South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 171-184

DOI: 10.56827/SEAJMMS.2022.1802.16 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

S_5-DECOMPOSITION OF KNESER GRAPHS

C. Sankari, R. Sangeetha and K. Arthi

Department of Mathematics,
A. V. V. M. Sri Pushpam College,
Poondi, Thanjavur - 613503, Tamil Nadu, INDIA

E-mail : sankari9791@gmail.com, jaisangmaths@yahoo.com,
arthi1505@gmail.com

(**Received:** *Apr. 28, 2021* **Accepted:** *Jul. 20, 2022* **Published:** *Aug. 30, 2022)*

Abstract: Let $A = \{1, 2, 3, ..., n\}$ and $\mathcal{P}_k(A)$ denotes the set of all k-element subsets of A. The Kneser graph $KG_{n,2}$ has the vertex set $V(KG_{n,2})= \mathcal{P}_2(A)$ and edge set $E(KG_{n,2}) = \{XY|X, Y \in \mathcal{P}_2(A)$ and $X \cap Y = \emptyset\}$. A star with k edges is denoted by S_k. In this paper, we show that the graph $KG_{n,2}$ can be decomposed into S_5 if and only if $n \geq 7$ and $n \equiv 0, 1, 2, 3(mod\ 5)$.

Keywords and Phrases: Decomposition, Tensor Product, Complete Bipartite Graph, Kneser Graph, Crown Graph, Star.

2020 Mathematics Subject Classification: 05C70, 05C76.

1. Introduction

All the graphs considered in this paper are finite. For a graph G, $G(\lambda)$ is the graph obtained from G by replacing each of its edges by λ parallel edges. If a graph G has no edges, then it is called a *null graph*. Let $K_{m,n}$ denote a *complete bipartite graph* with m and n vertices in the parts. A *star* with k edges is denoted by S_k and $S_k \cong K_{1,k}$. A *path* with k edges is denoted by P_k and a *cycle* with k edges is denoted by C_k. A *Hamilton cycle* of G is a cycle that contains every vertex of G. A graph G is *Hamiltonian* if it contains a Hamilton cycle. The degree of a vertex x of G, denoted by $deg_G x$ is the number of edges incident with x in G. Let k be a positive integer. A graph G is said to be *k-regular*, if each vertex in G is of degree k. If $H_1, H_2, ..., H_l$ are edge disjoint subgraphs of a graph G such that

$E(G) = \bigcup_{i=1}^{l} E(H_i)$, then we say that $H_1, H_2, ..., H_l$ *decompose* G and we denote it by $G = \oplus_{i=1}^{l} H_i$. If $H_i \cong S_k$ for $i = 1, 2, ..., l$, then we say that G is S_k-*decomposable* and we denote it by $S_k|G$. For positive integers l and n with $1 \leq l \leq n$, the crown $C_{n,l}$ is the bipartite graph with bipartition (A, B), where $A = \{a_0, a_1, ..., a_{n-1}\}$ and $B = \{b_0, b_1, ..., b_{n-1}\}$, and the edge set $\{a_i b_j | 1 \leq j - i \leq l$ with arithmetic modulo $n\}$. Note that $C_{n,n} \cong K_{n,n}$ and $C_{n,n-1} \cong K_{n,n} - I$, where I is a 1-factor of $K_{n,n}$. The *tensor product* of G and H, denoted by $G \times H$ has vertex set $V(G) \times V(H)$ in which two vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever $g_1 g_2 \in E(G)$ and $h_1 h_2 \in E(H)$. The *line graph* $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$ and $e_i e_j \in E(L(G))$ if and only if the edges e_i and e_j are incident with a common end vertex in G. The complete graph on n vertices is denoted by K_n. The line graph of the complete graph K_n is denoted by $L(K_n)$. Let $A = \{1, 2, 3, ..., n\}$ and $\mathcal{P}_k(A)$ denotes the set of all k-element subsets of A. The Kneser graph $KG_{n,2}$ is defined as follows: $V(KG_{n,2})= \mathcal{P}_2(A)$ and $E(KG_{n,2}) = \{XY | X, Y \in \mathcal{P}_2(A)$ and $X \cap Y = \emptyset\}$. Note that, the graph $KG_{n,2} \cong \overline{L(K_n)}$, where $\overline{L(K_n)}$ denotes the complement of the graph $L(K_n)$. Also, it is interesting to note that $KG_{5,2}$ is the Petersen graph. The Generalized Kneser Graph, $GKG_{n,k,r}$ is the graph whose vertices are the k-element subsets of some set of n elements, in which two vertices are adjacent if and only if they intersect in precisely r elements.

In 1955, M. Kneser [3] introduced the Kneser graph. In 2000, Chen [1] proved that $KG_{n,2}$ is Hamiltonian, when $n \geq 3k$, $k \geq 1$. In 2004, Shields and Savage [7] proved that all connected Kneser graphs (except $KG_{5,2}$) have Hamilton cyles, when $n \leq 27$ and the problem $KG_{n,2}(n \neq 5)$ is Hamiltonian is still open. In 2015, Rodger and Whitt [5] established the necessary and sufficient conditions for a P_3-decomposition of the Kneser graph $KG_{n,2}$ and the Generalized Kneser Graph $GKG_{n,3,1}$. In 2015, Whitt and Rodger [8] proved that the Kneser graph $KG_{n,2}$ is P_4-decomposable if and only if $n \equiv 0, 1, 2, 3 (mod\ 16)$. In 2018, Ganesamurthy and Paulraja [2] proved that if $n \equiv 0, 1, 2, 3 (mod\ 8k), k \geq 2$, then the Kneser graph $KG_{n,2}$ can be decomposed into paths of length $2k$. In the same paper they also proved that, for $k = 2^l$, $l \geq 1$, $KG_{n,2}$ has a P_{2k}-decomposition if and only if $n \equiv 0, 1, 2, 3 (mod\ 2^{l+3})$. Recently, the authors [6] proved that, $KG_{n,2}$ is claw-decomposable, for all $n \geq 6$. In this paper, we discuss S_5-decomposition of the Kneser graphs. It is obtained that $KG_{n,2}$ is S_5-decomposable if and only if $n \geq 7$ and $n \equiv 0, 1, 2, 3 (mod\ 5)$.

