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G- FRAMES AND THEIR STABILITY IN HILBERT SPACE

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Abstract: W. Sun in his paper [W. Sun, G-frames and g-Riesz bases. J. Math. Anal. Appl 322 (2006),437-452] has introduced g-frames which are generalized frames and cover many recent generalizations of frames such as bounded quasi-projections, fusion frames and pseudo-frames. We give a necessary and sufficient condition for a g-frame to be a dual to a given g-frame and obtain some sufficient conditions under which sequences are stable under small perturbations.

Keywords and Phrases: G-frames, dual g-frames, orthogonal g-frames, g-R-dual sequence.

2020 Mathematics Subject Classification: 42C15.

1. Introduction

Frames in Hilbert spaces have been introduced in 1952 by J. Duffin and A. C. Schaeffer [5] while studying non harmonic Fourier series. The work of Daubechies, Grossmann and Meyer [4] in 1986 reintroduced the frames.

In [11], W. Sun introduced the concept of generalized frames (or g-frames) in Hilbert spaces, which are generalizations of frames and cover many other recent generalizations of frames such as bounded quasi-projections, fusion frames, and pseudo frames. Study of stability of frames and g-frames under small perturbation is also important in applications. Finding the conditions under which a g-frame close to a given g-frame is also a g-frame is called stability problem. Stability of g-frames and dual g-frames has been given by W. Sun. [12] and subsequently developed by many other authors [1, 7, 8, 10]. In this paper we give a necessary and sufficient condition under which a g-frame can be a dual to a given g-frame and obtain some sufficient conditions under which g-frames are stable under small perturbations and also generalize the characterization of an alternate dual g-frame of a given g-frame.

2. Preliminaries

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces and $\{\mathscr{H}_i\}_{i\in I}$ is a sequence of closed subspaces of $\mathscr{K}$, where I is a subset of Z and $L(\mathscr{H}, \mathscr{H}_i)$ is the collection of all bounded linear operators from $\mathscr{H}$ into $\mathscr{H}_i$. And we denote by $I_{\mathscr{H}}$ the identity operator on $\mathscr{H}$.

Definition 2.1. [2] *A sequence $\{f_i : i \in I\}$ of elements in $\mathscr{H}$ is called a frame for $\mathscr{H}$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|\mathbf{f}\|^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|\mathbf{f}\|^2, \quad \forall f \in \mathscr{H}. \tag{1}$$

The constants A and B are called lower and upper frame bounds.

Definition 2.2. [12] $\left(\sum_{i\in \mathrm{I}} \oplus \mathscr{H}_i\right)_{l^2}$ *is a Hilbert space and is defined by*

$$\left(\sum_{i\in \mathrm{I}} \oplus \mathscr{H}_i\right)_{l^2} = \left\{\{f_i\}_{i\in I} : f_i \in \mathscr{H}_i, i \in I, \left\|\{f_i\}_{i\in I}\right\|^2 = \sum_{i\in \mathrm{I}} \|f_i\|^2 < \infty\right\}.$$

with the inner product defined by: $\quad \langle \{f_i\}, \{g_i\}\rangle = \sum_{i\in \mathrm{I}} \langle f_i, g_i\rangle$.

Definition 2.3. [11] *A sequence $\{\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i) : i \in I\}$ of bounded operators is said to be a generalized frame or simply a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$ if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|\mathbf{f}\|^2 \leq \sum_{i\in \mathrm{I}} \|\mathbf{\Lambda_i f}\|^2 \leq B\|\mathbf{f}\|^2, \quad \forall f \in \mathscr{H}. \tag{2}$$

we call A and B the lower and upper g-frame bounds, respectively.
We call $\{\Lambda_i\}_{i\in I}$ a tight g-frame if $A = B$ and a Parseval g-frame or a normalized tight g-frame if $A = B = 1$.
We call $\{\Lambda_i : i \in I\}$ an exact g-frame if it ceases to be a g-frame whenever any one of its element is removed.
We call $\{\Lambda_i : i \in I\}$ a g-frame for $\mathscr{H}$ whenever $\mathscr{H}_i = \mathscr{H}, \forall i \in I$.

