If $fx = gx = p$ for some $x \in X$, then x is called a coincidence point [9] of f and g and p is called a point of coincidence of f and g. In 1986, Jungck gave the concept of compatible mappings [9]. A pair (A, B) of self mappings of a metric space (X, d) is said to be compatible if $\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t \ for \ some \ t \in X.$$

Jungck introduced weakly compatible mappings [11] which is generalized concept of compatible mappings [9]. A pair (A, B) of self mappings of a metric space (X, d) is said to be weakly compatible, if $Ax = Bx$ for some $x \in X$ implies $ABx = BAx$. To prove the common fixed point of weakly compatible mappings, it was required that the space should be complete or the mappings should be continuous. In 2002, El Moutawakil and Aamri introduced the concept of (E.A) property [21]. Let (X, d) be a metric space and $f,\ h : X \to X$ be two self-maps. The pair (f, h) is said to satisfy the (E.A) property if there exists sequence $\{x_n\}$ in X and some $z \in X$ such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z \in X.$$

Using (E.A) property one can prove the common fixed point theorems including only the closeness condition of the space. The concept of (E.A) property has been generalized by Liu *et. al.* [20] to common (E.A) property [20]. Let (X, d) be a metric space and f, g, h and $J : X \to X$ be four self-maps. The pairs (f, h) and (g, J) satisfy the common (E.A) property if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t \in X.$$

In [4], we get the concept of subsequentially continuous mappings.
Let (X, d) be a metric space and $f, h : X \to X$ be self-maps. The pair (f, h) is called subsequentially continuous if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = z$$

for some $z \in X$ and

$$\lim_{n\to\infty} fhx_n = \lim_{n\to\infty} hfx_n = hz.$$

The concept compatibility and subsequential continuity together imply the existence of coincidence point of two mappings. Kumam and Sintunavarat introduced the following concept which does not require the condition closeness of the space. Let (X, d) be a metric space and $f, h : X \to X$ be self-maps. The pair (f, h) said to satisfy the common limit in the range [18] of h property if

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} hx_n = hx,$$

for some $x \in X$. One can observe that (E.A) property of (f, h) together with the closeness of h(X) implies common limit in the range of h property. Chauhan *et. al.* extended the concept of (CLR) property to the common (CLR) property [5]. Let

(X, d) be a metric space and f, g, h and $J : X \to X$ be four self-maps. The pairs (f, h) and (g, J) satisfy the common limit range property with respect to mappings h and J, denoted by $CLR_{(h,J)}$ if there exist two sequences $\{x_n\}$ and $\{y_n\}$ in X such that

$$\lim_{n\ to\infty} fx_n = \lim_{n\to\infty} hx_n = \lim_{n\to\infty} gy_n = \lim_{n\to\infty} Jy_n = t \in h(X) \cap J(X).$$

V. Popa [23] introduced a new class of mappings and proved some results for common fixed point theorem. Let (X, d) be a metric space and f, g and $T : X \to X$ be three self-maps. The pair (f, g) is said to satisfy common limit range property with respect to T [23], denoted $CLR_{(f,g),T}$ if there exists a sequence $\{x_n\}$ in X such that

$$\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t \in g(X) \cap T(X).$$

The author has given a remark that if (f, h) and (g, T) satisfy the common limit range property with respect to mappings h and T, then (f, h) satisfy the $CLR_{(f,h),T}$. But the converse is not true. Let f be a continuous selfmap of a metric space (X, d). A selfmap g of X is said to be f-contractive [24] if $d(gx, gy) < d(fx, fy)$ for each x, y in X for which $gx \neq gy$. The definitions quoted above have been used in the following theorems to obtain a common fixed point.

