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MULTIPLICATIVE ZAGREB INDICES OF FOUR NEW $\mathcal{F}$-SUMS OF GRAPHS

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Abstract: For molecular graph G, the first multiplicative Zagreb index is defined as the product of squares of degree of all vertices of graph and the second multiplicative Zagreb index is defined as $\prod_2(G) = \prod_{u \in V(G)} d_G(u)^{d_G(u)}$. In this paper, we obtain first and second multiplicative Zagreb indices of four new $\mathcal{F}$-Sums of graphs.

Keywords and Phrases: F-sums of graphs, multiplicative Zagreb indices, graph indices.

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1. Introduction

The topological indices are graph invariants which are numerical values associated with molecular graphs. In mathematical chemistry, molecular descriptors play a leading role specifically in the field of QSPR/QSAR modelling. The topological indices were initiated when the eminent chemist H. Wiener found the first topological index, known as Wiener index, the Zagreb indices belong to the well known and well researched molecular descriptors. It was firstly presented by Gutman and Trinajestič in [14, 15, 26], where they investigated how the total energy of π-electron depends on the structure of molecules. For more on topological indices, one can refer [3, 7, 10, 20, 18].

2. Definitions and Preliminaries

Let G be a finite undirected graph without loops and multiple edges on n vertices and m edges and is called (n, m) graph. We denote vertex set and edge set of graph G as $V(G)$ and $E(G)$, respectively. For a graph G, the *degree* of a vertex V is the number of edges incident to V and is denoted by $d_G(v)$. For undefined terms and notations refer [16].

In 1984, Narumi and Katayama [19] considered the product index as

$$NK(G) = \prod_{u \in V(G)} d_G(u).$$

Tomović and Gutman renamed this molecular structure descriptor as the Narumi-Katayama index [25]. In 2010, Todeshine et al. [22, 23] proposed the multiplicative variants of ordinary Zagreb indices, which are defined as follows

$$\prod_1(G) = \prod_{u \in V(G)} d_G(u)^2 = [NK(G)]^2$$

$$\prod_2(G) = \prod_{uv \in E(G)} d_G(u)d_G(v).$$

These two graph invariants are called first and second multiplicative Zagreb indices by Gutman [113]. Recently, Eliasi et al. [10] introduced a further multiplicative version of the first Zagreb index as

$$\prod_1^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)].$$

In [12, 27] the authors called it as multiplicative sum Zagreb index and modified first multiplicative Zagreb index respectively. The second multiplicative Zagreb index for any graph G can also be written as [13].

$$\prod_2(G) = \prod_{u \in V(G)} d_G(u)^{d_G(u)}.$$

Basavanagoud et al. [5] introduced a further multiplicative version of the second Zagreb index as

$$\prod_2^*(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)]^{d_G(u)+d_G(v)}.$$

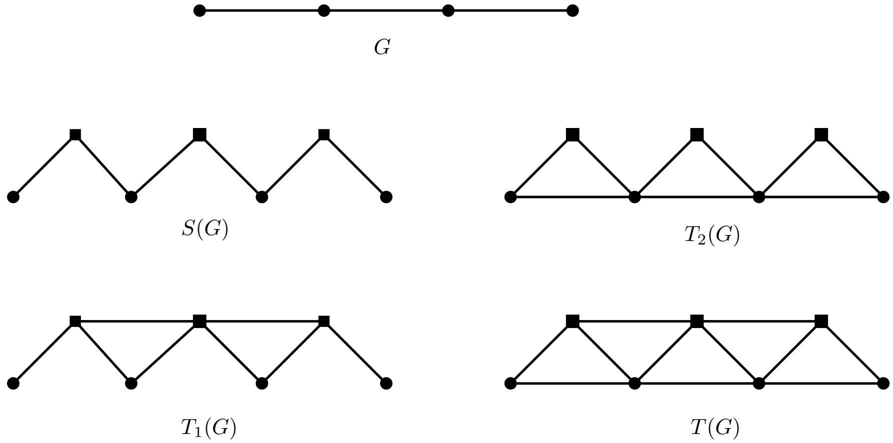

Figure 1: Graph G and its transformations $S(G)$, $T_2(G)$, $T_1(G)$, $T(G)$.

The main properties of multiplicative Zagreb indices are summarized in [7, 27]. For more on multiplicative Zagreb indices are in [1, 2, 9, 17, 24, 28].