The synthesis(g-pre frame) operator of $\{\Lambda_i\}_{i\in I}; T_\Lambda : \left(\sum_{i\in \mathrm{I}} \oplus \mathscr{H}_i\right)_{l^2} \to \mathscr{H}$ is defined by

$$T_\Lambda(\{f_i\}_{i\in I}) = \sum_{i\in I} \Lambda_i^* f_i.$$

We call the adjoint T_Λ^* , where $T_\Lambda^* : \mathscr{H} \to \left(\sum_{i\in \mathrm{I}} \oplus \mathscr{H}_i\right)_{l^2}$, of the synthesis operator, the analysis operator which is given by

$$T_\Lambda^* f = \{\Lambda_i f\}_{i\in I}, \quad \forall f \in \mathscr{H}.$$

By composing T_Λ and T_Λ^*, we obtain the g-frame operator $S_\Lambda : \mathscr{H} \to \mathscr{H}$ given by

$$S_\Lambda f = T_\Lambda T_\Lambda^* f = \sum_{i\in I} \Lambda_i^* \Lambda_i f \tag{3}$$

which is a bounded, positive, self adjoint, invertible operator and satisfies $AI_{\mathscr{H}} \leq S_\Lambda \leq BI_{\mathscr{H}}$. Then the following reconstruction formula takes place for all $f \in \mathscr{H}$

$$f = S_\Lambda^{-1} S_\Lambda f = S_\Lambda S_\Lambda^{-1} f.$$

$\{\Lambda_i S_\Lambda^{-1}\}_{i\in I}$ is also a g-frame for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$ with bounds B^{-1} and A^{-1} and it is said to be the canonical dual g-frame of $\{\Lambda_i\}_{i\in I}$.

Definition 2.4. [7] *A g-frame $\{\Theta_i\}_{i\in I}$ of $\mathscr{H}$ is called an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$ if it satisfies*

$$f = \sum_{i\in I} \Lambda_i^* \Theta_i f, \qquad \forall f \in \mathscr{H} \tag{4}$$

It is easy to show that if $\{\Theta_i\}_{i\in I}$ is an alternate dual g-frame of $\{\Lambda_i\}_{i\in I}$, then $\{\Lambda_i\}_{i\in I}$ will be an alternate dual g-frame of $\{\Theta_i\}_{i\in I}$.

Definition 2.5. [11] *Let $\Lambda_i \in \mathscr{L}(\mathscr{H}, \mathscr{H}_i), i \in I$.*
(1) If the right hand inequality of (2) holds, then we say that $\{\Lambda_i : i \in I\}$ is a g-Bessel sequence for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i \in I\}$.
(2) If $\{f : \Lambda_i f = 0, i \in I\} = \{0\}$, then we say that $\{\Lambda_i : i \in I\}$ is g-complete.
(3) If $\{\Lambda_i : i \in I\}$ is g-complete and there are positive constants A and B such that for any finite subset $I_1 \subset I$ and $g_i \in \mathscr{H}_i, i \in I_1$,

$$A \sum_{i\in I_1} \|g_i\|^2 \leq \left\| \sum_{i\in I_1} \Lambda_i^* g_i \right\|^2 \leq B \sum_{i\in I_1} \|g_i\|^2 \tag{5}$$

then we say that $\{\Lambda_i : i \in I\}$ is a g-Riesz basis for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i \in I\}$.
(4) We say $\{\Lambda_i : i \in I\}$ is a g-orthonormal basis for $\mathscr{H}$ with respect to $\{\mathscr{H}_i : i \in I\}$ if it satisfies the following:
$\left\langle \Lambda_{i_1}^* g_{i_1}, \Lambda_{i_2}^* g_{i_2} \right\rangle = \delta_{i_1,i_2} \left\langle g_{i_1}, g_{i_2} \right\rangle, \quad \forall i_1, i_2 \in I, g_{i_1} \in \mathscr{H}_{i_1}, g_{i_2} \in \mathscr{H}_{i_2},$
$\sum_{i\in I} \|\Lambda_i f^2\| = \|f\|^2, \quad \forall f \in \mathscr{H}.$

Definition 2.6. [9] *We call two g-Bessel sequences* $\{\Lambda_i\}_{i\in I}$ *and* $\{\Theta_i\}_{i\in I}$ *to be orthogonal if*

$$\sum_{i\in I}\Lambda_i^*\Theta_i f = 0 \quad or \quad \sum_{i\in I}\Theta_i^*\Lambda_i f = 0, \qquad \forall f \in \mathscr{H}. \tag{6}$$

