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RICHNESS OF A VERTEX IN A GRAPH

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Abstract: The stress of a vertex in a graph is the number of geodesics passing through it. The status of a vertex v in a graph is the sum of the distances from v to all other vertices. We define the richness of a vertex v in a graph as the status of v minus the stress of v. The total richness of a graph is the sum of richness of all the vertices in that graph. We made some observations, compute richness of vertices in some standard graphs and obtain some interesting results.

Keywords and Phrases: Geodesic, stress of a vertex, status of a vertex.

2020 Mathematics Subject Classification: 05C12.

1. Introduction

For basic terminologies we follow the text-book of Harary [3].

Let $G = (V, E)$ be a graph (finite, undirected, simple). The number of edges in a path P is its length $l(P)$. A shortest path between two vertices u and w in G is called a $u - w$ geodesic. The length of the $u - w$ geodesic is the distance $d(u, w)$ between u and w in G. We say that a geodesic P is passing through a vertex v in G if v is an internal vertex of P. Given two vertices u and v in G, $\rho_{G,v}(u)$ denotes the number of geodesics having v as an end vertex and passing through u.

The length of a longest geodesic in G is called the diameter of G, denoted by $d(G)$. Eccentricity $e(v)$ of a vertex v denotes the distance between v and a vertex farthest from v. For any vertex v, its open neighborhood is $N_G(v)$ (or simply $N(v)$) is the set of all vertices which are adjacent to v and the closed neighborhood of v is $N[v] = N(v) \cup \{v\}$. We say that a graph G is vertex transitive if the automorphism group of G acts transitively on $V(G)$.

The concept of stress of a vertex in a graph was defined by Alfonso Shimbel [15] in 1953. The concept has many applications in the study of biological networks, social networks etc. Some related works can be found in [5], [6], [7], [14] and [16]. Further, the concepts of stress number of a graph and stress regular graphs have been studied by K. Bhargava, N. N. Dattatreya, and R. Rajendra in [1].

In [8, 9, 10], the authors have studied the VL Temperature index, VL Status index, VL Reciprocal Status Index and Co-index for certain graphs. They have also presented the correlations between VL Reciprocal status index and some properties of Butane derivatives.

2. Definitions

Definition 2.1. *The Wiener index $W(G)$ of a connected graph G is defined to be the sum of distances between all vertex pairs in G (See* [17]*)*

$$W(G) = \sum_{\{u,v\} \subset V(G)} d(u, v). \tag{1}$$

Definition 2.2. (Alfonso Shimbel [15]) *Let G be a graph and v be a vertex in G. The stress of v, denoted by $str_G(v)$ or simply $str(v)$, is defined as the number of geodesics in G passing through v.*

Rajendra *et al.* [13] have introduced two topological indices of for graphs called first stress index and second stress index, using stresses of vertices. The first stress index $\mathcal{S}_1(G)$ and the second stress index $\mathcal{S}_2(G)$ of a simple graph G are defined as

$$\mathcal{S}_1(G) = \sum_{v\in V(G)} \text{str}(v)^2 \tag{2}$$

$$\mathcal{S}_2(G) = \sum_{uv\in E(G)} \text{str}(u)\text{str}(v). \tag{3}$$

Using the stress on vertices, Rajendra *et al.* [11, 12] have defined the stress-sum index and square root stress sum index.

We denote the maximum stress among all the vertices of G by Θ_G and minimum stress among all the vertices of G by θ_G.

Definition 2.3. [4] *The status of a vertex u in a graph G, denoted by $\sigma_G(u)$ or simply $\sigma(u)$ is the sum of the distances from v to all other vertices. That is*

$$\sigma(u) = \sum_{v\in V(G)} d(u,v) \tag{4}$$

We denote the maximum status among all the vertices of G by S_G and minimum status among all the vertices of G by s_G.