Theorem 1.1. [8] *Let f be a continuous selfmap of a complete metric space (X, d). Then f has a fixed point in X if and only if there exists an $\alpha \in (0, 1)$ and a mapping $g : X \to X$ which commutes with f and satisfies $g(X) \subset f(X)$ and $d(gx, gy) \leq \alpha d(f(x), f(y))$ for all $x, y \in X$. Indeed, f and g have a unique common fixed point.*

Theorem 1.2. [6] *Let S and T be continuous selfmaps of a complete metric space (X, d). Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $S(X) \cap T(X)$ such that $AS = SA, AT = TA$ and $d(Ax, Ay) \leq \alpha d(Sx, Ty)$ for all $x, y \in X$ and $0 < \alpha < 1$. Indeed, S, T and A have a unique common fixed point.*

Theorem 1.3. [17] *Let S and T be continuous selfmaps of a Hilbert space X. Then S and T have a common fixed point in X if and only if there exists a continuous mapping A of X into $S(X) \cap T(X)$ which commutes with S and T and satisfies the inequality*

$$\|Ax - Ay\| \leq \alpha\|Ax - Sx\| + \beta\|Ay - Ty\| + \gamma\|Sx - Ty\|$$

for all x, y in X, where $\alpha, \beta, \gamma \geq 0$ with $0 < \alpha + \beta + \gamma < 1$. Indeed S, T and A then have a unique common fixed point.

Theorem 1.4. [22] *A continuous selfmap f of a metric space (X, d) has a fixed point if and only if there exists an f-contractive map g, which commutes with f, a subset $M \subset X$, and a point $x_0 \in M$ such that*

$$d(fx, fx_0) - d(gx, gx_0) \geq 2d(fx_0, gx_0)$$

for every $x \in X/M$, and g maps M into a compact subset of X. Indeed, f and g have a unique common fixed point.

Theorem 1.5. [24] *Let f and g be continuous selfmaps of a complete metric space (X, d). Then f and g have a common fixed point in X if and only if there exists a continuous map $h : X \to f(X) \cap g(X)$ which is compitable with f and g which satisfies*

$$\begin{aligned} d(hx, hy) \ \leq \ & \max\Big\{d(hx, fy), d(hy, gy), d(fx, gy), \frac{d(hx, gy) + d(hy, fx)}{2}\Big\} \\ & -\omega\Big(\max\Big\{d(hx, fy), d(hy, gy), d(fx, gy), \frac{d(hx, gy) + d(hy, fx)}{2}\Big\}\Big) \end{aligned}$$

for all $x, y \in X$ where $\omega : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ is a continuous function such that $0 < \omega(r) < r$ for all $r > 0$. . Indeed f, g and h have a unique common fixed point.

Let $\mathbf{R}^+$ denote the set of nonnegative reals and $\omega : \mathbf{R}^+ \to \mathbf{R}^+$ a continuous function such that $0 < \omega(r) < r$ for all $r > 0$.

2. Main Result

Theorem 2.1. *Let, (X, d) be a complete metric space. Let A, B, S, T, P and Q be self mappings satisfying $P(X) \subset AB(X), Q(X) \subset ST(X)$,*

$$AB = BA, ST = TS, BQ = QB, PT = TP, TQ = QT$$

and

$$d(Px, Qy) \leq \max\Big\{d(STx, ABy), d(Px, STx), d(Qy, ABy), \frac{d(Px, ABy) + d(Qy, STx)}{2}\Big\}$$

$$-\omega\Big(\max\Big\{d(STx, ABy), d(Px, STx), d(Qy, ABy), \frac{d(Px, ABy) + d(Qy, STx)}{2}\Big\}\Big)$$

where $\omega : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be a continuous function which satisfies $0 < \omega(t) < t$.
Again (P, ST) and (Q, AB) are weakly compatible.
Then A, B, S, T, P and Q have a unique common fixed point if either P(X) or Q(X) is complete.

Proof. Since $P(x) \subset AB(x)$, then for any $x_0 \in X, Px_0 \in P(X)$ and there exists $x_1 \in X$ such that $Px_0 = ABx_1$.
Again $Q(x) \subset ST(X)$, then for Qx_1, there exists $x_2 \in X$ such that $Qx_1 = STx_2$.
Proceding in the similar way we can construct a sequence $\{y_n\}$ where

$$y_{2n} = Px_{2n} = ABx_{2n+1}$$

$$y_{2n+1} = Qx_{2n+1} = ST_{2n+2}, n = 0, 1, 2 \ldots$$

First we proof that $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.
Define: $d_n = d(y_n, y_{n+1})$ i.e. $d_{2n} = d(y_{2n}, y_{2n+1})$. Now

