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RELATION BETWEEN GENERAL RANDIĆ INDEX AND GENERAL SUM CONNECTIVITY INDEX

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Abstract: The general Randić index is the sum of weights of $(d(u).d(v))^k$ for every edge uv of a molecular graph G. On the other hand general Sum-Connectivity index is the sum of the weights $(d(u) + d(v))^k$ for every edge uv of G, where k is a real number and d(u) is the degree of vertex u. Both families of topological indices are well known and closely related. In fact the correlation coefficient value of these two families of indices for the trees representing the Octane Isomers vary between 0.915 to 0.998. In the recent years these families of indices have been extensively explored and studied. The major research on these indices mostly consists of the application in QSPR/QSAR analysis, computation of these indices for various molecular graphs and bounds of the indices for certain graphs, satisfying certain conditions. The main focus of this paper is a comparative study on these two families of indices for various families of graphs. We find a few algebraic relationships between general Randić index and general Sum-connectivity index of certain graphs.

Keywords and Phrases: General Randić index, general Sum-connectivity index, Path graph, Star graph, Tree graph, r-Regular graph, complete Bipartite graph.

2020 Mathematics Subject Classification: 05C92.

1. Introduction

In 1975, Milan Randić [6] proposed the first genuine degree-based topological index to measure the extent of branching of the carbon-atom skeleton of saturated hydrocarbons. Though Randić named the index as "branching index", it was soon re-named to "connectivity index". Nowadays, most authors refer to it after his name as the "Randić index". Certainly the Randić index is one of the most used topological indices in view of chemistry and chemical graph theory. It was defined as

$$R(G) = \sum_{uv\epsilon E} \frac{1}{\sqrt{d(u).d(v)}}$$

with summation going over all pairs of adjacent vertices of the molecular graph G. In 1998 Bollobas and Erdos [1] generalized this index by replacing the exponent -0.5 with any real number k, which is called the general Randić index. For a chemical graph G = (V,E), the general Randić index $R_k(G)$ is defined as the sum of $(d(u).d(v))^k$ over all edges uv of G, where d(u) denotes the degree of the vertex u of G, i.e.,

$$R_k(G) = \sum_{uv\epsilon E} (d(u).d(v))^k \text{ where k is an arbitrary real number.}$$

Following are the six variants of the general Randić index.

1.**Randić Index** [7] : $R_{-0.5}(G) = \sum_{uv\epsilon E} \frac{1}{\sqrt{(d(u).d(v))}}$

2.**Reciprocal Randić Index** [5] : $R_{0.5}(G) = \sum_{uv\epsilon E} \sqrt{(d(u).d(v))}$

3.**Second Reciprocal Zagreb Index** : $R_{-1}(G) = \sum_{uv\epsilon E} \frac{1}{(d(u).d(v))}$

4.**Second Zagreb Index** [2] : $R_1(G) = \sum_{uv\epsilon E} (d(u).d(v))$

which is 2 times the SK_1 Index [8].

5.**Second Reciprocal Hyper Zagreb Index** : $R_{-2}(G) = \sum_{uv\epsilon E} \frac{1}{(d(u).d(v))^2}$

6.**Second Hyper Zagreb Index** [9] : $R_2(G) = \sum_{uv\epsilon E} (d(u).d(v))^2$

On the other hand the Sum-Connectivity index is a recent invention by Zhou and Trinajstić [10]. They noticed that in the definition of Randić index there is no a priori reason for using the product d(u).d(v) of vertex degrees, and this term may be replaced by the sum d(u)+d(v) [4]. Hence the new index was defined as

$$SCI(G) = \sum_{uv\epsilon E} \frac{1}{\sqrt{d(u) + d(v)}}$$

In 2010 Zhou and Trinajstić [11] modified the concept of general Randić index and obtained a new index called the general Sum-Connectivity index and defined as follows:

$$SCI_k(G) = \sum_{uv\epsilon E} (d(u) + d(v))^k \text{ where k is an arbitrary real number.}$$

Following are the six variants of the general Sum-Connectivity index.