2. Preliminaries

Let G be a graph on n vertices and $\{1, 2, 3, ..., k\} \subset V(G)$. The notation $(1; 2, 3, ..., k)$ denotes a star with a center vertex 1 and $k-1$ pendent edges $12, 13$, ..., $1k$. Let X and Y be two disjoint subsets of $V(G)$. Then $E(X, Y)$ denotes the set of edges in G, whose one end vertex is in X and the other end vertex is in Y. The notation $\langle E(X,Y)\rangle$ denotes the graph induced by the edges of $E(X,Y)$. To prove our results we use the following:

Theorem 2.1. (Lin et al. [4]) *Let λ, k, l and n be positive integers. The graph $C_{n,l}(\lambda)$ is S_k-decomposable if and only if $k \le l$ and $\lambda nl \equiv 0(mod\ k)$.*

Theorem 2.2. (Yamamoto et al. [9]) *Let k, m and $n \in \mathbb{Z}_+$ with $m \le n$. There exists an S_k-decomposition of $K_{m,n}$ if and only if one of the following holds:*

(i) $k \le m$ and $mn \equiv 0(mod\ k)$;

(ii) $m < k \le n$ and $n \equiv 0(mod\ k)$.

Note that, the graphs $KG_{2,2}$ and $KG_{3,2}$ are null graphs. For $n \ge 5$, $|E(KG_{n,2})| = \frac{n(n-1)(n-2)(n-3)}{8}$, which is divisible by 5 only when $n \equiv 0,1,2,3(mod\ 5)$. We know that the graph $KG_{5,2}$ (Petersen graph) is 3-regular, hence doesn't admit an S_5-decomposition. In the following Lemma, we prove that the graph $KG_{6,2}$ can't be decomposed into S_5.

Lemma 2.1. *There doesn't exist an S_5-decomposition in $KG_{6,2}$.*
proof. (Necessity). For n=6, $|E(KG_{6,2})|$=45, which is divisible by 5.
(Sufficiency). Vertex set of $KG_{6,2}$ is $\{\{1,2\},\{1,3\},\{1,4\},\{1,5\},\{1,6\},\{2,3\},\{2,4\},\{2,5\},\{2,6\},\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\},\{5,6\}\}$ and $deg_{KG_{6,2}}v$=6, for all vertices $v \in V(KG_{6,2})$. Without loss of generality, we choose an S_5, centered at $\{1,2\}$ $i.e. S^1 : (\{1,2\};\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\})$. In G^1=$KG_{6,2} \setminus S^1$, the degree of the vertex $\{1,2\}$ is 1 which implies that the edge $\{1,2\}\{5,6\}$ can only be included in the star centered at $\{5,6\}$. Therefore, we choose $S^2 : (\{5,6\};\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\})$. In G^2=$G^1 \setminus S^2$, the degree of the vertex $\{5,6\}$ is 1 which implies that the edge $\{5,6\}\{3,4\}$ can only be included in the star centered at $\{3,4\}$. We choose $S^3 : (\{3,4\};\{5,6\},\{1,5\},\{1,6\},\{2,5\},\{2,6\})$. In G^3=$G^2 \setminus S^3$, the degrees of the vertices are $\{0,5,5,5,5,5,5,5,5,0,5,5,5,5,0\}$. Among these 12 vertices of degree 5, we can choose any vertex as a center vertex for the next star. Suppose we choose $\{1,3\}$ as a center vertex, then $S^4 : (\{1,3\};\{2,4\},\{2,5\},\{2,6\},\{4,5\},\{4,6\})$. In G^4=$G^3 \setminus S^4$, the degrees of the vertices become $\{0,0,5,5,5,5,4,4,4,0,5,5,4,4,0\}$. There are 6 vertices of degree 5. These vertices are $\{1,4\}$, $\{1,5\}$, $\{1,6\}$, $\{2,3\}$, $\{3,5\}$ and $\{3,6\}$. To choose the next star, we have the following cases.
Case (i).
Suppose we choose $\{1,4\}$ as a center vertex. Then S^5:$(\{1,4\};\{2,3\},\{2,5\},\{2,6\},\{3,5\},\{3,6\})$. In G^5=$G^4 \setminus S^5$, the degrees of the vertices become $\{0,0,0,5,5,4,4,3,3,0,4,4,4,4,0\}$. We note that there are only two vertices of degree 5. Hence it is impossible to choose four more stars.

Case (ii).
Suppose we choose $\{1,5\}$ as a center vertex. Then S^5:$(\{1,5\};\{2,3\},\{2,4\},\{2,6\},\{3,6\},\{4,6\})$. In $G^5{=}G^4\setminus S^5$, the degrees of the vertices become $\{0,0,5,0,5,4,3,4,3,0,5,4,4,3,0\}$. We note that there are only three vertices of degree 5. Hence it is impossible to choose four more stars.
Case (iii).
Suppose we choose $\{1,6\}$ as a center vertex. Then S^5:$(\{1,6\};\{2,3\},\{2,4\},\{2,5\},\{3,5\},\{4,5\})$. In $G^5{=}G^4\setminus S^5$, the degrees of the vertices become $\{0,0,5,5,0,4,3,3,4,0,4,5,3,4,0\}$. As there are only three vertices of degree 5, it is impossible to choose four more stars.
Case (iv).
Suppose we choose $\{2,3\}$ as a center vertex. Then S^5:$(\{2,3\};\{1,4\},\{1,5\},\{1,6\},\{4,5\},\{4,6\})$. In $G^5{=}G^4\setminus S^5$, the degrees of the vertices become $\{0,0,4,4,4,0,4,4,4,0,5,5,3,3,0\}$. We note that there are only two vertices of degree 5. Hence it is impossible to choose four more stars.
Case (v).
Suppose we choose $\{3,5\}$ as a center vertex. Then S^5:$(\{3,5\};\{1,4\},\{1,6\},\{2,4\},\{2,6\},\{4,6\})$. In $G^5{=}G^4\setminus S^5$, the degrees of the vertices become $\{0,0,4,5,4,5,3,4,3,0,0,5,4,3,0\}$. As there are only three vertices of degree 5, it is impossible to choose four more stars.
Case (vi).
Suppose we choose $\{3,6\}$ as a center vertex. Then S^5:$(\{3,6\};\{1,4\},\{1,5\},\{2,4\},\{2,5\},\{4,5\})$. In $G^5{=}G^4\setminus S^5$, the degrees of the vertices become $\{0,0,4,4,5,5,3,3,4,0,5,0,3,4,0\}$. We note that there are only three vertices of degree 5. Hence it is impossible to choose four more stars.
So, there doesn't exist an S_5-decomposition in $KG_{6,2}$.