Definition 2.4. *Let G be a graph and v be a vertex in G. The richness of u, denoted by $h_G(u)$ or simply $h(v)$, is defined as*

$$h(u) = \text{str}(u) - \sigma(u) \tag{5}$$

The richness of a vertex v is the number of geodesics passing through v minus the sum of distances from v to all other vertices in G.

Definition 2.5. *A graph is said to be*

(*i*) *k-stress regular if all of its vertices have stress k;*

(*ii*) *k-status regular if all of its vertices have status k; and*

(*iii*) *k-richness regular if all of its vertices have richness k.*

Definition 2.6. *Let $G = (V, E)$ be a graph. The total stress of G, denoted by $N_{str}(G)$, is defined as,*

$$N_{\text{str}}(G) = \sum_{u\in V} \text{str}(u) \tag{6}$$

Definition 2.7. *Let $G = (V, E)$ be a graph. The total richness of G, denoted by $\mathcal{H}(G)$, is defined as,*

$$\mathcal{H}(G) = \sum_{u\in V} h(u) \tag{7}$$

3. Some Observations

(i) For any vertex u in a graph G, we have

$$\theta_G - S_G \leq h(u) \leq \Theta_G - s_G.$$

(ii) For any vertex u in a graph G, we have $\mathrm{str}(u) \geq 0$ and $\sigma(u) \leq W(G)$, where $W(G)$ is the Wiener index of G. Therefore, by the Definition 2.4, we have

$$h(u) \geq -\sigma(u) \geq -W(G) \tag{8}$$

(iii) If N is the number of geodesics of length at least 2 in a graph G, then by the Definitions 2.2 and 2.4, for any vertex v in G, we have

$$-\sigma(v) \leq h(v) \leq N - \sigma(v)$$

and

$$-S_G \leq h(v) \leq N - s_G.$$

(iv) If there is no geodesic of length ≥ 2 passing through a vertex v in a graph G, then $\mathrm{str}(v) = 0$ and $r(v) = -\sigma(v)$. Hence for any vertex v in a complete graph K_n, we have $h(v) = 1 - n$.

(v) By the Definition 2.4, it follows that, a stress regular graph is richness regular if and only if it is status regular.

(vi) A regular graph may not be richness regular. For instance, it is easy to verify that the graph in Figure 1 is a 3-regular graph, but it is not richness regular. Also, this graph is neither stress regular nor status regular.

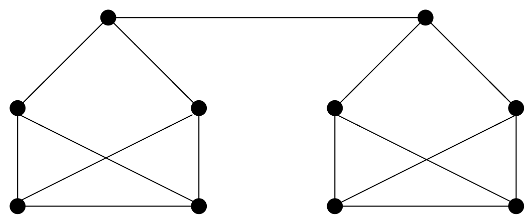

Figure 1: A regular graph which is not richness regular

(vii) A graph G is 0-richness regular if and only if $\mathrm{str}(v) = \sigma(v)$, $\forall v \in V(G)$.

(viii) If η is an automorphism of a graph G and v is any vertex in G, then $h(v) = h(\eta(v))$. Hence, it follows that, any vertex transitive graph is richness regular.

4. Some Results

Proposition 4.1. *Let G be a graph, v be a pendant vertex and u be any vertex in G. Then*

$$h_G(u) = h_{G-v}(u) + \rho_{G,v}(u) - d(u,v). \tag{9}$$

Proof. We have,

$$\begin{aligned} h_{G-v}(u) &= \mathrm{str}_{G-v}(u) - \sigma_{G-v}(u) \\ &= \mathrm{str}_G(u) - \rho_{G,v}(u) - [\sigma_G(u) - d(u,v)] \\ &= h_G(u) - \rho_{G,v}(u) + d(u,v), \end{aligned}$$

which gives (9).

Theorem 4.2. *Let G be any graph and let u be any vertex in G. Then*

(*i*) *$h(u) = -\sigma(u)$ if and only if the neighbors of u induce a complete subgraph.*

(*ii*) *$h(u) = str(u)$ if and only if G is a trivial graph.*

Proof.