$$\begin{aligned}
d(y_{2n}, y_{2n+1}) &= d(Px_{2n}, Qx_{2n+1}) \\
&\leq \max\Big\{d(STx_{2n}, ABx_{2n+1}), d(Px_{2n}, STx_{2n}), d(Qx_{2n+1}, ABx_{2n+1}), \\
&\quad \frac{d(Px_{2n}, ABx_{2n+1}) + d(Qx_{2n+1}, STx_{2n})}{2}\Big\} \\
&\quad -\omega\Big(\max\Big\{d(STx_{2n}, ABx_{2n+1}), d(Px_{2n}, STx_{2n}), d(Qx_{2n+1}, ABx_{2n+1}), \\
&\quad \frac{d(Px_{2n}, ABx_{2n+1}) + d(Qx_{2n+1}, STx_{2n})}{2}\Big\}\Big) \\
&= \max\Big\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \\
&\quad \frac{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{2}\Big\} \\
&\quad -\omega\Big(\max\Big\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), \\
&\quad \frac{d(y_{2n}, y_{2n}) + d(y_{2n+1}, y_{2n-1})}{2}\Big\}\Big)
\end{aligned}$$

If $d_{2n-1} < d_{2n}$ then from the above inequality we get

$$d_{2n} \leq d_{2n} - \omega(d_{2n}),\ \textit{where}\ \omega(t) > 0\ \textit{for}\ t > 0,\ \textit{which is a contradiction.}$$

Therefore $d_{2n} \leq d_{2n-1}$ and so

$$d_{2n} \leq d_{2n-1} - \omega(d_{2n-1})$$

$$i.e.\ \omega(d_{2n-1}) \leq d_{2n-1} - d_{2n}.$$

Similarly we get $w(d_{2n}) \leq d_{2n} - d_{2n+1}$.
Therefore $\sum_{i=0}^{n} \omega(d_i) \leq d_0 - d_{n+1} \leq d_0$.

Hence $\sum_{n=1}^{\infty}\omega(d_n)$ is convergent and so $\lim_{n\to\infty}\omega(d_n)=0$.
The sequence $\{d_n\}$ being a non-increasing sequence of +ve numbers is convergent.
Say the limit is p i.e. $\lim_{n\to\infty} d_n = p$.
If $p>0$, then $\lim_{n\to\infty}\omega(d_n)=\omega(p)=0$, since ω is continuous.
But for $p>0, \omega(p)>0$.
Therefore $\lim_{n\to\infty} d_n = \lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.
Claim. $\{y_n\}$ is Cauchy sequence in (X, d).
In order to show that $\{y_n\}$is Cauchy sequence, it is sufficient to show that $\{y_{2n}\}$ is Cauchy sequence since $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.
If possible let $\{y_{2n}\}$ is not a Cauchy sequence.
Then for every $\epsilon>0$ and every positive integer k there exists two positive integers 2m(k) and 2n(k) such that

$$d(y_{2m(k)}, y_{2n(k)}) > \epsilon \ \ for \ \ 2m(k) > 2n(k) > k.$$

Let 2m(k) be the least +ve integer for which $d(y_{2m(k)}, y_{2n(k)}) > \epsilon$ and $d(y_{2m(k)-2}, y_{2n(k)}) \leq \epsilon$.
Now,

$$\begin{aligned}\epsilon < d(y_{2m(k)}, y_{2n(k)}) &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)})\\ &\leq \epsilon + d_{2m(k)-2} + d_{2m(k)-1}.\end{aligned}$$

Taking the limit $n\to\infty$ in the above inequality, we get

$$\epsilon = \lim_{n\to\infty} d(y_{2m(k)}, y_{2n(k)}).$$

By triangle inequality, we get

$$|d(y_{2m(k)}, y_{2n(k)}) - d(y_{2m(k)}, y_{2n(k)+1})| \leq d(y_{2n(k)}, y_{2n(k)+1})$$

$$|d(y_{2m(k)}, y_{2n(k)+1}) - d(y_{2n(k)+1}, y_{2m(k)+1})| \leq d(y_{2m(k)}, y_{2m(k)+1})$$

$$|d(y_{2n(k)+1}, y_{2m(k)+1}) - d(y_{2m(k)+1}, y_{2n(k)+2})| \leq d(y_{2n(k)+1}, y_{2n(k)+2})$$

$$|d(y_{2m(k)}, y_{2n(k)+2}) - d(y_{2m(k)+1}, y_{2n(k)+2})| \leq d(y_{2m(k)}, y_{2m(k)+2}). \tag{1}$$