3. S_5-decomposition of $KG_{n,2}$

In this section, we prove that $KG_{n,2}$ is S_5-decomposable if and only if $n \geq 7$ and $n \equiv 0,1,2,3(mod\ 5)$.
Let $n \geq 10$, $n_1 \geq 3$ and $n_2 \geq 5$ be positive integers such that $n = n_1{+}n_2$. We define $V_1 = \{1,2,3,...,n_1\}$, $V_2 = \{n_1+1, n_1+2,...,n\}$ and $V(KG_{n,2}) = A_1 \cup A_2 \cup A_3$, where $A_1 = \mathcal{P}_2(V_1)$, $A_2 = \mathcal{P}_2(V_2)$ and $A_3 = \{\{i,j\}|\{i,j\} \in V_1 \times V_2\}$. For $i \in V_1$, $i \times V_2 = \{\{i,j\}|j \in V_2\}$ is called the i^{th} *layer* of the vertices of A_3 and we denote it by Z_i. We define the graphs $G_i, 1 \leq i \leq 6$ as follows:

$$V(G_1) = A_1 \quad ; \qquad E(G_1) = \{XY|X,Y \in A_1 \textit{ and } X \cap Y = \emptyset\}$$
$$V(G_2) = A_2 \quad ; \qquad E(G_2) = \{XY|X,Y \in A_2 \textit{ and } X \cap Y = \emptyset\}$$
$$V(G_3) = A_3 \quad ; \qquad E(G_3) = \{XY|X,Y \in A_3 \textit{ and } X \cap Y = \emptyset\}$$

$$V(G_4) = A_1 \cup A_2 \quad ; \qquad E(G_4) = \{XY | X \in A_1, Y \in A_2 \text{ and } X \cap Y = \emptyset\}$$
$$V(G_5) = A_1 \cup A_3 \quad ; \qquad E(G_5) = \{XY | X \in A_1, Y \in A_3 \text{ and } X \cap Y = \emptyset\}$$
$$V(G_6) = A_2 \cup A_3 \quad ; \qquad E(G_6) = \{XY | X \in A_2, Y \in A_3 \text{ and } X \cap Y = \emptyset\}$$

We observe that, $G_1 \cong KG_{n_1,2}$, $G_2 \cong KG_{n_2,2}$, $G_3 \cong K_{n_1} \times K_{n_2}$, $G_4 \cong K_{|A_1|,|A_2|}$, $G_5 \cong \langle E(A_1, A_3)\rangle$, $G_6 \cong \langle E(A_2, A_3)\rangle$ and $KG_{n,2} = \oplus_{i=1}^{6} G_i$.

Lemma 3.1. *The graph $KG_{7,2}$ is S_5-decomposable.*
Proof. An S_5-decomposition of $KG_{7,2}$ is as follows:
$(\{1,2\}; \{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{5,6\})$, $(\{1,3\}; \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\}, \{5,7\})$, $(\{1,4\}; \{2,3\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,5\})$, $(\{1,5\}; \{2,3\}, \{2,4\}, \{2,6\}, \{2,7\}, \{3,4\})$, $(\{1,6\}; \{2,3\}, \{2,4\}, \{2,5\}, \{2,7\}, \{3,4\})$, $(\{1,7\}; \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\})$, $(\{2,3\}; \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\})$, $(\{2,4\}; \{3,5\}, \{3,6\}, \{3,7\}, \{5,6\}, \{5,7\})$, $(\{2,5\}; \{3,4\}, \{3,6\}, \{3,7\}, \{4,6\}, \{4,7\})$, $(\{2,6\}; \{3,4\}, \{3,5\}, \{3,7\}, \{4,5\}, \{4,7\})$, $(\{2,7\}; \{3,4\}, \{5,6\}, \{3,6\}, \{4,5\}, \{4,6\})$, $(\{3,5\}; \{4,6\}, \{4,7\}, \{2,7\}, \{1,6\}, \{1,7\})$, $(\{3,6\}; \{5,7\}, \{4,7\}, \{1,4\}, \{1,5\}, \{1,7\})$, $(\{3,7\}; \{4,5\}, \{4,6\}, \{1,4\}, \{1,5\}, \{1,6\})$, $(\{4,5\}; \{3,6\}, \{1,6\}, \{1,7\}, \{1,3\}, \{1,2\})$, $(\{4,6\}; \{5,7\}, \{1,2\}, \{1,3\}, \{1,5\}, \{1,7\})$, $(\{4,7\}; \{5,6\}, \{1,2\}, \{1,3\}, \{1,5\}, \{1,6\})$, $(\{5,6\}; \{3,4\}, \{1,3\}, \{1,4\}, \{1,7\}, \{3,7\})$, $(\{5,7\}; \{1,2\}, \{3,4\}, \{1,4\}, \{1,6\}, \{2,6\})$, $(\{6,7\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\})$, $(\{6,7\}; \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\})$.

Lemma 3.2. *The graph $KG_{8,2}$ is S_5-decomposable.*
Proof. An S_5-decomposition of $KG_{8,2}$ is as follows:
$(\{1,3\}; \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\}, \{2,8\})$, $(\{1,4\}; \{2,3\}, \{2,5\}, \{2,6\}, \{2,7\}, \{2,8\})$, $(\{1,5\}; \{2,3\}, \{2,4\}, \{2,6\}, \{2,7\}, \{2,8\})$, $(\{1,6\}; \{2,3\}, \{2,4\}, \{2,5\}, \{2,7\}, \{2,8\})$, $(\{1,7\}; \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,8\})$, $(\{1,8\}; \{2,3\}, \{2,4\}, \{2,5\}, \{2,6\}, \{2,7\})$, $(\{1,6\}; \{3,7\}, \{3,8\}, \{4,5\}, \{4,7\}, \{4,8\})$, $(\{1,7\}; \{3,6\}, \{3,8\}, \{4,5\}, \{4,6\}, \{4,8\})$, $(\{1,8\}; \{3,6\}, \{3,7\}, \{4,5\}, \{4,6\}, \{4,7\})$, $(\{2,4\}; \{5,6\}, \{5,7\}, \{5,8\}, \{6,7\}, \{6,8\})$, $(\{2,5\}; \{4,6\}, \{4,7\}, \{4,8\}, \{6,7\}, \{6,8\})$, $(\{2,6\}; \{3,4\}, \{3,5\}, \{3,7\}, \{3,8\}, \{4,5\})$, $(\{2,7\}; \{3,4\}, \{3,5\}, \{3,6\}, \{3,8\}, \{4,5\})$, $(\{2,8\}; \{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{4,5\})$, $(\{3,4\}; \{5,6\}, \{5,7\}, \{5,8\}, \{6,7\}, \{6,8\})$, $(\{3,5\}; \{4,6\}, \{4,7\}, \{4,8\}, \{6,7\}, \{6,8\})$, $(\{3,6\}; \{4,5\}, \{4,7\}, \{4,8\}, \{5,7\}, \{5,8\})$, $(\{3,7\}; \{4,5\}, \{4,6\}, \{4,8\}, \{5,6\}, \{5,8\})$, $(\{3,8\}; \{4,5\}, \{4,6\}, \{4,7\}, \{5,6\}, \{5,7\})$, $(\{4,5\}; \{6,7\}, \{6,8\}, \{7,8\}, \{1,2\}, \{1,3\})$, $(\{4,6\}; \{5,7\}, \{5,8\}, \{7,8\}, \{1,2\}, \{1,3\})$, $(\{4,7\}; \{5,6\}, \{5,8\}, \{6,8\}, \{1,2\}, \{1,3\})$, $(\{4,8\}; \{5,6\}, \{5,7\}, \{6,7\}, \{1,2\}, \{1,3\})$, $(\{5,6\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,7\}, \{1,8\})$, $(\{5,7\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,6\}, \{1,8\})$, $(\{5,8\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,6\}, \{1,7\})$, $(\{6,7\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,8\}, \{5,8\})$, $(\{6,8\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,7\}, \{5,7\})$, $(\{1,2\}; \{3,4\}, \{3,5\}, \{3,6\}, \{3,7\}, \{3,8\})$, $(\{7,8\}; \{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{1,6\})$, $(\{7,8\}; \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{5,6\})$ and $(\{1,5\}; \{4,6\}, \{4,7\}, \{4,8\}, \{6,7\}, \{6,8\})$. Now, consider the subgraph G_1 obtained by deleting all these stars from $KG_{8,2}$.