In terms of synthesis operators

$$T_\Lambda T_\Theta^* = 0 \quad or \quad T_\Theta T_\Lambda^* = 0. \tag{7}$$

where T_Λ *and* T_Θ *are the synthesis operators for* $\{\Lambda_i\}_{i\in I}$ *and* $\{\Theta_i\}_{i\in I}$ *respectively.*

Definition 2.7. [6] *Let* $\{\Xi_i\}_{i\in I}$ *and* $\{\Psi_i\}_{i\in I}$ *be g-orthonormal basis for* $\mathscr{H}$ *with respect to* $\{\mathscr{W}_i\}_{i\in I}$ *and* $\{\mathscr{H}_i\}_{i\in I}$*, respectively. Let* $\{\Lambda_i \in \mathscr{L}(\mathscr{H},\mathscr{H}_i) : i\in I\}$*be such that the series* $\sum_{i\in I}\Lambda_i^* g_i'$ *is convergent for all* $\{g_i'\}_{i\in I} \in \left(\sum_{i\in \mathrm{I}}\oplus\mathscr{H}_i\right)_{l^2}$.
The g-R-dual sequence for the sequence $\{\Lambda_i\}_{i\in I}$ *is* $\Gamma_j^\Lambda : \mathscr{H} \to W_j$ *which is defined as*
$\Gamma_j^\Lambda = \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i, \quad \forall j\in I.$

The following results which are referred to in this paper are listed in the form of lemmas.

Lemma 2.8. [10] *Let* $\{\Theta_i \in L(\mathscr{H},\mathscr{H}_i) : i\in I\}$ *be a g-orthonormal basis for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i : i\in I\}$ *and* $\{\Lambda_i \in L(\mathscr{H},\mathscr{H}_i) : i\in I\}$ *be a g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i : i\in I\}$*. Then there is a bounded and onto operator* $V : \mathscr{H} \to \mathscr{H}$ *such that* $\Lambda_i = \Theta_i V^*$*, for all* $i\in I$.

Lemma 2.9. [2] *Let* $T : K \to H$ *be a bounded surjective operator. Then there exists a bounded operator (called the pseudoinverse of T)* $T^\dagger : H \to K$ *for which* $TT^\dagger f = f, \quad \forall f \in \mathscr{H}$.

Lemma 2.10. [6] *Let* $\{\Lambda_i\}_{i\in I}$ *and* $\{\Omega_i\}_{i\in I}$ *be g-frames for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in I}$*. Then* $\{\Omega_i\}_{i\in I}$ *is a dual g-frame of* $\{\Lambda_i\}_{i\in I}$ *if and only if g-R-dual sequences* $\left\{\Gamma_j^\Lambda\right\}_{j\in I}$ *and* $\left\{\Gamma_j^\Omega\right\}_{j\in I}$ *are g-biorthogonal; that is,*
$\Gamma_i^\Lambda(\Gamma_j^\Omega)^* g_j = \Gamma_i^\Omega(\Gamma_j^\Lambda)^* g_j = \delta_{ij} g_j, \quad \forall i,j\in I, g_j \in W_j$

Lemma 2.11. [3] *Let* $\{\Lambda_i \in L(\mathscr{H},\mathscr{H}_i) : i\in I\}$ *be a g-frame for* $\mathscr{H}$ *with g-frame operator S and bounds A and B. Let L be a bounded linear operator on* $\mathscr{H}$*. Then* $\{\Lambda_i L\}_{i\in I} \subseteq (\mathscr{H},\mathscr{H}_i)$ *is a g-frame for* $\mathscr{H}$ *if and only if L is invertible on* $\mathscr{H}$*. Moreover, in this case, the g-frame operator for* $\{\Lambda_i L\}_{i\in I}$ *is* L^*SL *and new bounds are* $A\|L^{-1}\|^2$ *and* $B\|L\|^2$.