Taking limit $n\to\infty$ in the above inequalities, we get

$$\begin{aligned}\epsilon &= \lim_{n\to\infty} d(y_{2m(k)}, y_{2n(k)+1}),\\ &= \lim_{n\to\infty} d(y_{2n(k)+1}, y_{2m(k)+1}),\\ &= \lim_{n\to\infty} d(y_{2m(k)+1}, y_{2n(k)+2}).\end{aligned}$$

Now, $d(y_{2n(k)+2}, y_{2m(k)+1})$

$$\begin{aligned}
&= d(Px_{2n(k)+2}, Qx_{2m(k)+1}) \\
&\leq \max\Big\{d(STx_{2n(k)+2}, ABx_{2m(k)+1}), d(Px_{2n(k)+2}, STx_{2n(k)+2}), \\
&\quad d(Qx_{2m(k)+1}, ABx_{2m(k)+1}), \\
&\quad \frac{d(Px_{2n(k)+2}, ABx_{2m(k)+1}) + d(STx_{2n(k)+2}, Qx_{2m(k)+1})}{2}\Big\} - \\
&\quad \omega\Big((\max\Big\{d(STx_{2n(k)+2}, ABx_{2m(k)+1}), d(Px_{2n(k)+2}, STx_{2n(k)+2}), \\
&\quad d(Qx_{2m(k)+1}, ABx_{2m(k)+1}), \\
&\quad \frac{d(Px_{2n(k)+2}, ABx_{2m(k)+1}) + d(STx_{2n(k)+2}, Qx_{2m(k)+1})}{2}\Big\}\Big) \\
&= \max\Big\{d(y_{2n(k)+1}, y_{2m(k)}), d(y_{2n(k)+2}, d_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)}), \\
&\quad \frac{d(y_{2n(k)+2}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)+1})}{2}\Big\} - \\
&\quad \omega\Big(\max\Big\{d(y_{2n(k)+1}, y_{2m(k)}), d(y_{2n(k)+2}, d_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)}), \\
&\quad \frac{d(y_{2n(k)+2}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)+1})}{2}\Big\}\Big) \\
&= \max\Big\{d(y_{2n(k)+1}, y_{2m(k)}), d(y_{2n(k)+2}, y_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)}), \\
&\quad \frac{d(y_{2n(k)+2}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)+1})}{2}\Big\} - \\
&\quad \omega\Big(\max\Big\{d(y_{2n(k)+1}, y_{2m(k)}), d(y_{2n(k)+2}, y_{2n(k)+1}), d(y_{2m(k)+1}, y_{2m(k)}), \\
&\quad \frac{d(y_{2n(k)+2}, y_{2n(k)+1}) + d(y_{2n(k)+1}, y_{2n(k)}) + d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)+1})}{2}\Big\}\Big)
\end{aligned}$$

Taking limit $n \to \infty$, we get

$$\begin{aligned}
\epsilon &\leq \max\Big\{\epsilon, 0, 0, \frac{0+0+\epsilon+\epsilon}{2}\Big\} - \omega\Big(\max\Big\{\epsilon, 0, 0, \frac{0+0+\epsilon+\epsilon}{2}\Big\}\Big) \\
&= \epsilon - \omega(\epsilon), \textit{which is a contraction.}
\end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence.
Since (X, d) is complete. So $\{y_n\}$ is convergent in (X, d), say $\{y_n\}$ converges to z, $z \in X$.
Then $\{Px_{2n}\}, \{Qx_{2n+1}\}, \{ABx_{2n+1}\}, \{STx_{2n+2}\}$ begin a subsequences of $\{y_n\}$ also

converge to z, i.e.
$\lim_{n\to\infty} Px_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} ABx_{2n+1} = \lim_{n\to\infty} STx_{2n+2} = z.$
Case 1. Let P(X) is complete.
Then $z \in P(X)$. Since $P(X) \subset AB(X)$. Then there exists a point $u \in AB$ such that $z = ABu$.
We shall prove that $Qu = z$.
If possible that $z \neq Qu$.