In G_1, the degree of the vertex $\{2,3\}$ is exactly 10. Let $6 \le j_1 \le 8$ and $4 \le j_2 \le 8$, then the degree of the vertices $\{2, j_1\}$ and $\{3, j_2\}$ is exactly 5 in G_1. Now, by fixing these vertices as center vertices, we get an S_5-decomposition in G_1.

Lemma 3.3. *The graph $KG_{10,2}$ is S_5-decomposable.*
Proof. Let n_1, n_2=5. In $G_1 \cup G_2 \cup G_4$, consider the following stars: $S^1 : (\{1,2\}; \{3,4\}, \{3,5\}, \{4,5\}, \{7,8\}, \{7,9\})$, $S^2 : (\{1,3\}; \{2,4\}, \{2,5\}, \{4,5\}, \{7,8\}, \{7,9\})$, $S^3 : (\{1,4\}; \{2,3\}, \{2,5\}, \{3,5\}, \{6,7\}, \{7,9\})$, $S^4 : (\{1,5\}; \{2,3\}, \{2,4\}, \{3,4\}, \{6,10\}, \{7,10\})$, $S^5 : (\{2,3\}; \{4,5\}, \{6,7\}, \{6,8\}, \{6,9\}, \{6,10\})$, $S^6 : (\{2,4\}; \{3,5\}, \{6,7\}, \{6,8\}, \{6,9\}, \{6,10\})$, $S^7 : (\{2,5\}; \{3,4\}, \{6,7\}, \{6,8\}, \{6,9\}, \{6,10\})$, $S^8 : (\{6,7\}; \{8,9\}, \{8,10\}, \{9,10\}, \{1,3\}, \{3,4\})$, $S^9 : (\{6,8\}; \{7,9\}, \{7,10\}, \{9,10\}, \{1,2\}, \{3,4\})$, $S^{10} : (\{6,9\}; \{7,8\}, \{7,10\}, \{8,10\}, \{1,4\}, \{1,5\})$, $S^{11} : (\{6,10\}; \{7,8\}, \{7,9\}, \{8,9\}, \{1,2\}, \{1,3\})$, $S^{12} : (\{7,8\}; \{9,10\}, \{1,4\}, \{1,5\}, \{3,4\}, \{2,5\})$, $S^{13} : (\{7,9\}; \{8,10\}, \{1,5\}, \{2,3\}, \{2,4\}, \{3,4\})$ and $S^{14} : (\{7,10\}; \{8,9\}, \{1,2\}, \{1,3\}, \{1,4\}, \{3,4\})$.

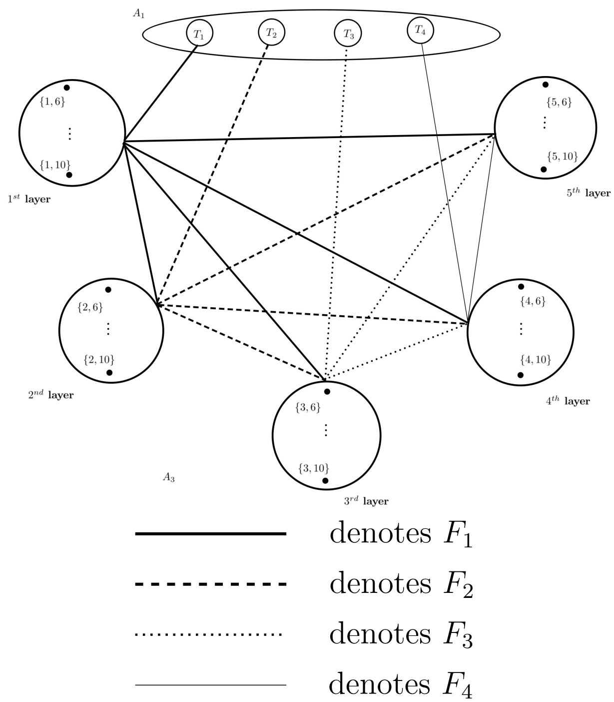

Figure 1: The subgraph F' of $G_3 \cup G_5$

In $(G_1 \cup G_2 \cup G_4) \setminus E(\bigcup_{i=1}^{14} S^i)$, the degree of the vertices $\{3,5\}$ and $\{4,5\}$ is exactly 10, the degree of each vertex of $A_1 \setminus \{\{3,5\},\{4,5\}\}$ is (where $V(G_1 \cup G_2 \cup G_4)$=$A_1 \cup A_2$) exactly 5. Now, by fixing each vertex of A_1 as a center vertex, we get an S_5-decomposition in $(G_1 \cup G_2 \cup G_4) \setminus E(\bigcup_{i=1}^{14} S^i)$. In $G_3 \cup G_5$, we partition the vertex set $A_1 = \bigcup_{i=1}^{4} T_i$, where $T_1 = \{\{2,3\},\{2,4\},\{2,5\},\{3,4\}\}$, $T_2 = \{\{1,3\},\{3,5\},\{4,5\}\}$, $T_3 = \{\{1,2\},\{1,4\}\}$ and $T_4 = \{\{1,5\}\}$. For $1 \leq i \leq 4$, $i < j \leq 5$, we define F_i=$\langle E(\{i,y\},\{j,y'\})\rangle \cup \langle E(\{i,y\},\{a_i,b_i\})\rangle$, where $6 \leq y \neq y' \leq 10$ and for all $\{a_i,b_i\} \in T_i$. Consider the subgraph $F' = \bigcup_{i=1}^{4} F_i$, see Figure 1. Note that, the degree of the vertex $\{i,y\}$ is exactly $5(5-i), 1 \leq i \leq 4$ in F_i. In $(G_3 \cup G_5) \setminus E(F')$, the degree each vertex of A_1 is exactly 10. Now, by fixing each vertex of A_3 and A_1 as a center vertex, we get an S_5-decomposition in F' and $(G_3 \cup G_5) \setminus E(F')$. In G_6, the degree of each vertex of A_2 is exactly 15 and by fixing each vertex of A_2 as a center vertex (ofcourse, 3 times), we get an S_5-decomposition in G_6.