For a graph G with vertex set $V(G)$ and edge set $E(G)$. Let $L(G)$ be the line graph of G. There are four related graphs as shown in Figure 1 and defined as follows.

- The subdivision graph $S = S(G)$ [16]; is the graph obtained by inserting a new vertex onto each edge of G.
- Semitotal-point graph $T_2 = T_2(G)$ [21]; $V(T_2) = V(G) \cap E(G)$ and $E(G) = E(S) \cap E(G)$.
- Semitotal-line graph $T_1 = T_1(G)$ [21]; $V(T_1) = V(G) \cap E(G)$ and $E(T_1) = E(S) \cap E(L)$.
- Total graph $T = T(G)$ [6]; $V(T) = V(G) \cap E(G)$ and $E(T) = E(S) \cap E(G) \cap E(L)$. Here $L = L(G)$ is the line graph of G.

In the recent paper [11], Eliasi and Taeri introduced four new operations on graphs defined as follows.

Definition 1. [11] *Let $\mathcal{F} \in S, T_2, T_1, T$. The $\mathcal{F}$- sums of G_1 and G_2, denoted by $G_1 +_F G_2$, is a graph with the set of vertices $V(G_1 +_F G_2) = ((V(G_1) \cup E(G_1)) \times V(G_2))$ and two vertices (u_1, u_2) and (v_1, v_2) of $G_1 +_F G_2$ are adjacent if and only if $[u_1 = v_1 \in V(G_1)$ and $u_2v_2 \in E(G_2)]$ or $[u_2 = v_2 \in V(G_2)$ and $u_1v_1 \in E(F(G_1))]$.*

Thus, authors in [11] obtained four new graph operations namely $G_1 +_S G_2$, $G_1 +_{T_2} G_2$, $G_1 +_{T_1} G_2$ and $G_1 +_T G_2$ as depicted in Figure 2 and studied the Wiener indices of these graphs.

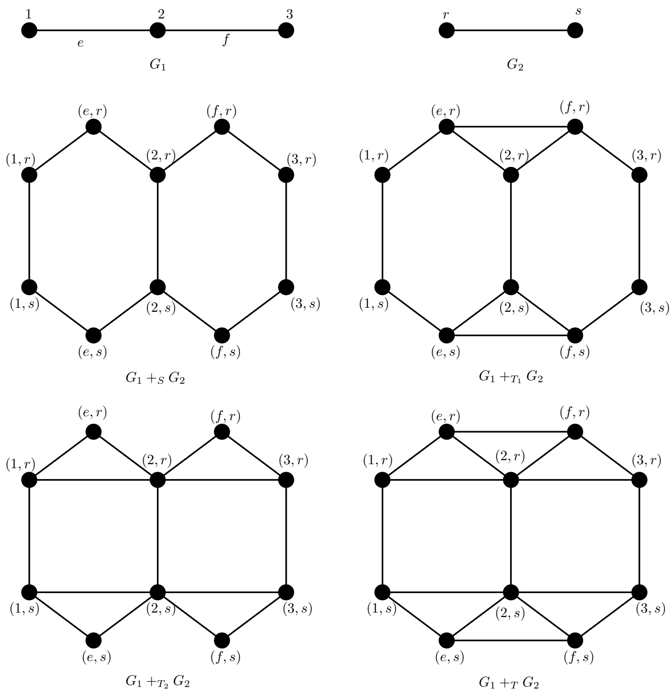

Figure 2: Four new F-sums of graphs

In the recent papers, Deng et al. [8] gave expressions for first and second Zagreb indices of these new graphs. Basavanagoud et al. [4], studied first neighbourhood Zagreb index of these graphs. Motivated by this, we obtain explicit formulae for finding first and second multiplicative Zagreb indices of four new $\mathcal{F}$-sums of graphs.

Before, we proceed to the purpose of the paper, we mention some existing results.

Theorem 2.1. [5] *Let G be a graph of order n and size m, then*
(i) $\prod_1(S) = 4^m \prod_1(G)$,
(ii) $\prod_1(T_2) = 4^{n+m} \prod_1(G)$,
(iii) $\prod_1(T_1) = \prod_1(G)[\prod_1^*(G)]^2$,
(iv) $\prod_1(T) = 4^n \prod_1(G)[\prod_1^*(G)]^2$.