1.**Sum Connectivity Index** [10] : $SCI_{-0.5}(G) = \sum_{uv\epsilon E} \frac{1}{\sqrt{(d(u) + d(v))}}$

2.**Reciprocal Sum Connectivity Index** : $SCI_{0.5}(G) = \sum_{uv\epsilon E} \sqrt{(d(u) + d(v))}$

3.**First Reciprocal Zagreb Index** : $SCI_{-1}(G) = \sum_{uv\epsilon E} \frac{1}{(d(u) + d(v))}$

4.**First Zagreb Index** [3] : $SCI_1(G) = \sum_{uv\epsilon E} (d(u) + d(v))$

which is 2 times the SK Index [8].

5.**First Reciprocal Hyper Zagreb Index** : $SCI_{-2}(G) = \sum_{uv\epsilon E} \frac{1}{(d(u) + d(v))^2}$

6.**First Hyper Zagreb Index** [9] : $SCI_2(G) = \sum_{uv\epsilon E} (d(u) + d(v))^2$

which is 4 times the SK_2 Index [8].

The general Randić and general Sum-Connectivity indices are both well explored and extensively studied topological indices. Apart from the quantitative structure–property relationship (QSPR) and quantitative structure–activity relationship (QSAR), the interest in these topological indices is mainly related to the mathematical properties they posses.
The main focus of this paper is on the algebraic relation between general Randić index and general Sum-Connectivity index of certain families of graphs. A few results may be found in the mathematico-chemical literature based on the algebraic relation between the Randić and Sum-Connectivity indices such as;
1. $SCI(G) \geq R(G)$ for all graphs G without pendant vertices [10].
2. $SCI(P_n) < R(P_n)$ for all path graphs P_n with n vertices [10].
We extend these and propose new results relating general Randić and general Sum-Connectivity indices.

2. Motivation

In the recent years, one of the prominent areas of research in chemical graph theory has been finding the bounds of topological descriptors. For example Zhou B and Wei Luo [12] obtained a lower bound of general Randić index for the graphs with n vertices and m $\geq$ 1 edges, given as

$$R_k(G) \geq 4^k n^{-2k} m^{1+2k} \text{ where k} \geq 1$$

Suppose $SCI_k(G) \geq R_k(G)$ is an inequality for the same family of graphs with n vertices and m $\geq$ 1 edges, then one could easily extend the lower bound of the general Randić index to the general Sum-connectivity index by stating,

$$SCI_k(P_k) \geq 4^k n^{-2k} m^{1+2k} \text{ where k} \geq 1$$

The inequalities derived in this paper help in extending the existing bounds to the new topological descriptors.

3. Preliminaries

A molecular graph is a simple graph without loops and multiple edges representing the carbon-atom skeleton of an organic molecule (usually, of a hydrocarbon). Thus, the vertices of a molecular graph represent the carbon atoms, and its edges are the carbon-carbon bonds. Let G=(V,E) be a molecular graph, where V = V(G) is a non-empty set of elements called vertices or points and E = E(G) is a set of unordered pairs of distinct elements of V (G) called edges or lines. Two vertices of G, connected by an edge, are said to be "adjacent". If two vertices u and v are adjacent, they form an edge denoted as uv. The number of vertices of G, adjacent to a given vertex u, is called the degree of u, and is denoted by d(u). The concept of degree in graph theory is closely related (but not identical) to the concept of valence in chemistry [4]. Depending upon the number of vertices, number of edges, interconnectivity, and their overall structure graphs can be classified as path graph, star graph, regular graph, bipartite graph etc. Let P_n and S_n denote the Path and the Star graphs of order n respectively. A complete Bipartite graph, denoted as $K_{m,n}$, is a graph whose vertex-set V can be partitioned into two subsets V_1 and V_2 such that every vertex of V_1 is adjacent to every vertex of V_2 [11].