$$\begin{aligned} d(Qu,z) &\leq d(Qu,Px_{2n(k)+2}) + d(Px_{2n(k)+2},z) \\ &\leq \max\Big\{d(STx_{2n(k)+2},ABu), d(Px_{2n(k)+2},STx_{2n(k)+2}), d(Qu,ABu), \\ &\quad \frac{d(Px_{2n(k)+2},ABu)+d(Qu,STx_{2n(k)+2})}{2}\Big\} - \\ &\quad \omega\Big(\max\Big\{d(STx_{2n(k)+2},ABu), d(Px_{2n(k)+2},STx_{2n(k)+2}), d(Qu,ABu), \\ &\quad \frac{d(Px_{2n(k)+2},ABu)+d(Qu,STx_{2n(k)+2})}{2}\Big\}\Big) + d(Px_{2n(k)+2},z). \end{aligned}$$

Taking limit $n \to \infty$ we get

$$\begin{aligned} d(Qu,z) &\leq \max\Big\{d(z,z), d(z,z), d(Qu,z), \frac{d(z,z)+d(Qu,z)}{2}\Big\} - \\ &\quad \omega\Big(\max\Big\{d(z,z), d(z,z), d(Qu,z), \frac{d(z,z)+d(Qu,z)}{2}\Big\}\Big) \\ &= \max\Big\{0,0,d(Qu,z),\frac{d(Qu,z)}{2}\Big\} - \omega\Big(\max\Big\{0,0,d(Qu,z),\frac{d(Qu,z)}{2}\Big\}\Big) \\ &= d(Qu,z) - w(d(Qu,z)), \ \textit{a contradiction.} \end{aligned}$$

Hence $Qu = z$ and so we have $Qu = ABu = z$.
Since $Q(X) \subset ST(X)$, and $z \in Q(X)$. Then there exists v in X such that $z = STv$.
Now we shall prove that $Pv = z$.
If possible let $Pv \neq z$.

$$\begin{aligned} & d(Pv,z) = d(Pv,Qu) \\ & \leq \max\Big\{d(STv,ABu), d(Pv,STv), d(Qu,ABu), \frac{d(Pv,ABu)+d(Qu,STv)}{2}\Big\} - \\ & \omega\Big(\max\Big\{d(STv,ABu), d(Pv,STv), d(Qu,ABu), \frac{d(Pv,ABu)+d(Qu,STv)}{2}\Big\}\Big) \\ & = \max\Big\{d(z,z), d(Pv,z), d(z,z), \frac{d(Pv,z)+d(z,z)}{2}\Big\} - \end{aligned}$$

$$\omega\bigg(\max\Big\{d(z,z), d(Pv,z), d(z,z), \frac{d(Pv,z)+d(z,z)}{2}\Big\}\bigg)$$
$$= d(Pv,z) - w(d(Pv,z)),\ \textit{a contradiction.}$$

Therefore we have $Pv = z$ and so $Qu = ABu = Pv = STv = z$.
Case 2. If Q(X) is complete the similarly we have,
$Qu = ABu = Pv = STv = z$.
Since (P, ST) is weakly compatible and
$Pv = STv = z$.
So, $ST(Pv) = P(STv)$ i.e. $STz = Pz$.
Again, since (Q, AB) is weakly compatible and $Qu = ABu = z$ So, $Q(ABu) = AB(Qu)$ i.e. $Qz = ABz$.
Now, we show that z is fixed point of P.
If $Pz = z$, then

$$\begin{aligned}
& d(Pz,z) = d(Pz,Qu) \\
& \leq \max\Big\{d(STz,ABu), d(Pz,STz), d(Qu,ABu), \frac{d(Pz,ABu)+d(Qu,STz)}{2}\Big\} - \\
& \omega\bigg(\max\Big\{d(STz,ABu), d(Pz,STz), d(Qu,ABu), \frac{d(Pz,ABu)+d(Qu,STz)}{2}\Big\}\bigg) \\
& = \max\Big\{d(Pz,z), 0, 0, \frac{d(Pz,z)+d(z,Pz)}{2}\Big\} - \\
& \omega\bigg(\max\Big\{d(Pz,z), 0, 0, \frac{d(Pz,z)+d(z,Pz)}{2}\Big\}\bigg) \\
& = d(Pz,z) - w(d(Pz,z)),\ \textit{a contradiction.}
\end{aligned}$$

So, $Pz = z$.
Hence $STz = Pz = z$.
Claim: Tz = z
If possible let $Tz \neq z$. Let $x = Tz, y = x_{2n+1}$.