Lemma 3.4. *The graph $KG_{11,2}$ is S_5-decomposable.*

Proof. Let n_1=5 and n_2=6. In G_3, the vertex set A_3 has 5 layers and each layer has 6 vertices. Note that, each subgraph $\langle E(Z_i,Z_j)\rangle$, $1 \leq i < j \leq 5$ of G_3 form a crown graph $C_{6,5}$. By Theorem 2.1, the graph G_3 is S_5-decomposable. In $G_1 \cup G_4$, we choose the following stars: $S^1 : (\{1,2\};\{3,4\},\{3,5\},\{4,5\},\{6,7\},\{6,8\})$, $S^2 : (\{1,3\};\{2,4\},\{2,5\},\{4,5\},\{6,7\},\{6,8\})$, $S^3 : (\{1,4\};\{2,3\},\{2,5\},\{3,5\},\{6,9\},\{6,10\})$, $S^4 : (\{1,5\};\{2,3\},\{2,4\},\{3,4\},\{6,9\},\{6,10\})$, $S^5 : (\{2,3\};\{4,5\},\{6,7\},\{6,8\},\{6,9\},\{6,10\})$, $S^6 : (\{2,4\};\{3,5\},\{6,7\},\{6,8\},\{6,9\},\{6,10\})$ and $S^7 : (\{2,5\};\{3,4\},\{6,7\},\{6,8\},\{6,9\},\{6,10\})$. Let $B' = \{\{6,7\},\{6,8\},\{6,9\},\{6,10\}\}$, $B'' = \{\{6,11\}\{7,8\}\}$. Then $B', B'' \subset A_2 \subset V((G_1 \cup G_4) \setminus E(\bigcup_{i=1}^{7} S^i))$. We write $[(G_1 \cup G_4) \setminus E(\bigcup_{i=1}^{7} S^i)] \cup G_5$=$F_1 \cup F_2 \cup F_3$ where $F_1 = \langle E(A_1,A_3)\rangle \cup \langle E(A_1,B'')\rangle$, $F_2 = \langle E(A_1,B')\rangle$ and $F_3 = \langle E(A_1, A_2 \setminus (B' \cup B''))\rangle$, see Figure 2. Note that, the degree of each vertex of A_1 is 20 in F_1, B' is 5 in F_2 and $A_2 \setminus (B' \cup B'')$ is 10 in F_3. By fixing each vertex of A_1, B' and $A_2 \setminus (B' \cup B'')$ as a center vertex, we get an S_5-decomposition in $[(G_1 \cup G_4) \setminus E(\bigcup_{i=1}^{7} S^i)] \cup G_5$. In $G_2 \cup G_6$, consider the following stars: $S^1 : (\{6,7\};\{8,9\},\{8,10\},\{8,11\},\{9,10\},\{9,11\})$, $S^2 : (\{6,8\};\{7,9\},\{7,10\},\{7,11\},\{9,10\},\{9,11\})$, $S^3 : (\{6,9\};\{7,8\},\{7,10\},\{7,11\},\{8,10\},\{8,11\})$, $S^4 : (\{6,10\};\{7,8\},\{7,9\},\{7,11\},\{8,9\},\{8,11\})$, $S^5 : (\{6,11\};\{7,8\},\{7,9\},\{7,10\},\{8,9\},\{8,10\})$, $S^6 : (\{10,11\};\{6,7\},\{6,8\},\{6,9\},\{7,8\},\{7,9\})$, $S^7 : (\{8,9\};\{7,10\},\{7,11\},\{10,11\},\{1,6\},\{1,7\})$, $S^8 : (\{8,10\};\{7,9\},\{7,11\},\{9,11\},\{1,6\},\{1,7\})$, $S^9 : (\{8,11\};\{7,9\},\{7,10\},\{9,10\},\{1,6\},\{1,7\})$, $S^{10} : (\{9,10\};\{6,11\},\{7,8\},\{7,11\},\{1,6\},\{1,7\})$ and $S^{11} : (\{9,11\};\{6,10\},\{7,8\},\{7,10\},\{1,6\},\{1,7\})$. In $(G_2 \cup G_6) \setminus E(\bigcup_{i=1}^{11} S^i)$, The degree of the vertices $\{1,6\}$ and $\{1,7\}$ is exactly 5. The degree of each vertex of $A_3 \setminus \{\{1,6\},\{1,7\}\}$ is exactly 10. By fixing each vertex of A_3 as a center vertex, we get an S_5-decomposition in $(G_2 \cup G_6) \setminus E(\bigcup_{i=1}^{11} S^i)$.

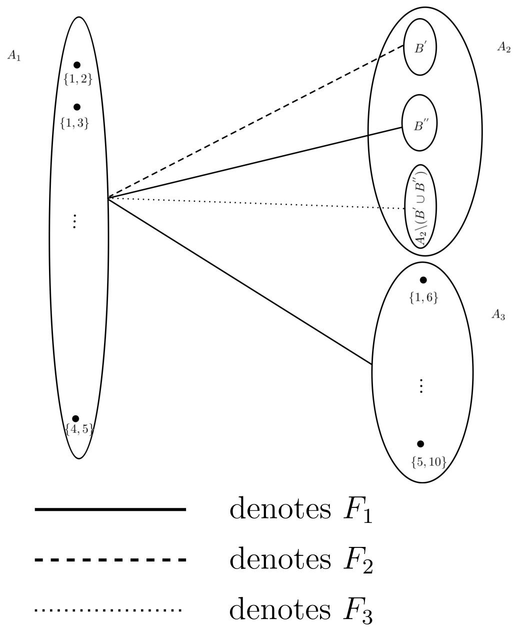

Figure 2: The induced subgraph $F_1 \cup F_2 \cup F_3$

Lemma 3.5. *The graph $KG_{15,2}$ is S_5-decomposable.*
Proof. Let n_1=5 and n_2=10. The graph G_2 is S_5-decomposable, by Lemma 3.3. In G_3, the vertex set A_3 has 5 layers and each layer has 10 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \leq i < j \leq 5$ of G_3 form a crown graph $C_{10,9}$, see Figure 3.