4. Main Result

Lemma 1. *Let R:$\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ and S:$\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ be two functions defined as*

R(x,y)=$(xy)^k$ and S(x,y)=$(x+y)^k$ for all x, y $\in \mathbb{N}$ *and for any fixed k*$\in \mathbb{R}^+$. *Then*

1.$R(x,y) = S(x,y)$ *for* $x,\ y = 2$
2.$R(x,y) < S(x,y)$ *for* ($x = 1$ *and* $y \geq 1$) *or* ($x \geq 1$ *and* $y = 1$)
3.$R(x,y) \geq S(x,y)$ *for all* $x,\ y \geq 2$

Proof.
Case 1: when x, y = 2
It is an obvious case as $(2.2)^k = (2{+}2)^k$
Hence R(x,y) = S(x, y).
Case 2: when (x = 1 and y $\geq$ 1) or (x $\geq$ 1 and y = 1)
Without loss of generality let x = n and y = 1 for all n $\in \mathbb{N}$.
Consider

$$\begin{aligned} S(x,y) &= (n+1)^k \\ &= \sum_{r=0}^{\infty} \frac{k(k-1)(k-2)....(k-r+1)}{r!} n^{k-r} 1^r \\ &= (n^k + kn^{k-1} + \frac{k(k-1)}{2!} n^{k-2}........) \\ &> n^k \\ &= R(x,y) \\ \text{Hence } S(x,y) &> R(x,y). \end{aligned}$$

Case 3: when x, y $\geq$ 2
Let x = 1 + n and y = 1 + m for all n, m $\in \mathbb{N}$
Consider

$$\begin{aligned} R(x,y) &= ((1+n).(1+m))^k \\ &= (1+n+m+nm)^k \\ &= ((2+n+m) + (nm-1))^k \\ &\geq (2+n+m)^k \\ &= S(x,y) \\ \text{Hence } R(x,y) &\geq S(x,y) \end{aligned}$$

The equality holds when m=n=1.

Lemma 2. *Let R*:$\mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ *and S*:$\mathbb{N} \times \mathbb{N} \to \mathbb{Q}$ *be two functions defined as R(x,y)*

$=\frac{1}{(x.y)^k}$ *and* $S(x,y)=\frac{1}{(x+y)^k}$ *for all* $x,\ y \in \mathbb{N}$ *and for any fixed* $k\in \mathbb{R}^+$. *Then*

1.$R(x, y) = S(x, y)$ *for* $x,\ y = 2$

2.$R(x, y) > S(x, y)$ *for* ($x = 1$ *and* $y \geq 1$) *or* ($x \geq 1$ *and* $y = 1$)

3.$R(x, y) \leq S(x, y)$ *for all* $x,\ y \geq 2$

Proof.
Case 1: when x, y =2
It is an obvious case as $\frac{1}{(2.2)^k} = \frac{1}{(2+2)^k}$
Hence R(x, y) = S(x, y)
Case 2: when (x = 1 and y $\geq$ 1) or (x $\geq$ 1 and y = 1)
Without loss of generality let x = n and y = 1 for all n $\in \mathbb{N}$
Consider

$$R(x, y) = \frac{1}{(1.n)^k} > \frac{1}{(n^k + kn^{k-1} + \frac{k(k-1)}{2!}n^{k-2}........)} = \frac{1}{(n+1)^k} = S(x, y)$$

Case 3: when x, y $\geq$ 2
Let x = 1 + n and y = 1 + m for all n, m $\in \mathbb{N}$
Consider

$$\begin{aligned} S(x, y) &= \frac{1}{((1+n)+(1+m))^k} \\ &= \frac{1}{((2+n+m))^k} \\ &\geq \frac{1}{((1+n+m+nm))^k} \\ &= \frac{1}{((1+n).(1+m))^k} \\ &= R(x, y) \end{aligned}$$

The equality holds when m=n=1.

In the following theorems 1 to 5 we will be using the above lemmas to find the algebraic relationships between the general Randić index and general Sum-Connectivity index of certain graphs in which we will assume x = d(u) and y = d(v) for every edge uv, where d(u) and d(v) are the degree of vertices u and v respectively.