$$\begin{aligned}
d(P(Tz), Qx_{2n+1}) \ \leq \ & \max\Big\{d(ST(Tz), ABx_{2n+1}), d(P(Tz), ST(Tz)), \\
& d(Qx_{2n+1}, ABx_{2n+1}), \frac{d(P(Tz), ABx_{2n+1}) + d(Qx_{2n+1}, ST(Tz))}{2}\Big\} - \\
& \omega\bigg(\max\Big\{d(ST(Tz), ABx_{2n+1}), d(P(Tz), ST(Tz)), d(Qx_{2n+1}, ABx_{2n+1}), \\
& \frac{d(P(Tz), ABx_{2n+1}) + d(Qx_{2n+1}, ST(Tz))}{2}\Big\}\bigg).
\end{aligned}$$

Taking $n \to \infty$

$$\begin{aligned} d(P(Tz),z) &\leq \max\Big\{d(Tz,z),0,0,\frac{d(Tz,z)+d(Tz,z)}{2}\Big\} \\ &\quad -\omega\Big(\max\Big\{d(Tz,z),0,0,\frac{d(Tz,z)+d(Tz,z)}{2}\Big\}\Big) \\ \Rightarrow d(Tz,z) &\leq d(Tz,z)-w(d(Tz,z)),\ a\ contradiction. \end{aligned}$$

Therefore $Tz = z$.
Hence $Pz = ABz = Sz = Tz = z$.
Claim Bz =z.
If $Bz \neq z$ then let $x = z, y = Bz$. d(Pz, Q(Bz))

$$\begin{aligned} &\leq \max\Big\{d(STz,AB(Bz)),d(Pz,STz),d(Q(Bz),AB(Bz)), \\ &\quad \frac{d(Pz,AB(Bz))+d(Q(Bz),STz)}{2}\Big\} - \\ &\quad \omega\Big(\max\Big\{d(STz,AB(Bz)),d(Pz,STz),d(Q(Bz),AB(Bz)), \\ &\quad \frac{d(Pz,AB(Bz))+d(Q(Bz),STz)}{2}\Big\}\Big) \\ &= \max\Big\{d(z,Bz),0,0,d(z,Bz)\Big\} - \omega\Big(\max\Big\{d(z,Bz),0,0,d(z,Bz)\Big) \\ &= d(Bz,z)-w(d(Bz,z)),\ a\ contradiction. \end{aligned}$$

Therefore $Bz = z$.
Hence $Pz = Qz = Sz = Tz = Az = Bz = z$.
Uniqueness.
Let $z' \in X$ be such that $Pz' = Qz' = Az' = Bz' = Sz' = Tz' = z'$ and $z' \neq z$.

$$\begin{aligned} &d(z,z') = d(Pz,Qz') \\ &\leq \max\Big\{d(STz,ABz'),d(Pz,STz),d(Qz',ABz'),\frac{d(Pz,ABz')+d(Qz,STz)}{2}\Big\} - \\ &\omega\Big(\max\Big\{d(STz,ABz'),d(Pz,STz),d(Qz',ABz'),\frac{d(Pz,ABz')+d(Qz,STz)}{2}\Big\}\Big) \\ &= \max\Big\{d(z,z'),0,0,d(z,z')\Big\} - \omega\Big(\max\Big\{d(z,z'),0,0,d(z,z')\Big\}\Big) \\ &= d(z,z')-w(d(z,z')),\ a\ contradiction. \end{aligned}$$

Hence $z = z'$.

Corollary 2.1. *Let, (X, d) be a complete metric space. Let A, B, P and Q be self mappings satisfying $P(X) \subseteq A(X), Q(X) \subseteq S(X)$, and*

$$d(Px, Qy) \leq \max\Big\{d(Sx, Ay), d(Px, Sx), d(Qy, Ay), \frac{d(Px, Ay) + d(Qy, Sx)}{2}\Big\}$$
$$-\omega\Big(\max\Big\{d(Sx, Ay), d(Px, Sx), d(Qy, Ay), \frac{d(Px, Ay) + d(Qy, Sx)}{2}\Big\}\Big)$$

where $\omega : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be a continuous function which satisfies $0 < \omega(t) < t$.
Again (P, S) and (Q, A) are weakly compatible.
Then A, B, P and Q have a unique common fixed point if either P(X) or Q(X) is complete.
Proof. This result easily follows from Theorem 2.1 by taking B= T= I, the identity map.