Theorem 2.2. [5] *Let G be a graph of order n and size m, then*
(i) $\prod_2(S) = 4^m \prod_2(G)$,

(ii) $\prod_2(T_2) = 64^m \prod_1(G)\prod_2(G)$,
(iii) $\prod_2(T_1) = \prod_2(G)\prod_2^*(G)$,
(iv) $\prod_2(T) = 16^m \prod_2^*(G)[\prod_2(G)]^2$.

3. Main Results

Let $G_i = (V_i, E_i)$ with $|V_i| = n_i$ and $|E_i| = m_i$ be two graphs for i=1, 2. In this section, we obtain first and second multiplicative Zagreb indices of four new $\mathcal{F}$-sums of graphs.

3.1. First Multiplicative Zagreb Index of Four New $\mathcal{F}$-sums of Graphs

In this section, we proceed to obtain first multiplicative Zagreb index of four new $\mathcal{F}$-sums of graphs, where $\mathcal{F} \in \{S, T_2, T_1, T\}$.

The following theorem gives the first multiplicative Zagreb index of four new $\mathcal{F}$-sums of graphs G_1 and G_2.

Theorem 3.1. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_1(G_1 +_S G_2) = 4^{m_1 n_2}[\prod_1(G_2)]^{m_1} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^2.$$

Proof. By the definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod_1(G_1 +_S G_2) &= \prod_{(u,v)\in V(G_1+_S G_2)} d^2_{G_1+_S G_2}(u,v) \\ &= \prod_{u\in(S(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{S(G_1)}(u) + d_{G_2}(v)]^2 \times \\ &\prod_{v\in V(G_2)} \prod_{e\in(S(G_1))\cap E(G_1)} [d_{S(G_1)}(e) d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (S(G_1)) \cap V(G_1)$, $d_{S(G_1)}(u) = d_{G_1}(u)$ and $e \in (S(G_1)) \cap E(G_1)$, $d_{S(G_1)}(e) = 2$. Therefore,

$$\begin{aligned}\prod_1(G_1 +_S G_2) &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^2 \prod_{v\in V(G_2)} \prod_{e\in E(G_1)} [2d_{G_2}(v)]^2 \\ &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^2 \prod_{v\in V(G_2)} \prod_{e\in E(G_1)} [4d^2_{G_2}(v)] \\ &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^2 \times 4^{m_1 n_2}[\prod_1(G_2)]^{m_1} \\ &= 4^{m_1 n_2}[\prod_1(G_2)]^{m_1} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^2.\end{aligned}$$

Theorem 3.2. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_1 (G_1 +_{T_2} G_2) = 4^{m_1 n_2} \, [\prod_1 (G_2)]^{m_1} \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} (2d_{G_1}(u) + d_{G_2}(v))^2.$$

Proof. By the definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod_1 (G_1 +_{T_2} G_2) &= \prod_{(u,v) \in V(G_1 +_{T_2} G_2)} d^2_{G_1 +_{T_2} G_2}(u, v) \\ &= \prod_{u \in (T_2(G_1)) \cap V(G_1)} \prod_{v \in V(G_2)} [d_{T_2(G_1)}(u) + d_{G_2}(v)]^2 \times \\ &\quad \prod_{v \in V(G_2)} \prod_{e \in (T_2(G_1)) \cap E(G_1)} [d_{T_2(G_1)}(e) d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (T_2(G_1)) \cap V(G_1)$, $d_{T_2(G_1)}(u) = 2d_{G_1}(u)$ and $e \in (T_2(G_1)) \cap E(G_1)$, $d_{T_2(G_1)}(e) = 2$. Therefore,

$$\begin{aligned}\prod_1 (G_1 +_{T_2} G_2) &= \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^2 \prod_{v \in V(G_2)} \prod_{e \in E(G_1)} [2d_{G_2}(v)]^2 \\ &= \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^2 \prod_{v \in V(G_2)} \prod_{e \in E(G_1)} [4d^2_{G_2}(v)] \\ &= \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^2 \times 4^{m_1 n_2} [\prod_1 (G_2)]^{m_1} \\ &= 4^{m_1 n_2} \, [\prod_1 (G_2)]^{m_1} \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} (2d_{G_1}(u) + d_{G_2}(v))^2.\end{aligned}$$

Theorem 3.3. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_1 (G_1 +_{T_1} G_2) = [\prod_1 (G_2)]^{m_1} [\prod_1^{*} (G_1)]^{2n_2} \prod_{u \in V(G_1)} \prod_{v \in V(G_2)} (d_{G_1}(u) + d_{G_2}(v))^2.$$