Lemma 2.12. [9] *Let* $\{\Lambda_i\}_{i\in I}$ *be a g-frame for* $\mathscr{H}$ *with respect to* $\{\mathscr{H}_i\}_{i\in I}$ *and* S_Λ

be the g-frame operator of $\{\Lambda_i\}_{i\in I}$. *Then* $\{\Omega_i\}_{i\in I}$ *is an alternate dual g-frame of* $\{\Lambda_i\}_{i\in I}$ *if and only if* $\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\}_{i\in I}$ *and* $\{\Lambda_i S_\Lambda^{-1} - \Omega_i\}_{i\in I}$ *are orthogonal.*

3. Main Result

Let $\{\Lambda_i : i \in I\}$ be a g-frame for $\mathscr{H}$ and $\{\Gamma_i : i \in I\}$ be a g-Bessel sequence. Here we discuss the condition under which this g-Bessel sequence is a dual of $\{\Lambda_i : i \in I\}$ when we already know one of the dual g-frame.

Theorem 3.1. *Let* $\{\Lambda_i : i \in I\}$ *be a g-frame for* $\mathscr{H}$ *with bounds* A, B *and frame operator* S_Λ *and* $\{\Omega_i : i \in I\}$*be a dual g-frame of* $\{\Lambda_i : i \in I\}$. *Let* $\{\Gamma_i : i \in I\}$ *be a g-Bessel sequence. Then* $\{\Gamma_i : i \in I\}$ *is also a dual g-frame of* $\{\Lambda_i : i \in I\}$ *if and only if* $\left\{\Lambda_i - \Lambda_i S_\Lambda^{-1} : i \in I\right\}$ *is orthogonal to* $\{\Gamma_i - \Omega_i : i \in I\}$.

Proof. Let $\{\Gamma_i : i \in I\}$ be a dual g-frame of $\{\Lambda_i : i \in I\}$. Then $\{\Gamma_i - \Omega_i : i \in I\}$ is a g-Bessel sequence. Now

$$\begin{aligned}
\sum_{i\in I}(\Lambda_i - \Lambda_i S_\Lambda^{-1})^*(\Gamma_i - \Omega_i)f &= \sum_{i\in I}(\Lambda_i^* - S_\Lambda^{-1}\Lambda_i^*)(\Gamma_i - \Omega_i)f \\
&= (\sum_{i\in I}\Lambda_i^*\Gamma_i - \sum_{i\in I}\Lambda_i^*\Omega_i - S_\Lambda^{-1}\sum_{i\in I}\Lambda_i^*\Gamma_i + S_\Lambda^{-1}\sum_{i\in I}\Lambda_i^*\Omega_i)f \\
&= (I_{\mathscr{H}} - I_{\mathscr{H}} - S_\Lambda^{-1} + S_\Lambda^{-1})f \\
&= 0
\end{aligned}$$

which implies that $\left\{\Lambda_i - \Lambda_i S_\Lambda^{-1} : i \in I\right\}$ is orthogonal to $\{\Gamma_i - \Omega_i : i \in I\}$. Conversely, let $\left\{\Lambda_i - \Lambda_i S_\Lambda^{-1} : i \in I\right\}$ be orthogonal to $\{\Gamma_i - \Omega_i : i \in I\}$. Then,

$$\begin{aligned}
\sum_{i\in I}\left(\Lambda_i - \Lambda_i S_\Lambda^{-1}\right)^*(\Gamma_i - \Omega_i)f &= 0 \\
\Rightarrow \sum_{i\in I}\left(\Lambda_i^* - S_\Lambda^{-1}\Lambda_i^*\right)(\Gamma_i - \Omega_i)f &= 0 \\
\Rightarrow \sum_{i\in I}\Lambda_i^*\Gamma_i - \sum_{i\in I}\Lambda_i^*\Omega_i - S_\Lambda^{-1}\sum_{i\in I}\Lambda_i^*\Gamma_i + S_\Lambda^{-1}\sum_{i\in I}\Lambda_i^*\Omega_i f &= 0 \\
\Rightarrow \sum_{i\in I}\Lambda_i^*\Gamma_i - I_{\mathscr{H}} - S_\Lambda^{-1}\sum_{i\in I}\Lambda_i^*\Gamma_i + S_\Lambda^{-1})f &= 0 \\
\Rightarrow \left(I_{\mathscr{H}} - S_\Lambda^{-1}\right)\sum_{i\in I}\Lambda_i^*\Gamma_i f - \left(I_{\mathscr{H}} - S_\Lambda^{-1}\right)f &= 0 \\
\Rightarrow \left(I_{\mathscr{H}} - S_\Lambda^{-1}\right)\left(\sum_{i\in I}\Lambda_i^*\Gamma_i - I_{\mathscr{H}}\right)f &= 0
\end{aligned}$$