(*i*) We have

$$\begin{aligned} h(u) = -\sigma(u) &\iff \mathrm{str}(u) = 0 && \text{(By Definition 2.4)} \\ &\iff \text{the neighbors of } u \text{ induce a complete subgraph.} \end{aligned}$$

(*ii*) We have

$$\begin{aligned} h(u) = \mathrm{str}(u) &\iff \sigma(u) = 0 && \text{(By Definition 2.4)} \\ &\iff d(u,v) = 0,\ \forall v \in V(G) \\ &\iff V(G) = \{u\} \\ &\iff G \text{ is a trivial graph.} \end{aligned}$$

The following corollaries are immediate from the Theorem 4.2.

Corollary 4.3. *For a pendant vertex v in a graph G, $h(v) = -\sigma(v)$.*

Corollary 4.4. *The complete graph K_n on n vertices is $(1-n)$-richness regular.*

Proposition 4.5. *For any graph G, the total richness of G, is given by*

$$\mathcal{H}(G) = N_{str}(G) - 2W(G), \tag{10}$$

where W(G) is the Wiener index of G.
Proof. By the Definition 2.7, we have

$$\begin{aligned}\mathcal{H}(G) = \sum_{u\in V} h(u) &= \sum_{u\in V} \mathrm{str}(u) - \sigma(u)\\ &= \sum_{u\in V} \mathrm{str}(u) - \sum_{u\in V} \sigma(u)\\ &= N_{\mathrm{str}}(G) - \sum_{u\in V}\sum_{v\in V} d(u,v)\\ &= N_{\mathrm{str}}(G) - 2W(G).\end{aligned}$$

Theorem 4.6. *Let G be a graph with at least 2 vertices and $u \in V(G)$. Then $h(u) = -W(G)$ if and only if G is the complete graph on 2 vertices.*
Proof. If G is a complete graph on 2 vertices, then $h(u) = -1 = -W(G)$. Conversely suppose that $h(u) = -W(G)$. Then by (8),

$$\begin{aligned}-\sigma(u) = -W(G) &\Longrightarrow \sigma(u) = W(G)\\ &\Longrightarrow \sum_{v\in V(G)} d(u,v) = \sum_{\{w,v\}\subset V(G)} d(w,v)\\ &\Longrightarrow \sum_{\substack{\{w,v\}\subset V(G)\\ w,v\neq u}} d(w,v) = 0\\ &\Longrightarrow d(w,v) = 0 \text{ for any } w,v \neq u \text{ in } V(G).\\ &\Longrightarrow w = v \text{ for any } w,v \neq u \text{ in } V(G).\\ &\Longrightarrow \text{there is only one vertex other than } u \text{ in } G.\\ &\Longrightarrow G \text{ is the complete graph on 2 vertices.}\end{aligned}$$

Proposition 4.7. *In a complete bipartite $K_{m,n}$, if A and B are the partite sets of $K_{m,n}$ with $|A| = m$ and $|B| = n$, then*

$$h(v) = \begin{cases} \dfrac{n(n-1)}{2} - (2m+n-2), & \text{if } v \in A;\\ \dfrac{m(m-1)}{2} - (2n+m-2), & \text{if } v \in B \end{cases} \tag{11}$$

Proof. In a complete bipartite $K_{m,n}$, if A and B are the partite sets of $K_{m,n}$ with $|A| = m$ and $|B| = n$, then

$$\operatorname{str}(v) = \begin{cases} \dfrac{n(n-1)}{2}, & \text{if } v \in A; \\ \dfrac{m(m-1)}{2}, & \text{if } v \in B. \end{cases} \tag{12}$$

Let $A = \{u_1, \ldots, u_m\}$ and $B = \{v_1, \ldots, v_n\}$ be the partite sets of $K_{m,n}$. Then

$$\begin{aligned} \sigma(u_i) &= \sum_{v \in V(K_{m,n})} d(u,v) \\ &= \sum_{u_j, u_j \neq u_i} 2 + \sum_{vj} 1 \\ &= (m-1) \cdot 2 + n \cdot 1 \\ &= 2m + n - 2. \end{aligned} \tag{13}$$

Similarly, we compute

$$\sigma(v_j) = 2n + m - 2 \tag{14}$$

Using (12), (13) and (14) in (5), we get (11).