Corollary 2.2. *Let, (X, d) be a complete metric space. Let P and Q be onto mappings on (X, d) and*

$$d(x, y) \leq \max\Big\{d(Qx, Py), d(x, Qx), d(y, Py), \frac{d(x, Py) + d(y, Qx)}{2}\Big\}$$
$$-\omega\Big(\max\Big\{d(x, y), d(x, Qx), d(y, Py), \frac{d(x, Py) + d(y, Qx)}{2}\Big\}\Big)$$

where $\omega : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be a continuous function which satisfies $0 < \omega(t) < t$.
Then P and Q have a unique common fixed point.
Proof. Set P = Q = I, the identity mapping and $A \to P$ and $B \to Q$ in the Corollary 2.1.

Corollary 2.3. *Let, (X, d) be a complete metric space. Let P be onto mapping on (X, d) and*

$$d(x, y) \leq \max\Big\{d(Px, Py), d(x, Px), d(y, Py), \frac{d(x, Py) + d(y, Px)}{2}\Big\}$$
$$-\omega\Big(\max\Big\{d(Px, Py), d(x, Px), d(y, Py), \frac{d(x, Py) + d(y, Px)}{2}\Big\}\Big)$$

where $\omega : \boldsymbol{R}^+ \to \boldsymbol{R}^+$ be a continuous function which satisfies $0 < \omega(t) < t$.
Then P has a unique common fixed point.
Proof. Let P= Q in Corollary 2.2.

To validate the conditions used in theorem 2.1. we exhibit the following example:

Example 1. Let $X = [0, 1]$ be a set with the metric $d : X \times X \to \mathbf{R}$ given by $d(x, y) = 0$, if $x = y$ and $d(x, y) = \frac{x+y}{2}$, if $x \neq y$ and $\omega : \mathbf{R}^+ \to \mathbf{R}^+$, defined by $\omega(t) = \frac{t}{4}$ be a continuous function which satisfies $0 < \omega(t) < t$.
Define $P, A, B, Q, S, T : X \times X \to X$ by
$P(x) = \frac{x}{4}, A(x) = x, B(x) = \frac{x}{2}, Q(x) = \frac{x}{12}, S(x) = \frac{x}{3}, T(x) = \frac{x}{2}$.
Now $AB(x) = A(\frac{x}{2}) = \frac{x}{2}$ and $ST(x) = S(\frac{x}{2}) = \frac{x}{6}$. So $P(X) = [0, \frac{1}{4}] \subset [0, \frac{1}{2}] = AB(X)$ and $Q(X) = [0, \frac{1}{12}] \subset [0, \frac{1}{6}] = ST(X)$.
One can easily verify that $AB(x) = \frac{x}{2} = BA(x)$, $ST(x) = \frac{x}{6} = TS(x)$, $BQ(x) = \frac{x}{12} = QB(x)$, $PT(x) = \frac{x}{8} = TP(x)$, and $TQ(x) = \frac{x}{24} = QT(x)$.
Now