By [9, Corollary 3.2] $\{\Lambda_i - \Lambda_i S_\Lambda^{-1} : i \in I\} = \left\{\Lambda_i(I_{\mathscr{H}} - S_\Lambda^{-1})\right\}_{i\in I}$ is a g-frame. Therefore by Lemma (2.11), $\left(I_{\mathscr{H}} - S_\Lambda^{-1}\right)$ is invertible and we have $\sum_{i\in I}\Lambda_i^*\Gamma_i = I_{\mathscr{H}}$. Which implies that $\{\Gamma_i : i \in I\}$ is a dual g-frame of $\{\Lambda_i : i \in I\}$

Now we give a condition under which the difference of the g-frame $\{\Lambda_i : i \in I\}$ and g-Bessel sequence $\{\Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.

Theorem 3.2. *Let $\{\Lambda_i : i \in I\}$ be a g-frame for Hilbert space $\mathscr{H}$ with bounds A, B. Let $\{\Gamma_i : i \in I\}$ be a g-Bessel sequence with synthesis operator T_Γ. If $2\left\|T_\Gamma\right\|^2 < A$, then $\{\Lambda_i - \Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.*

Proof. For any $f \in \mathscr{H}$, we have

$$\begin{aligned}\sum_{i\in I}\left\|(\Lambda_i - \Gamma_i)f\right\|^2 &\leq 2\left(\sum_{i\in I}\|\Lambda_i f\|^2 + \sum_{i\in I}\left\|\Gamma_i f\right\|^2\right)\\ &\leq 2\Big(B\|f\|^2 + \|T_\Gamma^* f\|^2\Big)\\ &\leq 2\Big(B + \|T_\Gamma\|^2\Big)\|f\|^2.\end{aligned}$$

Since $\sum\limits_{i\in I}\|\Lambda_i f\|^2 = \sum\limits_{i\in I}\|(\Lambda_i - \Gamma_i + \Gamma_i)f\|^2 \leq 2(\sum\limits_{i\in I}\|(\Lambda_i - \Gamma_i)f\|^2 + \sum\limits_{i\in I}\|\Gamma_i f\|^2)$, we have

$$\begin{aligned}\sum_{i\in I}\|(\Lambda_i - \Gamma_i)f\|^2 &\geq \frac{1}{2}\sum_{i\in I}\|\Lambda_i f\|^2 - \sum_{i\in I}\|\Gamma_i f\|^2\\ &\geq \left(\frac{A}{2}\|f\|^2 - \|T_\Gamma^* f\|^2\right)\\ &\geq \left(\frac{A}{2} - \|T_\Gamma\|^2\right)\|f\|^2\end{aligned}$$

If $(\frac{A}{2} - \|T_\Gamma\|^2\Big) > 0$ or $A > 2\|T_\Gamma\|^2$, then $\{\Lambda_i - \Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.

Here we give an alternate proof of Theorem 15 [7].

Theorem 3.3. *Let $\{\Lambda_i : i \in I\}$ be a g-frame for $\mathscr{H}$ and $\{\Theta_i : i \in I\}$ be a g-orthonormal basis for $\mathscr{H}$ with respect to $\{\mathscr{H}_i\}_{i\in I}$. Let g-preframe operator associated with $\{\Lambda_i : i \in I\}$ be T ; that is, $\Lambda_i = \Theta_i T^*$ for any $i \in I$. Then $\{\Omega_i : i \in I\}$ is a dual g-frame of $\{\Lambda_i : i \in I\}$ if and only if $\Omega_i = \Theta_i V^*$ for any $i \in I$, where V is a bounded left inverse of T^*.*