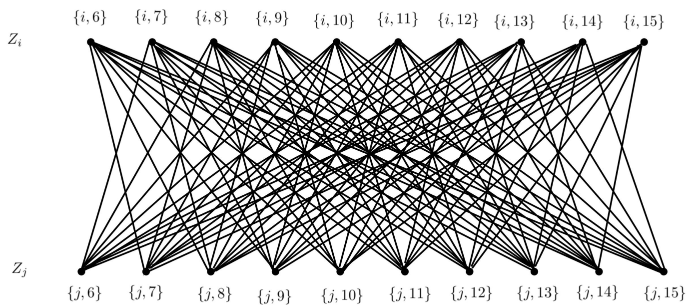

Figure 3: The subgraph $\langle E(Z_i, Z_j)\rangle$ of $KG_{15,2}$

By Theorem 2.1, the graph G_3 is S_5-decomposable. In $G_1 \cup G_4$, consider the following stars: $S^1 : (\{1,2\}; \{3,4\}, \{3,5\}, \{4,5\}, \{6,7\}, \{6,8\})$, $S^2 : (\{1,3\}; \{2,4\}, \{2,5\}$ $, \{4,5\}, \{6,7\}, \{6,8\})$, $S^3 : (\{1,4\}; \{2,3\}, \{2,5\}, \{3,5\}, \{6,9\}, \{6,10\})$, $S^4 : (\{1,5\};$ $\{2,3\}, \{2,4\}, \{3,4\}, \{6,9\}, \{6,10\})$, $S^5 : (\{2,3\}; \{4,5\}, \{6,7\}, \{6,8\}, \{6,9\}, \{6,10\})$,

S^6 : $(\{2,4\};\{3,5\},\{6,7\},\{6,8\},\{6,9\},\{6,10\})$, S^7 : $(\{2,5\};\{3,4\},\{6,7\},\{6,8\},\{6,9\},\{6,10\})$. Let B'=$\{\{6,7\},\{6,8\},\{6,9\},\{6,10\}\} \subset A_2$. In $(G_1 \cup G_4) \setminus E(\bigcup_{i=1}^{7} S^i)$, the degree of each vertex of $A_2 \setminus B'$ and B' is exactly 10 and 5, respectively. Now, by fixing each vertex of A_2 as a center vertex, we get an S_5-decomposition in $(G_1 \cup G_4) \setminus E(\bigcup_{i=1}^{7} S^i)$. In G_5 and G_6, the degree of each vertex of A_1 and A_2 is exactly 30 and 40 respectively. So, by fixing each vertex of A_1 and A_2 as a center vertex, we get an S_5-decomposition in G_5 and G_6.

Lemma 3.6. *The graph $KG_{16,2}$ is S_5-decomposable.*
Proof. Let n_1=6 and n_2=10. The graph G_2 is S_5-decomposable, by Lemma 3.3. In G_3, the vertex set A_3 has 6 layers and each layer has 10 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \leq i < j \leq 6$ of G_3 form a crown graph $C_{10,9}$. By Theorem 2.1, the graph G_3 is S_5-decomposable. Let T'=$\{\{1,2\},\{1,3\}\} \subset A_1 \subset V(G_4)$. In $\langle E(A_1 \setminus T', A_2)\rangle$, the degree of each vertex of $A_1 \setminus T'$ is exactly 45. By fixing each vertex of $A_1 \setminus T'$ as a center vertex, we get an S_5-decomposition in $\langle E(A_1 \setminus T', A_2)\rangle$. In $[G_4 \setminus \langle E(A_1 \setminus T', A_2)\rangle]$, the degree of each vertex of A_2 is exactly two. In G_6, the degree of each vertex of A_2 is exactly 48. In $[G_4 \setminus \langle E(A_1 \setminus T', A_2)\rangle] \cup G_6$, the degree of each vertex of A_2 is exactly 50. Now, by fixing each vertex of A_2 as a center vertex, we get an S_5-decomposition in $[G_4 \setminus \langle E(A_1 \setminus T', A_2)\rangle] \cup G_6$. In $G_1 \cup G_5$, consider the following stars: S^1 : $(\{1,2\};\{3,4\},\{3,5\},\{3,6\},\{4,5\},\{4,6\})$, S^2 : $(\{1,3\};\{2,4\},\{2,5\},\{2,6\},\{4,5\},\{4,6\})$, S^3 : $(\{1,4\};\{2,3\},\{2,5\},\{2,6\},\{3,5\},\{3,6\})$, S^4 : $(\{1,5\};\{2,3\},\{2,4\},\{2,6\},\{3,4\},\{3,6\})$, S^5 : $(\{1,6\};\{2,3\},\{2,4\},\{2,5\},\{3,4\},\{3,5\})$, S^6 : $(\{5,6\};\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\})$, S^7 : $(\{3,4\};\{1,7\},\{1,8\},\{2,5\},\{2,6\},\{5,6\})$, S^8 : $(\{3,5\};\{1,7\},\{1,8\},\{2,4\},\{2,6\},\{4,6\})$, S^9 : $(\{3,6\};\{1,7\},\{1,8\},\{2,4\},\{2,5\},\{4,5\})$, S^{10} : $(\{4,5\};\{1,6\},\{1,7\},\{1,8\},\{2,3\},\{2,6\})$ and S^{11} : $(\{4,6\};\{1,5\},\{1,7\},\{1,8\},\{2,3\},\{2,5\})$. In $(G_1 \cup G_5) \setminus E(\bigcup_{i=1}^{11} S^i)$, the degree of each vertex of $A_3 \setminus \{\{1,7\},\{1,8\}\}$ is 10. The degree of the vertices $\{1,7\}$ and $\{1,8\}$ is exactly 5. Now, by fixing each vertex of A_3 as a center vertex, we get an S_5-decomposition in $(G_1 \cup G_5) \setminus E(\bigcup_{i=1}^{11} S^i)$.

Lemma 3.7. *If $n \in \{12,17\}$, then $KG_{n,2}$ is S_5-decomposable.*
Proof. Let N_1=$\{\{1,y\}|2 \leq y \leq n\}$ and N_2=$\{\{x,y\}|2 \leq x < y \leq n\}$, we partition the vertex set $V(KG_{n,2}) = N_1 \cup N_2$. We write, $KG_{n,2} = \langle E(N_1)\rangle \cup \langle E(N_2)\rangle \cup \langle E(N_1, N_2)\rangle$, where $\langle E(N_1)\rangle$ and $\langle E(N_2)\rangle$ denote the graphs induced by the vertices of N_1 and N_2 respectively. The graph $\langle E(N_1)\rangle$ is a null graph. The graph $\langle E(N_2)\rangle \cong KG_{n-1,2}$ is S_5-decomposable by Lemma 3.4, if n=12 and Lemma 3.6, if n=17. In $\langle E(N_1, N_2)\rangle$, the degree of each vertex of N_1 is exactly 45 (if n=12) or 105 (if n=17). By fixing each vertex of N_1 as a center vertex, we get an S_5-decomposition in $\langle E(N_1, N_2)\rangle$.