$$\begin{aligned}
d(Px, Qy) \leq\ & \max\Big\{d(STx, ABy), d(Px, STx), d(Qy, ABy), \\
& \frac{d(Px, ABy) + d(Qy, STx)}{2}\Big\} - \omega\bigg(\max\Big\{d(STx, ABy), d(Px, STx), \\
& d(Qy, ABy), \frac{d(Px, ABy) + d(Qy, STx)}{2}\Big\}\bigg) \\
d(\frac{x}{4}, \frac{y}{2}) \leq\ & \max\Big\{d(\frac{x}{6}, \frac{y}{2}), d(\frac{x}{4}, \frac{x}{6}), d(\frac{y}{12}, \frac{y}{2}), \frac{d(\frac{x}{2}, \frac{y}{2}) + d(\frac{y}{12}, \frac{x}{6})}{2}\Big\} \\
& -\omega\bigg(\max\Big\{d(\frac{x}{6}, \frac{y}{2}), d(\frac{x}{4}, \frac{x}{6}), d(\frac{y}{12}, \frac{y}{2}), \frac{d(\frac{x}{2}, \frac{y}{2}) + d(\frac{y}{12}, \frac{x}{6})}{2}\Big\}\bigg) \\
\frac{1}{2}\Big(\frac{x}{4} + \frac{y}{12}\Big) \leq\ & \max\Big\{\frac{1}{2}\Big(\frac{y}{6} + \frac{x}{2}\Big), \frac{1}{2}\Big(\frac{x}{4} + \frac{x}{6}\Big), \frac{1}{2}\Big(\frac{y}{2} + \frac{y}{12}\Big), \frac{1}{2.2}\Big(\frac{x}{4} + \frac{y}{2} + \frac{y}{12} + \frac{x}{6}\Big)\Big\} \\
& -\omega\bigg(\max\Big\{\frac{1}{2}\Big(\frac{y}{6} + \frac{x}{2}\Big), \frac{1}{2}\Big(\frac{x}{4} + \frac{x}{6}\Big), \frac{1}{2}\Big(\frac{y}{2} + \frac{y}{12}\Big), \frac{1}{2.2}\Big(\frac{x}{4} + \frac{y}{2} + \frac{y}{12} + \frac{x}{6}\Big)\Big\}\bigg) \\
\frac{3x + y}{24} \leq\ & \max\Big\{\frac{3y + x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x + 6y}{48}\Big\} \\
& -\omega\bigg(\max\Big\{\frac{3y + x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x + 6y}{48}\Big\}\bigg).
\end{aligned}$$

Case 1. If $x = y$, then $\frac{3x+y}{24} = \frac{x}{6}$ and $\max\Big\{\frac{3y+x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x+6y}{48}\Big\}$ $-\omega\bigg(\max\Big\{\frac{3y+x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x+6y}{48}\Big\}\bigg) = \max\Big\{\frac{x}{3}, \frac{5x}{24}, \frac{7x}{24}, \frac{11x}{48}\Big\} - \omega\bigg(\max\Big\{\frac{x}{3}, \frac{5x}{24}, \frac{7x}{24}, \frac{11x}{48}\Big\}\bigg) = \frac{x}{3} - \omega(\frac{x}{3}) = \frac{x}{3} - \frac{x}{12} = \frac{x}{4}$.
The inequality holds.

Case 2. If $x < y$, then $\max\Big\{\frac{3y+x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x+6y}{48}\Big\} - \omega\bigg(\max\Big\{\frac{3y+x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x+6y}{48}\Big\}\bigg) \leq$

$\max\left\{\frac{y}{3}, \frac{5y}{24}, \frac{7y}{24}, \frac{y}{48}\right\} - \omega\left(\max\left\{\frac{y}{3}, \frac{5y}{24}, \frac{7y}{24}, \frac{y}{48}\right\}\right) = \frac{y}{3} - \omega(\frac{y}{3}) = \frac{y}{4}$.
and $\frac{3x+y}{24} \leq \frac{y}{4}$ i.e. $3x \leq 5y$, which is true. Thus the inequality holds.
Case 3. If $y < x$, then $\max\left\{\frac{3y+x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x+6y}{48}\right\} - \omega\left(\max\left\{\frac{3y+x}{12}, \frac{5x}{24}, \frac{7y}{24}, \frac{5x+6y}{48}\right\}\right) \leq$ $\max\left\{\frac{x}{3}, \frac{5x}{24}, \frac{7x}{24}, \frac{x}{48}\right\} - \omega\left(\max\left\{\frac{x}{3}, \frac{5x}{24}, \frac{7x}{24}, \frac{x}{48}\right\}\right) = \frac{x}{3} - \omega(\frac{x}{3}) = \frac{x}{4}$.
and $\frac{3x+y}{24} \leq \frac{x}{4}$ i.e. $y \leq 3x$, which is true. Thus the inequality holds.
All the necessary conditions of theorem 2.1. have been satisfied. So all these six functions, A, B, P, Q, S and T have unique common fixed point which is 0.

3. Conclusion

The results of the paper give generalisation of the works which are quoted in the introduction. The results of the paper can be applied to obtain common fixed point for point valued and set valued mappings using Hausdorff metric.

Acknowledgement

The authors are thankful to the learned referees for their kind suggestions for the improvement of the paper. The first author is thankful to the CSIR, India for granting the JRF, vide letter no.- 09/1224(0002)/2019-EMR-I, Date: 10/10/2019.

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