Theorem 1. *For all path graphs* P_n *of order* $n \geq 2$

$$SCI_k(P_n) > R_k(P_n) \text{ when } k > 0$$
$$SCI_k(P_n) < R_k(P_n) \text{ when } k < 0$$

Proof. Let E_{ij} denote the set of edges uv such that d(u) = i and d(v) = j.
Case 1: when k > 0
Any path graph P_n of order n ≥ 2 has edges belonging to either E_{12} and E_{22} or E_{11}.
Consider uv ∈ E_{12}
By Lemma 1, (d(u)+d(v))^k > (d(u).d(v))^k
Consider uv ∈ E_{22}
By Lemma 1, (d(u)+d(v))^k = (d(u).d(v))^k
Consider uv ∈ E_{11}
By Lemma 1, (d(u)+d(v))^k > (d(u).d(v))^k
Hence it is clear that
$SCI_k(P_n) > R_k(P_n)$
Case 2: when k < 0
Any path graph P_n of order n ≥ 2 has edges belonging to either E_{12} and E_{22} or E_{11}.
Consider uv ∈ E_{12}
By Lemma 2, (d(u)+d(v))k < (d(u).d(v))k
Consider uv ∈ E_{22}
By Lemma 2, (d(u)+d(v))k = (d(u).d(v))k
Consider uv ∈ E_{11}
By Lemma 2, (d(u)+d(v))k < (d(u).d(v))k
Hence it is clear that
$SCI_k(P_n) < R_k(P_n)$

Theorem 2. *For all Star Graphs S_n of order $n > 1$*

$$SCI_k(S_n) > R_k(S_n) \quad when\ k > 0$$
$$SCI_k(S_n) < R_k(S_n) \quad when\ k < 0$$

Proof. Let E_{ij} denote the set of edges uv such that d(u) = i and d(v) = j.
Any Star graph S_n of order n > 1 has edges belonging to $E_{1(n-1)}$.
Case 1: when k > 0
Consider uv ∈ $E_{1(n-1)}$
By Lemma 1, $(d(u) + d(v))^k > (d(u).d(v))^k$
Hence it is clear that $SCI_k(S_n) > R_k(S_n)$
Case 2: when $k < 0$
Consider uv ∈ $E_{1(n-1)}$
By Lemma 2, $(d(u) + d(v))^k < (d(u).d(v))^k$
Hence it is clear that $SCI_k(S_n) < R_k(S_n)$

In Theorem 3 if G is a disconnected graph, then $G = G_1 \cup G_2 \cup ... \cup G_m$ such that

$$SCI_k(G) = SCI_k(G_1) + SCI_k(G_2) + ... + SCI_k(G_m).$$
$$R_k(G) = R_k(G_1) + R_k(G_2) + ... + R_k(G_m).$$

Theorem 3. *For all r-Regular graphs G of order $n \geq 2$ and $r \geq 1$.*

$$SCI_k(G) > R_k(G) \text{ when } k > 0 \text{ and } r = 1$$
$$SCI_k(G) = R_k(G) \text{ when } k > 0 \text{ and } r = 2$$
$$SCI_k(G) < R_k(G) \text{ when } k > 0 \text{ and } r \geq 3$$
$$SCI_k(G) < R_k(G) \text{ when } k < 0 \text{ and } r = 1$$
$$SCI_k(G) = R_k(G) \text{ when } k < 0 \text{ and } r = 2$$
$$SCI_k(G) > R_k(G) \text{ when } k < 0 \text{ and } r \geq 3$$