Proof. By the definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod_1 (G_1 +_{T_1} G_2) &= \prod_{(u,v) \in V(G_1 +_{T_1} G_2)} d^2_{G_1 +_{T_1} G_2}(u, v) \\ &= \prod_{u \in (T_1(G_1)) \cap V(G_1)} \prod_{v \in V(G_2)} [d_{T_1(G_1)}(u) + d_{G_2}(v)]^2 \times \\ &\quad \prod_{v \in V(G_2)} \prod_{e \in (T_1(G_1)) \cap E(G_1)} [d_{T_1(G_1)}(e) d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (T_1(G_1)) \cap V(G_1), d_{T_1(G_1)}(u) = d_{G_1}(u)$ and $e \in (T_1(G_1)) \cap E(G_1)$, $d_{T_1(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$. Therefore,

$$\begin{aligned}\prod_1(G_1 +_{T_1} G_2) &= \prod_{u\in V(G_1)}\prod_{v\in V(G_2)}[d_{G_1}(u)+d_{G_2}(v)]^2\times \\ &\prod_{v\in V(G_2)}\prod_{xy\in E(G_1)}[(d_{G_1}(x)+d_{G_1}(y))\ d_{G_2}(v)]^2 \\ &= \prod_{u\in V(G_1)}\prod_{v\in V(G_2)}(d_{G_1}(u)+d_{G_2}(v))^2\times \\ &\prod_{v\in V(G_2)}\prod_{xy\in E(G_1)}[d_{G_1}(x)+d_{G_1}(y)]^2\ d^2_{G_2}(v) \\ &= \prod_{u\in V(G_1)}\prod_{v\in V(G_2)}(d_{G_1}(u)+d_{G_2}(v))^2\ \ [\prod_1(G_2)]^{m_1}]\ \ [\prod_1^{*}(G_1)]^{2n_2} \\ &= [\prod_1(G_2)]^{m_1}\ \ [\prod_1^{*}(G_1)]^{2n_2}\prod_{u\in V(G_1)}\prod_{v\in V(G_2)}(d_{G_1}(u)+d_{G_2}(v))^2.\end{aligned}$$

Theorem 3.4. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_1(G_1 +_T G_2) = [\prod_1(G_2)]^{m_1}\ \ [\prod_1^{*}(G_1)]^{2n_2}\prod_{u\in V(G_1)}\prod_{v\in V(G_2)}(2d_{G_1}(u)+d_{G_2}(v))^2.$$

Proof. By the definition of first multiplicative Zagreb index, we have

$$\begin{aligned}\prod_1(G_1 +_T G_2) &= \prod_{(u,v)\in V(G_1+_TG_2)} d^2_{G_1+_TG_2}(u,v) \\ &= \prod_{u\in(T(G_1))\cap V(G_1)}\prod_{v\in V(G_2)}[d_{T(G_1)}(u)+d_{G_2}(v)]^2\times \\ &\prod_{v\in V(G_2)}\prod_{e\in(T(G_1))\cap E(G_1)}[d_{T(G_1)}(e)d_{G_2}(v)]^2.\end{aligned}$$

For $u \in (T(G_1)) \cap V(G_1)$, $d_{T(G_1)}(u) = 2d_{G_1}(u)$ and $e \in (T(G_1)) \cap E(G_1)$, $d_{T(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$. Therefore,

$$\begin{aligned}\prod_1(G_1 +_T G_2) &= \prod_{u\in V(G_1)}\prod_{v\in V(G_2)}[2d_{G_1}(u)+d_{G_2}(v)]^2\times \\ &\prod_{v\in V(G_2)}\prod_{xy\in E(G_1)}[(d_{G_1}(x)+d_{G_1}(y))\ d_{G_2}(v)]^2\end{aligned}$$

$$= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^2 \prod_{v\in V(G_2)} \prod_{xy\in E(G_1)} [d_{G_1}(x) + d_{G_1}(y)]^2 \; d^2_{G_2}(v)$$

$$= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^2 \; [\prod_1 (G_2)]^{m_1} \;\; [\prod_1^* (G_1)]^{2n_2}$$

$$= [\prod_1 (G_2)]^{m_1} \;\; [\prod_1^* (G_1)]^{2n_2} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} (2d_{G_1}(u) + d_{G_2}(v))^2.$$

3.2. Second Multiplicative Zagreb Index of Four New $\mathcal{F}$-sums of Graphs

In this section, we proceed to obtain second multiplicative Zagreb index of four new $\mathcal{F}$-sums of graphs, where $\mathcal{F} \in \{S, T_2, T_1, T\}$.