Lemma 3.8. *If $n \in \{13, 18\}$, then $KG_{n,2}$ is S_5-decomposable.*
Proof. Let n_2=10. Then n_1=3 if n=13 and n_1=8 if n=18. If n_1=3, the graph G_1 is a null graph and if n_1=8, the graph G_1 is S_5-decomposable, by Lemma 3.2. The graph G_2 is S_5-decomposable, by Lemma 3.3. In G_3, the vertex set A_3 has n_1 layers and each layer has 10 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \leq i < j \leq n_1$ of G_3 form a crown graph $C_{10,9}$. By Theorem 2.1, the graph G_3 is S_5-decomposable. In G_5, the degree of each vertex of A_1 is exactly 10 (if n=13) or 60 (if n=18). Now, by fixing each vertex of A_1 as a center vertex, we get an S_5-decomposition in G_5. In G_4, let $T^{'} = \{\{1,2\},\{1,3\}\} \subset A_1$. In $\langle E(T^{'}, A_2)\rangle$, the degree of each vertex of $T^{'}$ is exactly 45. Now, by fixing each vertex of $T^{'}$ as a center vertex, we get an S_5-decomposition in $\langle E(T^{'}, A_2)\rangle$. In $G_4 \setminus \langle E(T^{'}, A_2)\rangle$, the degree of each vertex of A_2 is exactly 1 (if n=13) or 26 (if n=18). In G_6, the degree of each vertex of A_2 is exactly 24 (if n=13) or 64 (if n=18). In $[G_4 \setminus \langle E(T^{'}, A_2)\rangle] \cup G_6$, the degree of each vertex of A_2 is exactly 25 (if n=13) or 90 (if n=18). By fixing each vertex of A_2 as a center vertex, we get an S_5-decomposition in $[G_4 \setminus \langle E(T^{'}, A_2)\rangle] \cup G_6$.

Theorem 3.1. *If $n \equiv 0, 1, 3 (mod\ 5)$, then $KG_{n,2}$ is S_5-decomposable.*
Proof. Let $l \geq 1$ be positive integer and let

$$n = \begin{cases} 5l & \text{if } n \equiv 0 (mod\ 5) \\ 5l+1 & \text{if } n \equiv 1 (mod\ 5) \\ 5l+3 & \text{if } n \equiv 3 (mod\ 5) \end{cases}$$

If l=1, then $n \in \{5, 6, 8\}$. Clearly, there doesn't exist an S_5-decomposition in $KG_{5,2}$. By Lemma 2.1, there doesn't exist an S_5-decomposition in $KG_{6,2}$. By Lemma 3.2, there exists an S_5-decomposition in $KG_{8,2}$. If $l = 2, 3$, then $n \in \{10, 11, 13, 15, 16, 18\}$. The graph $KG_{n,2}$ is S_5-decomposable, by Lemma 3.3, 3.4, 3.5, 3.6 and 3.8. Hence, the result is true for $l = 1, 2, 3$. We apply mathematical induction on l. Assume that the result is true for all $4 \leq l < k$. Now, we prove that the result is true for $l = k$, $l \geq 4$. Let n_2=10. Then $n_1 = n - n_2$. The graph G_1 is S_5-decomposable, by our assumption and G_2 is S_5-decomposable, by Lemma 3.3. In G_3, the vertex set A_3 has n_1 layers and each layer has 10 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \leq i < j \leq n_1$ of G_3 form a crown graph $C_{10,9}$. By Theorem 2.1, the graph G_3 is S_5-decomposable. It is enough to prove that the graph G_4, G_5 and G_6 are S_5-decomposable. Now, we divide the proof into the following three cases:
Case 1. Let $n \equiv 0(mod\ 5)$. By Theorem 2.2, the graph G_4 is S_5-decomposable. In G_5 and G_6, the degree of each vertex of A_1 and A_2 is exactly $10(n_1 - 2)$ and

$8n_1$, note that $n_1 \equiv 0(mod\ 5)$. So, by fixing each vertex of A_1 and A_2 as a center vertex, we get an S_5-decomposition in G_5 and G_6.
Case 2. Let $n \equiv 1(mod\ 5)$. By Theorem 2.2, the graph G_4 is S_5-decomposable. In G_5, we define three induced subgraphs F_1, F_2 and F_3 as follows: For $n_1+1 \leq y \leq n$,

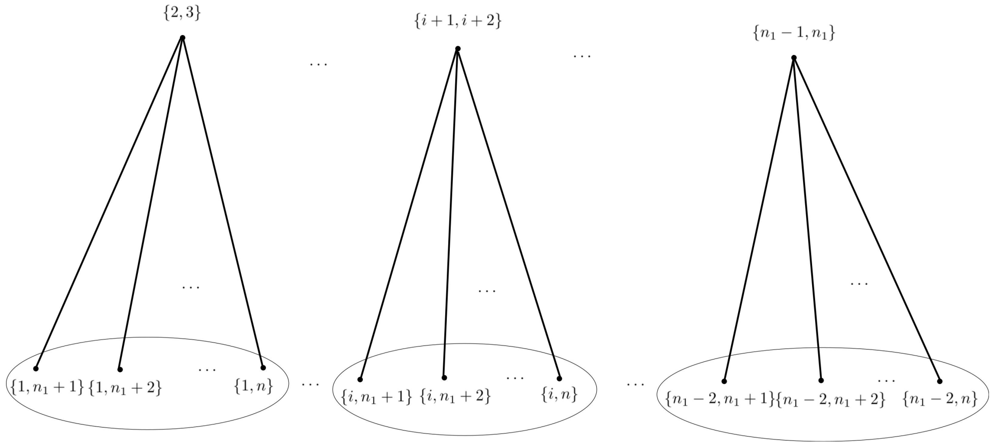

Figure 4: The induced subgraph F_1 of G_5

- $F_1 = \bigcup_{i=1}^{n_1-2} E_i$, where $E_i = \langle E(\{i+1, i+2\}, \{i, y\})\rangle$, see Fig 4.
- $F_2 = \langle E(\{1, n_1\}, \{n_1 - 1, y\})\rangle$.
- $F_3 = \langle E(\{1, 2\}, \{n_1, y\})\rangle$.