Proposition 4.8. *In a cycle C_n on n vertices, for any vertex v,*

$$h(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8} - \dfrac{n^2-1}{4}, & \text{if } n \text{ is odd;} \\ \dfrac{n(n-2)}{8} - \dfrac{n^2}{4}, & \text{if } n \text{ is even.} \end{cases} \tag{15}$$

Proof. For any vertex v in a cycle C_n, we have

$$\operatorname{str}(v) = \begin{cases} \dfrac{(n-1)(n-3)}{8}, & \text{if } n \text{ is odd;} \\ \dfrac{n(n-2)}{8}, & \text{if } n \text{ is even.} \end{cases} \tag{16}$$

and

$$\sigma(v) = \begin{cases} \dfrac{n^2-1}{4}, & \text{if } n \text{ is odd;} \\ \dfrac{n^2}{4}, & \text{if } n \text{ is even.} \end{cases} \tag{17}$$

Using (16) and (17) in (5), we get (15).

Proposition 4.9. *Let $Wd(n,m)$ denote the windmill graph [2] constructed for $n \geq 2$ and $m \geq 2$ by joining m copies of the complete graph K_n at a shared universal vertex v. Then*

$$h(v) = \frac{m(m-1)(n-1)^2}{2} - m(n-1) \tag{18}$$

and

$$h(w) = -(n-1)(2m-1), \tag{19}$$

where w is any vertex $\neq v$ in $Wd(n,m)$.
Proof. In the windmill $Wd(n,m)$, for the shared universal vertex v,

$$\text{str}(v) = m(m-1)(n-1)^2/2, \quad \sigma(v) = m(n-1) \tag{20}$$

and for any vertex $w \neq v$,

$$\text{str}(w) = 0, \quad \sigma(w) = (n-1)(2m-1) \tag{21}$$

Using (20) and (21) in (5), we get (18) and (19), respectively.

Proposition 4.10. *In the path P_n on n vertices $v_1, \ldots, v_n$ (shown in Figure 2),*

$$h(v_i) = (i-1)(n-i) - \frac{(i-1)i}{2} + \frac{(n-i)(n-i+1)}{2}. \tag{22}$$

Proof. We have

Figure 2: The path P_n on n vertices.

$$\text{str}(v_i) = (i-1)(n-i) \text{ and } \sigma(v_i) = \frac{(i-1)i}{2} + \frac{(n-i)(n-i+1)}{2}.$$

Using these in (5), we get (22).

Proposition 4.11. *Let v be an internal vertex of a tree T; $A_1, \ldots, A_m$ be the* *components of $T - v$ and let $l_i(v)$ be the number of vertices at level i (i.e., at distance i) from v. Then*

$$h(v) = \sum_{i<j} |A_i||A_j| - \sum_{1 \leq i \leq e(v)} i\, l_i(v), \tag{23}$$

where $e(v)$ is the eccentricity of v in T.
Proof. It is easy to verify that

$$\mathrm{str}(v) = \sum_{i<j} |A_i||A_j| \tag{24}$$

and

$$\sigma(v) = \sum_{1 \leq i \leq e(v)} i\, l_i(v) \tag{25}$$

Using (24) and (25) in (5), we get (23).

Theorem 4.12. *Let $G = (V, E)$ be a connected graph with at least 3 vertices. In G, $h(v) = -\sigma(v)$ for all vertices v except for one if and only if G is a graph with a unique cut-vertex such that all its blocks are complete subgraphs of G.*
Proof. In [1], it is proved that the graph G has all vertices of zero stress except for one if and only if G is a graph with a unique cut-vertex such that all its blocks are complete subgraphs of G. Hence the proof follows by Definition 2.4.