The following theorem gives the second multiplicative Zagreb index of four new $\mathcal{F}$-sums of two graphs G_1 and G_2.

Theorem 3.5. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_2 (G_1 +_S G_2) = 16^{m_1 m_2^2} \; [\prod_2 (G_2)]^{2m_1} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)}.$$

Proof. By the definition of second multiplicative Zagreb index, we have

$$\prod_2 (G_1 +_S G_2) = \prod_{(u,v)\in V(G_1+_S G_2)} [d_{G_1+_S G_2}(u,v)]^{d_{G_1+_S G_2}(u,v)}$$

$$= \prod_{u\in(S(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{S(G_1)}(u) + d_{G_2}(v)]^{d_{S(G_1)}(u)+d_{G_2}(v)} \times$$

$$\prod_{v\in V(G_2)} \prod_{e\in(S(G_1))\cap E(G_1)} [d_{S(G_1)}(e) d_{G_2}(v)]^{d_{S(G_1)}(e) d_{G_2}(v)}$$

For $u \in (S(G_1)) \cap V(G_1)$, $d_{S(G_1)}(u) = d_{G_1}(u)$ and $e \in (S(G_1)) \cap E(G_1)$, $d_{S(G_1)}(e) = 2$. Therefore,

$$\prod_2 (G_1 +_S G_2) = \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)} \times$$

$$\prod_{v\in V(G_2)} \prod_{e\in E(G_1)} [2d_{G_2}(v)]^{2d_{G_2}(v)}$$

$$= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) d_{G_2}(v)]^{d_{G_1}(u) d_{G_2}(v)} \times 16^{m_1 m_2^2} \; [\prod_2 (G_2)]^{2m_1}$$

$$= 16^{m_1 m_2^2} \; [\prod_2 (G_2)]^{2m_1} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)}.$$

Theorem 3.6. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*
$$\prod_2 (G_1 +_{T_2} G_2) = 16^{m_1 m_2^2}[\prod_2 (G_2)]^{2m_1} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u)+d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)}.$$

Proof. By the definition of second multiplicative Zagreb index, we have
$$\begin{aligned}\prod_2(G_1 +_{T_2} G_2) &= \prod_{(u,v)\in V(G_1+_{T_2}G_2)} [d_{G_1+_{T_2}G_2}(u,v)]^{d_{G_1+_{T_2}G_2}(u,v)} \\ &= \prod_{u\in(T_2(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{T_2(G_1)}(u) + d_{G_2}(v)]^{d_{T_2(G_1)}(u)+d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{e\in(S(G_1))\cap E(G_1)} [d_{T_2(G_1)}(e) d_{G_2}(v)]^{d_{T_2(G_1)}(e)d_{G_2}(v)}.\end{aligned}$$

For $u \in (T_2(G_1)) \cap V(G_1)$, $d_{T_2(G_1)}(u) = 2d_{G_1}(u)$ and $e \in (T_2(G_1)) \cap E(G_1)$, $d_{T_2(G_1)}(e) = 2$. Therefore,
$$\begin{aligned}\prod_2(G_1 +_{T_2} G_2) &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{e\in E(G_1)} [2d_{G_2}(v)]^{2d_{G_2}(v)} \\ &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)} 16^{m_1 m_2^2}[\prod_2 (G_2)]^{2m_1} \\ &= 16^{m_1 m_2^2}[\prod_2 (G_2)]^{2m_1} \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)}.\end{aligned}$$

Theorem 3.7. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*
$$\begin{aligned}\prod_2 (G_1 +_{T_1} G_2) = [\prod_2^{*} (G_1)]^{n_2 d_{G_2}(v)} [\prod_2 (G_2)]^{d_{G_1}(x)+d_{G_1}(y)} \times \\ \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)}.\end{aligned}$$

Proof. By the definition of second multiplicative Zagreb index, we have
$$\begin{aligned}\prod_2(G_1 +_{T_1} G_2) &= \prod_{(u,v)\in V(G_1+_{T_1}G_2)} [d_{G_1+_{T_1}G_2}(u,v)]^{d_{G_1+_{T_1}G_2}(u,v)} \\ &= \prod_{u\in(T_1(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{T_1(G_1)}(u) + d_{G_2}(v)]^{d_{T_1(G_1)}(u)+d_{G_2}(v)} \times \\ &\quad \prod_{v\in V(G_2)} \prod_{e\in(T_1(G_1))\cap E(G_1)} [d_{T_1(G_1)}(e) d_{G_2}(v)]^{d_{T_1(G_1)}(e)d_{G_2}(v)}.\end{aligned}$$