Proof. Let E_{ij} denote the set of edges uv such that d(u) = i and d(v) = j.
Case 1: when $k > 0$ and r = 1
Let G be a 1-regular graph of order $n \geq 2$. Now either $G = e_{11}$ or $G = G_1 \cup G_2 \cup ... \cup G_{\frac{n}{2}}$ such that every G_i for i = 1, 2,...$\frac{n}{2}$ is e_{11}. Hence every edge of G belonging to E_{11}.
Consider $uv \in E_{11}$
By Lemma 1, $(d(u) + d(v))^k > (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) > R_k(G)$
Case 2: when $k > 0$ and r=2
Let G be a 2-regular graph of order $n \geq 2$. Now either G is a Cycle or $G = G_1 \cup G_2 \cup ... \cup G_m$ such that every G_i for i = 1, 2,...m is a Cycle. Hence every edge of G belonging to E_{22}.
Consider $uv \in E_{22}$
By Lemma 1, $(d(u) + d(v))^k = (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) = R_k(G)$
Case 3: when $k > 0$ and $r \geq 3$
Let G be a r-regular graph of order $n \geq 2$. Now irrespective of G is connected or disconnected, every edge of G belongs to E_{rr}.
Consider $uv \in E_{rr}$
By Lemma 1, $(d(u) + d(v))^k < (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) < R_k(G)$
Case 4: when $k < 0$ and r = 1
Let G be a 1-regular graph of order $n \geq 2$. Now either $G = e_{11}$ or $G = G_1 \cup G_2 \cup$

$... \cup G_{\frac{n}{2}}$ such that every G_i for i = 1, 2,...$\frac{n}{2}$ is e_{11}. Hence every edge of G belonging to E_{11}.
Consider $uv \in E_{11}$
By Lemma 2, $(d(u)+d(v))^k < (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) < R_k(G)$
Case 5: when $k < 0$ and r= 2
Let G be a 2-regular graph of order $n \geq 2$. Now either G is a Cycle or $G = G_1 \cup G_2 \cup ... \cup G_m$ such that every G_i for i = 1, 2,...m is a Cycle. Hence every edge of G belonging to E_{22}.
Consider $uv \in E_{22}$
By Lemma 2, $(d(u)+d(v))^k = (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) = R_k(G)$
Case 6: when $k < 0$ and $r \geq 3$
Let G be a r-regular graph of order $n \geq 2$. Now irrespective of G is connected or disconnected, every edge of G belongs to E_{rr}.
Consider $uv \in E_{rr}$
By Lemma 2, $(d(u)+d(v))^k > (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) > R_k(G)$

Theorem 4. *For all complete Bipartite graphs $K_{m,n}$*

$$SCI_k(K_{m,n}) > R_k(K_{m,n}) \text{ when } k > 0,\ m = 1 \text{ and } n \geq 1$$
$$SCI_k(K_{m,n}) < R_k(Km,n) \text{ when } k < 0,\ m = 1 \text{ and } n \geq 1$$
$$SCI_k(K_{m,n}) \leq R_k(K_{m,n}) \text{ when } k > 0, \text{ and } m,n \geq 2$$
$$SCI_k(K_{m,n}) \geq R_k(K_{m,n}) \text{ when } k < 0, \text{ and } m,n \geq 2$$

Proof. Let E_{ij} denote the set of edges uv such that d(u) = i and d(v) = j.
Case 1: when $k > 0$, m = 1 and n $\geq$ 1
Suppose m = 1 and n = p for all p $\in \mathbb{N}$
A Complete Bipartite graph $K_{1,p}$ has edges belonging to E_{1p}.
Consider $uv \in E_{1p}$
By Lemma 1, $(d(u)+d(v))^k > (d(u).d(v))^k$
Hence it is clear that $SCI_k(K_{m,n}) > R_k(K_{m,n})$
Case 2: when $k < 0$, m = 1 and n $\geq$ 1
Suppose m = 1 and n = p for all p $\in \mathbb{N}$
A complete bipartite graph $K_{1,p}$ has edges belonging to E_{1p}
Consider uv $\in E_{1p}$
By Lemma 2, $(d(u)+d(v))^k < (d(u).d(v))^k$
Hence it is clear that $SCI_k(K_{m,n}) < R_k(K_{m,n})$