By Theorem 4.12, the following Corollary is immediate:

Corollary 4.13. *Let G be a connected graph on $n + 1$ vertices. Then $G = K_{1,n}$ if and only if G has exactly one vertex of richness $n(n - 3)/2$ with the richness of remaining vertices equal to $1 - 2n$.*

Theorem 4.14. *For any vertex v in a graph G of diameter 2, $h(v)$ equals the number of unordered pairs of non-adjacent vertices in $N(v)$ minus $\sigma(v)$.*
Proof. For any vertex v in a graph G of diameter 2, $\mathrm{str}(v)$ equals the number of unordered pairs of non-adjacent vertices in $N(v)$ (See [1]). Hence by Definition 2.4 the result follows.

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References

[1] Bhargava K., Dattatreya N. N., and Rajendra R., On stress of a vertex in a graph, Palest. J. Math., accepted for publication.

[2] Gallian J. A., A Dynamic Survey of Graph Labeling, The Electron. J. Combin., DS6, (2019), 1-535.

[3] Harary F., Graph theory, Narosa Publishing house, New Delhi, 1969.

[4] Harary F., Status and Contrastatus, Sociometry, 22 (1) (1959), 23-43.

[5] Indhumathy M., Arumugam S., Baths V. and Singh T., Graph Theoretic Concepts in the Study of Biological Networks, Applied Analysis in Biological and Physical Sciences, Springer Proceedings in Mathematics & Statistics, 186 (2016), 187-200.

[6] Kang C., Molinaro C., Kraus S., Shavitt Y. and Subrahmanian V., Diffusion centrality in social networks, In: Proceedings of the 2012 international conference on advances in social networks analysis and mining (ASONAM 2012), IEEE Computer Society, (2012), 558-564.

[7] Koschützki D., Lehmann K. A., Peeters L., Richter S., Tenfelde-Podehl D. and Zlotowski O., Centrality Indices, In: U. Brandes, T. Erlebach (eds), Network Analysis, Lecture Notes in Computer Science, Springer, Berlin, Heidelberg, Vol. 3418 (2005), 16-61.

[8] Lokesha V., Suvarna S., Manjunath M. and Zeba Yasmeen K., VL Temperature Index of Certain Archimedean Lattice, South East Asian J. Math. Math. Sci., 17(1) (2021), 213-222.

[9] Lokesha V., Suvarna S. and Sinan Cevik A., VL status index and co-index of connected graphs, Proceedings of the Jangjeon Math. Soc., 24 (3) (2021), 285-295.

[10] Lokesha V., Suvarna S., Sinan Cevik A. and Cangul I. N., VL Reciprocal Status Index and Co-Index of Graphs, Journal of Mathematics, Volume 2021, Article ID 5529080, 10 Pages.

[11] Mahesh K. B., Rajendra R. and Reddy P. S. K., Square Root Stress Sum Index for Graphs, Proyecciones, 40 (4) (2021), 927-937.

[12] Rajendra R., Reddy P. S. K. and Harshavardhana C. N., Stress-Sum Index for Graphs, Sci. Magna, 15 (1) (2020), 94-103.

[13] Rajendra R., Reddy P. S. K. and Cangul I. N., Stress Indices of Graphs, Advn. Stud. Contemp. Math., 31 (2) (2021), 163-173.

[14] Scardoni G., Petterlini M. and Laudanna C., Analyzing Biological Network Parameters with CentiScaPe., Bioinformatics, 25 (2009), 2857-2859.

[15] Shimbel A., Structural Parameters of Communication Networks, Bulletin of Mathematical Biophysics, 15 (1953), 501-507.

[16] Shinozaki K., Kazuko Y. S. and Motoaki S., Regulatory Network of Gene Expression in the Drought and Cold Stress responses, Curr. Opin. Plant Biol., 6 (5) (2003), 410-417.

[17] Wiener H., Structural determination of paraffin boiling points, J. Amer. Chem. Soc., 69 (1) (1947), 17-20.