For $u \in (T_1(G_1)) \cap V(G_1)$, $d_{T_1(G_1)}(u) = d_{G_1}(u)$ and $e \in (T_1(G_1)) \cap E(G_1)$, $d_{T_1(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$. Therefore,

$$\begin{aligned}\prod_2(G_1 +_{T_1} G_2) &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)} \times \\ &\prod_{v\in V(G_2)} \prod_{xy\in E(G_1)} [(d_{G_1}(x) + d_{G_1}(y))d_{G_2}(v)]^{(d_{G_1}(x)+d_{G_1}(y))d_{G_2}(v)} \\ &= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)} \times \\ &[\prod_2^* (G_1)]^{n_2 d_{G_2}(v)} [\prod_2 (G_2)]^{d_{G_1}(x)+d_{G_1}(y)} \\ &= [\prod_2^* (G_1)]^{n_2 d_{G_2}(v)} [\prod_2 (G_2)]^{d_{G_1}(x)+d_{G_1}(y)} \times \\ &\prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [d_{G_1}(u) + d_{G_2}(v)]^{d_{G_1}(u)+d_{G_2}(v)}.\end{aligned}$$

Theorem 3.8. *If G_1 and G_2 are (n_1, m_1) and (n_2, m_2) graphs, respectively, then*

$$\prod_2 (G_1 +_T G_2) = [\prod_2^* (G_1)]^{n_2 d_{G_2}(v)} [\prod_2 (G_2)]^{d_{G_1}(x)+d_{G_1}(y)} \times$$

$$\prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)}.$$

Proof. By the definition of second multiplicative Zagreb index, we have

$$\begin{aligned}\prod_2(G_1 +_T G_2) &= \prod_{(u,v)\in V(G_1+_T G_2)} [d_{G_1+_T G_2}(u, v)]^{d_{G_1+_T G_2}(u,v)} \\ &= \prod_{u\in(T(G_1))\cap V(G_1)} \prod_{v\in V(G_2)} [d_{T(G_1)}(u) + d_{G_2}(v)]^{d_{T(G_1)}(u)+d_{G_2}(v)} \times \\ &\prod_{v\in V(G_2)} \prod_{e\in(S(G_1))\cap E(G_1)} [d_{T(G_1)}(e) d_{G_2}(v)]^{d_{T(G_1)}(e) d_{G_2}(v)}.\end{aligned}$$

For $u \in (T(G_1)) \cap V(G_1), d_{T(G_1)}(u) = 2d_{G_1}(u)$ and for $e \in (T(G_1)) \cap E(G_1)$,

$d_{S(G_1)}(e) = d_{G_1}(x) + d_{G_1}(y)$ where $e = xy \in E(G_1)$ Therefore,

$$\prod_2(G_1 +_T G_2) = \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{(G_1)}(u)+d_{G_2}(v)} \times$$

$$\prod_{v\in V(G_2)} \prod_{xy\in E(G_1)} [\big(d_{G_1}(x) + d_{G_1}(y)\big)d_{G_2}(v)]^{(d_{G_1}(x)+d_{G_1}(y))d_{G_2}(v)}$$

$$= \prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)} \times$$

$$[\prod_2^* (G_1)]^{n_2 d_{G_2}(v)} [\prod_2 (G_2)]^{d_{G_1}(x)+d_{G_1}(y)}$$

$$= [\prod_2^* (G_1)]^{n_2 d_{G_2}(v)} [\prod_2 (G_2)]^{d_{G_1}(x)+d_{G_1}(y)} \times$$

$$\prod_{u\in V(G_1)} \prod_{v\in V(G_2)} [2d_{G_1}(u) + d_{G_2}(v)]^{2d_{G_1}(u)+d_{G_2}(v)}.$$

4. Conclusion

The results presented in this paper are first and second multiplicative Zagreb indices of four new $\mathcal{F}$-sums of graphs is calculated explicitly for each case $\mathcal{F} \in \{S, T_2, T_1, T\}$. The topological indices are used to study quantitative structure relationship-property/activity (QSPR/QSAR). By using these indices we can predict the physio-chemical properties of chemical compounds. Further one can investigate different topological indices of these graphs.

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