Proof. Let $\{\Xi_i : i \in I\}$ and $\{\Psi_i : i \in I\}$ be g-orthonormal bases for $\mathscr{H}$ with respect to $\{\mathscr{W}_j : j \in I\}$ and $\{\mathscr{V}_j : j \in I\}$. Let g-R-dual sequences of $\{\Lambda_i : i \in I\}$ and $\{\Omega_i : i \in I\}$ be denoted by Γ_j^Λ and Γ_j^Ω which are given by

$$\begin{aligned}\Gamma_j^\Lambda &= \sum_{i\in I}\Xi_j\Lambda_i^*\Psi_i \quad \forall j \in I\\ &= \sum_{i\in I}\Xi_j\left(\Theta_i T^*\right)^*\Psi_i\\ &= \sum_{i\in I}\Xi_j T\Theta_i^*\Psi_i\end{aligned}$$

and

$$\begin{aligned}\Gamma_j^\Omega &= \sum_{i\in I}\Xi_j\Omega_i^*\Psi_i \quad \forall j \in I\\ &= \sum_{i\in I}\Xi_j\left(\Theta_i V^*\right)^*\Psi_i = \sum_{i\in I}\Xi_j V\Theta_i^*\Psi_i\end{aligned}$$

First, let $TV^* = VT^* = I_{\mathscr{H}}$. For every $i, j \in I$ and $g_j \in \mathscr{W}_j$ we have,

$$\begin{aligned}
\Gamma_i^\Lambda \left(\Gamma_j^\Omega\right)^* g_j &= \sum_{k\in I} \Xi_i T\Theta_k^* \Psi_k \left(\sum_{m\in I} \Xi_j V\Theta_m^* \Psi_m\right)^* g_j \\
&= \sum_{k\in I}\sum_{m\in I} \Xi_i T\Theta_k^* \Psi_k \Psi_m^* \Theta_m V^* \Xi_j^* g_j \\
&= \sum_{k\in I} \Xi_i T\Theta_k^* \Theta_k V^* \Xi_j^* g_j \\
&= \Xi_i \Xi_j^* g_j \\
&= \delta_{i,j} g_j
\end{aligned}$$

therefore, by Lemma (2.10) $\{\Omega_i : i \in I\}$ is a dual g-frame of $\{\Lambda_i : i \in I\}$.
Next, let $\{\Omega_i : i \in I\}$ be a dual g-frame of $\{\Lambda_i : i \in I\}$. Therefore, by Lemma (2.12) $\left\{\Lambda_i - \Lambda_i S_\Lambda^{-1}\right\}_{i\in I}$ is orthogonal to $\left\{\Lambda_i S_\Lambda^{-1} - \Omega_i\right\}_{i\in I}$. We have

$$\begin{aligned}
\sum_{i\in I} (\Lambda_i - \Lambda_i S_\Lambda^{-1})^* (\Lambda_i S_\Lambda^{-1} - \Omega_i)) f &= 0 \\
\sum_{i\in I} \left(\Theta_i T^* - \Theta_i T^* S_\Lambda^{-1}\right)^* \left(\Theta_i T^* S_\Lambda^{-1} - \Theta_i V^*\right) f &= 0 \\
\sum_{i\in I} \left(T\Theta_i^* - S_\Lambda^{-1} T\Theta_i^*\right) \left(\Theta_i T^* S_\Lambda^{-1} - \Theta_i V^*\right) f &= 0 \\
\sum_{i\in I} \left(T\Theta_i^*\Theta_i T^* S_\Lambda^{-1} - T\Theta_i^*\Theta_i V^* - S_\Lambda^{-1} T\Theta_i^*\Theta_i T^* S_\Lambda^{-1} + S_\Lambda^{-1} T\Theta_i^*\Theta_i V^*\right) f &= 0.
\end{aligned}$$

Since $\{\Theta_i : i \in I\}$ is a g-orthonormal basis

$$\begin{aligned}
(TT^* S_\Lambda^{-1} - TV^* - S_\Lambda^{-1} TT^* S_\Lambda^{-1} + S_\Lambda^{-1} TV^*) f &= 0 \\
\left(I_{\mathscr{H}} - TV^* - S_\Lambda^{-1} + S_\Lambda^{-1} TV^*\right) f &= 0 \\
I_{\mathscr{H}} - TV^* - S_\Lambda^{-1} \left(I_{\mathscr{H}} - TV^*\right) f &= 0 \\
\left(I_{\mathscr{H}} - S_\Lambda^{-1}\right) \left(I_{\mathscr{H}} - TV^*\right) f &= 0
\end{aligned}$$

Since $\left(I_{\mathscr{H}} - S_\Lambda^{-1}\right)$ is invertible, we have

$$\begin{aligned}
(I_{\mathscr{H}} - TV^*) f = &\ 0 \\
\Rightarrow\ TV^* f = &\ f
\end{aligned}$$

Thus, $\{\Omega_i : i \in I\}$ is a dual g-frame of $\{\Lambda_i : i \in I\}$.