Case 3: when k > 0, and m,n ≥ 2
Any Complete Bipartite graph $K_{m,n}$ with m, n ≥ 2 has edges belonging to E_{mn}.
Consider uv $\in E_{mn}$
By Lemma 1, $(d(u) + d(v))^k \leq (d(u).d(v))^k$
Hence it is clear that $SCI_k(K_{m,n}) \leq R_k(K_{m,n})$
Case 4: when k < 0, and m,n ≥ 2
Any Complete Bipartite graph $K_{m,n}$ with m, n ≥ 2 has edges belonging to E_{mn}.
Consider uv $\in E_{mn}$
By Lemma 2, $(d(u) + d(v))^k \geq (d(u).d(v))^k$
Hence it is clear that $SCI_k(K_{m,n}) \geq R_k(K_{m,n})$

Theorem 5. *For any connected graph G of order $n \geq 3$ having no pendent vertices,*

$$SCI_k(G) \leq R_k(G) \text{ when } k > 0$$
$$SCI_k(G) \geq R_k(G) \text{ when } k < 0$$

Proof. Let E_{ij} denote the set of edges uv such that d(u) = i and d(v) = j.
In a connected graph G of order n ≥ 3 having no pendent vertices, every edge uv is such that d(u), d(v) ≥ 2.
Case 1: When k > 0
Consider uv $\in E_{ij}$
By Lemma 1, $(d(u) + d(v))^k \leq (d(u).d(v))^k$
Hence it is clear that $SCI^k(G) \leq R^k(G)$
Case 2: When k < 0
Consider uv $\in E_{ij}$
By Lemma 2, $(d(u) + d(v))^k \geq (d(u).d(v))^k$
Hence it is clear that $SCI_k(G) \geq R_k(G)$

Theorem 6. *For any tree T, in which every vertex other than the pendent vertices, is of degree 3 and has n number of vertices with degree 3,*

$$R_k(T) > SCI_k(T) \text{ when } k \geq 1 \text{ and } n \geq 3$$
$$R_k(T) \geq SCI_k(T) \text{ when } k \leq -0.5 \text{ and } n \geq 1$$

Proof. Consider a tree T, in which every vertex other than the pendent vertices, is of degree 3. Let n be the number of vertices of degree 3. In the tree T, every edge uv belong to E_{13} or E_{33}, where $|E_{13}| = (n + 2)$ and $|E_{33}| = (n - 1)$.
Case 1: When k ≥ 1

$$\begin{aligned} &\text{Consider } R_k(T) - SCI_k(T) \\ &\qquad = \big(|E_{13}|(1.3)^k + |E_{33}|(3.3)^k\big) - \big(|E_{13}|(1+3)^k + |E_{33}|(3+3)^k\big) \end{aligned}$$

$= \big((n+2)3^k + (n-1)9^k\big) - \big((n+2)4^k + (n-1)6^k\big)$
$= (n-1)(9^k - 6^k) - (n+2)(4^k - 3^k)$
$= (n-1)(9^k - 6^k) - (n-1)(4^k - 3^k) - 3(4^k - 3^k)$
$= (n-1)((9^k - 6^k) - (4^k - 3^k)) - 3(4^k - 3^k)$
$\geq (n-1)(3(4^k - 3^k) - (4^k - 3^k)) - 3(4^k - 3^k)$ $[\because 9^k - 6^k \geq 3(4^k - 3^k)$ for k $\geq 1]$
$= (n-1)(2(4^k - 3^k)) - 3(4^k - 3^k)$
$= (2n-5)(4^k - 3^k) > 0$ for all $n \geq 3$