Note that, the degree of the vertices $\{i+1, i+2\}, 1 \leq i \leq n_1 - 2$, $\{1, n_1\}$ and $\{1, 2\}$ is exactly 10 in F_1, F_2 and F_3, respectively. Now, by fixing $\{i+1, i+2\}$, $1 \leq i \leq n_1 - 2$, $\{1, n_1\}$ and $\{1, 2\}$ as center vertices, we get an S_5-decomposition in F_1, F_2 and F_3, respectively. In $G_5 \smallsetminus E(\bigcup_{i=1}^{3} F_i)$, the degree of each vertex of A_3 is $\frac{1}{2}[(n_1 - 1)(n_1 - 2) - 2]$. In G_6, the degree of each vertex of A_3 is exactly 36. In $[G_5 \smallsetminus E(\bigcup_{i=1}^{3} F_i)] \cup G_6$, the degree of each vertex of A_3 is $\frac{1}{2}[(n_1-1)(n_1-2)-2]+36= \frac{1}{2}[(n_1 - 1)(n_1 - 2)] + 35$. Now, by fixing each vertex of A_3 as a center vertex, we get an S_5-decomposition in $[G_5 \smallsetminus E(\bigcup_{i=1}^{3} F_i)] \cup G_6$.
Case 3. Let $n \equiv 3(mod\ 5)$. In G_4, let $T' = \{\{1, 2\}, \{1, 3\}\} \subset A_1$. In $\langle E(T', A_2)\rangle$, the degree of each vertex of T' is exactly 45, see Fig 5. Now, by fixing each vertex of T' as a center vertex, we get an S_5-decomposition in $\langle E(T', A_2)\rangle$. In $G_4 \smallsetminus \langle E(T', A_2)\rangle$, the degree of each vertex of A_2 is $\binom{n_1}{2} - 2$. In G_6, the degree of each vertex of A_2 is exactly $8n_1$. In $[G_4 \smallsetminus \langle E(T', A_2)\rangle] \cup G_6$, the degree of each vertex of A_2 is $\binom{n_1}{2} - 2 + 8n_1= \frac{1}{2}[n_1(n_1 + 15) - 4]$. Now, by fixing each vertex of A_2

as a center vertex, we get an S_5-decomposition in $[G_4 \setminus \langle E(T', A_2)\rangle] \cup G_6$. In G_5, the degree of each vertex of A_1 is exactly $10(n_1 - 2)$. Now, by fixing each vertex of A_1 as a center vertex, we get an S_5-decomposition in G_5. By the principle of mathematical induction, the graph $KG_{n,2}$ is S_5-decomposable.

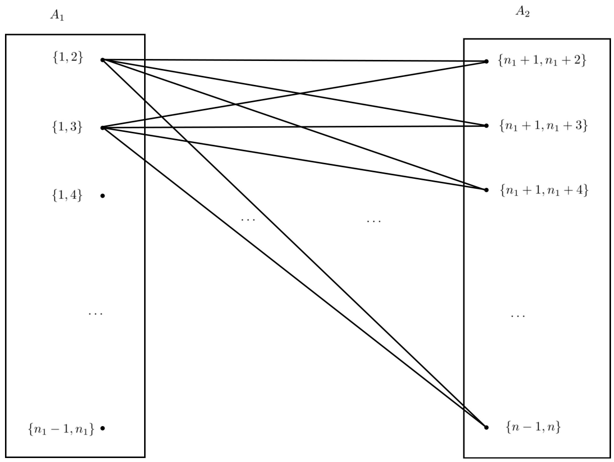

Figure 5: The induced subgraph $\langle E(T', A_2)\rangle$ of G_4

Lemma 3.9. *If $n \equiv 2(mod\ 5)$, then $KG_{n,2}$ is S_5-decomposable.*

Proof. Let $l \geq 1$ be positive integer and let n=$5l + 2$. If l=1,2,3, then $n \in \{7, 12, 17\}$. The graph $KG_{n,2}$ is S_5-decomposable, by Lemma 3.1 and 3.7. Now, we prove that the result is true for all $l \geq 4$. Let n_2=11. Then $n_1 = n - n_2$. By Lemma 3.4, the graph $KG_{11,2}$ is S_5-decomposable. By Theorem 3.1, the graph G_1 is S_5-decomposable. In G_3, the vertex set A_3 has n_1 layers and each layer has 11 vertices. Note that, each subgraph $\langle E(Z_i, Z_j)\rangle$, $1 \leq i < j \leq n_1$ of G_3 form a crown graph $C_{11,10}$. By Theorem 2.1, the graph G_3 is S_5-decomposable. By Theorem 2.2, the graph G_4 is S_5-decomposable. In G_5 and G_6, the degree of each vertex of A_3 is exactly $\binom{n_1}{2} - (n_1 - 1)$=$\frac{(n_1-1)(n_1-2)}{2}$ and 45 respectively. Now, by fixing each vertex of A_3 as a center vertex, we get an S_5-decomposition in G_5 and G_6.

By combining the Lemmas 3.1 to 3.9 and Theorem 3.1, we get the following:

Theorem 3.2. *The graph $KG_{n,2}$ is S_5-decomposable if and only if $n \geq 7$ and $n \equiv 0, 1, 2, 3(mod\ 5)$.*

Acknowledgment

The authors thank the anonymous referee for the valuable comments and suggestions, which improved the quality of the paper.

References

[1] Chen, Y., Kneser graphs are Hamiltonian for $n \geq 3k$, J. Combin. Theory Ser. B, 80 (2000), 69-79.

[2] Ganesamurthy, S., Paulraja, P., Existence of a P_{2k+1}-decomposition in the Kneser graph $KG_{n,2}$, Discrete Math., 341 (2018), 2113-2116.

[3] Kneser, M., Aufgabe, Jahresbericht der Deutschen Mathematiker-Vereinigung, 2, Abteilung 58 (1955), 27.

[4] Lin, C., Lin, J.-J., and Shyu, T.-W., Isomorphic star decomposition of multicrowns and the power of cycles, Ars Combin., 53 (1999), 249-256.

[5] Rodger, C. A., Whitt III, T. R., Path decompositions of Kneser and Generalized Kneser Graphs, Canad. Math. Bull., 58 (3) (2015), 610-619.

[6] Sankari, C., Sangeetha, R., and Arthi, K., Claw-decomposition of Kneser Graphs, Trans. Comb., 11 (1) (2022), 53-61.

[7] Shields, I., Savage, C. D., A note on Hamilton cycles in Kneser graphs, Bull. Inst. Combin. Appl., 40 (2004), 13-22.

[8] Whitt, T. R., Rodger, C. A., Decomposition of the Kneser graph into paths of length four, Discrete Math., 338 (2015), 1284-1288.

[9] Yamamoto, S., Ikeda, H., Shige-eda, S., Ushio, K., and Hamada, N., On claw decomposition of complete graphs and complete bipartite graphs, Hiroshima Math. J., 5(1) (1975), 33-42.