Hence $R_k(T) > SCI_k(T)$ for all $n \geq 3$.
Case 2: When k $\leq$ -0.5

Consider $R_k(T) - SCI_k(T)$

$$= \big(|E_{13}|(1.3)^k + |E_{33}|(3.3)^k\big) - \big(|E_{13}|(1+3)^k + |E_{33}|(3+3)^k\big)$$
$$= \big((n+2)3^k + (n-1)9^k\big) - \big((n+2)4^k + (n-1)6^k\big)$$
$$= (n+2)(3^k - 4^k) - (n-1)(6^k - 9^k)$$
$$= 3(3^k - 4^k) + (n-1)(3^k - 4^k) - (n-1)(6^k - 9^k)$$
$$= 3(3^k - 4^k) + (n-1)((3^k - 4^k) - (6^k - 9^k) \geq 0$$
$$[\because 3^k - 4^k \geq 0 \text{ and } (3^k - 4^k) - (6^k - 9^k) \geq 0 \text{ for all k} \geq \text{-0.5}]$$

Hence $R_k(T) \geq SCI_k(T)$ for all $n \geq 1$

Theorem 7. *For any tree T, in which every vertex other than the pendent vertices, is of degree 4 and has n number of vertices with degree 4,*

$$R_k(T) > SCI_k(T) \text{ when } k \geq 1 \text{ and } n \geq 2$$
$$R_k(T) \geq SCI_k(T) \text{ when } k \leq -0.5 \text{ and } n \geq 1$$

Proof. Consider a tree T, in which every vertex other than the pendent vertices, is of degree 4. Let n be the number of vertices of degree 4. In the tree T, every edge uv belong to E_{14} or E_{44}, where $|E_{14}| = (2n+2)$ and $|E_{33}| = (n-1)$.
Case 1: When k $\geq$ 1

Consider $R_k(T) - SCI_k(T)$

$$= \big(|E_{14}|(1.4)^k + |E_{44}|(4.4)^k\big) - \big(|E_{14}|(1+4)^k + |E_{44}|(4+4)^k\big)$$
$$= \big((2n+2)4^k + (n-1)16^k\big) - \big((2n+2)5^k + (n-1)8^k\big)$$
$$= (n-1)(16^k - 8^k) - (2n+2)(5^k - 4^k)$$
$$\geq (n-1)8(5^k - 4^k) - (2n+2)(5^k - 4^k) \quad [\because 16^k - 8^k \geq 8(5^k - 4^k) \text{ for k} \geq 1]$$

$$= (5^k - 4^k)(8n - 8 - 2n - 2)$$
$$= (5^k - 4^k)(6n - 10) > 0 \text{ for all n} \geq 2$$
Hence $R_k(T) > SCI_k(T)$ for all $n \geq 2$

Case 2: When k $\leq$ -0.5

Consider $R_k(T) - SCI_k(T)$

$$= \big(|E_{14}|(1.4)^k + |E_{44}|(4.4)^k\big) - \big(|E_{14}|(1+4)^k + |E_{44}|(4+4)^k\big)$$
$$= \big((2n+2)4^k + (n-1)16^k\big) - \big((2n+2)5^k + (n-1)8^k\big)$$
$$= (2n+2)(4^k - 5^k) - (n-1)(8^k - 16^k)$$
$$= (n+3)(4^k - 5^k) + (n-1)(4^k - 5^k) - (n-1)(8^k - 16^k)$$
$$= (n+3)(4^k - 5^k) + (n-1)((4^k - 5^k) - (8^k - 16^k))$$
$$= 4(4^k - 5^k) + (n-1)(2(4^k - 5^k) - (8^k - 16^k)) \geq 0 \text{ for all n} \geq 1$$
$$[\because (4^k - 5^k) \geq 0 \;\text{ and }\; 2(4^k - 5^k) - (8^k - 16^k) \geq 0 \text{ for all k} \leq \text{-0.5}]$$

Hence $R_k(T) \geq SCI_k(T)$ for all $n \geq 1$

5. Conclusion

We obtained a few inequalities relating the general Randić index and general Sum-Connectivity index for a few families of graphs. Similar inequalities can be derived for nanostuctures such as 2D-lattice, nanotube and nanotorus. In fact, there is still scope to work in the same line on many other general families of graphs such as acyclic graphs, bipartite graphs, line graphs, subdivision graphs and so on. One could also explore the algebraic relationships between other topological descriptors for the above discussed families of graphs.

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