4. The Stability of g-frames

Let $\{\Lambda_i : i \in I\}$ be a g-frame for Hilbert space $\mathscr{H}$ and $\{\Gamma_i : i \in I\}$ be a sequence such that $\{\Lambda_i - \Gamma_i : i \in I\}$ is a g-Bessel sequence. We give a condition under which $\{\Gamma_i : i \in I\}$ is a g- frame.

Theorem 4.1. *Let $\{\Lambda_i : i \in I\}$ be a g-frame for Hilbert space $\mathscr{H}$ with bounds A, B*

and frame operator S_Λ. Let $\{\Gamma_i : i \in I\}$ be a sequence for $\mathscr{H}$.
If $D = \sum_{i\in I}\|\Lambda_i - \Gamma_i\|^2 \leq \Big(\frac{A^2}{2}\|S_\Lambda\|^{-1}\Big)$, then $\{\Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.

Proof. For any $f \in \mathscr{H}$, we have

$$\begin{aligned}\sum_{i\in I}\|\Gamma_i f\|^2 &= \sum_{i\in I}\|(\Gamma_i - \Lambda_i)f + \Lambda_i f\|^2 \\ &\leq 2(\sum_{i\in I}\|(\Gamma_i - \Lambda_i)f\|^2 + \sum_{i\in I}\|\Lambda_i f\|^2) \\ &\leq 2(\sum_{i\in I}\|(\Lambda_i - \Gamma_i)f\|^2 + \sum_{i\in I}\|\Lambda_i f\|^2) \\ &\leq 2(D+B)\|f\|^2\end{aligned}$$

As $B^{-1}I_{\mathscr{H}} \leq {S_\Lambda}^{-1} \leq A^{-1}I_{\mathscr{H}}$

$$\begin{aligned}\sum_{i\in I}\|\Lambda_i f\|^2 &= \sum_{i\in I}\|(\Lambda_i - \Gamma_i)f + \Gamma_i f\|^2 \\ &\leq 2(\sum_{i\in I}\|(\Lambda_i - \Gamma_i)f\|^2 + \sum_{i\in I}\|\Gamma_i f\|^2)\end{aligned}$$

which implies that

$$\begin{aligned}\sum_{i\in I}\|\Gamma_i f\|^2 &\geq \frac{1}{2}\sum_{i\in I}\|\Lambda_i f\|^2 - \sum_{i\in I}\|(\Lambda_i - \Gamma_i)f\|^2 \\ &\geq \frac{A}{2}\|f\|^2 - D\|f\|^2 \\ &\geq \Big(\frac{A^2}{2}\|S_\Lambda^{-1}\| - D\Big)\|f\|^2\end{aligned}$$

$\Big(\frac{A^2}{2}\|S_\Lambda^{-1}\| - D\Big) > 0 \Rightarrow D < \Big(\frac{A^2}{2}\|S_\Lambda\|^{-1}\Big)$

Thus $\{\Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$ if $D < \Big(\frac{A^2}{2}\|S_\Lambda\|^{-1}\Big)$.

Theorem 4.2. *Let $\{\Lambda_i : i \in I\}$ be a g-frame for Hilbert space $\mathscr{H}$ with bounds A and B. Let $\{\Gamma_i : i \in I\}$ be a sequence for $\mathscr{H}$. Then $\{\Gamma_i : i \in I\}$ is a g-Bessel sequence for $\mathscr{H}$ if and only if there exists a λ such that $\sum_{i\in I}\|(\Lambda_i + \Gamma_i)f\|^2 \leq \lambda\sum_{i\in I}\|\Lambda_i f\|^2$, $\forall f \in \mathscr{H}$. Moreover, if T_Λ is the synthesis operator of $\{\Lambda_i : i \in I\}$ and $\lambda < \frac{A\|T_\Lambda^\dagger\|^{-2}}{2B}$, then $\{\Gamma_i : i \in I\}$ is a g-frame for $\mathscr{H}$.*

Proof. Let $\sum_{i\in I}\|(\Lambda_i + \Gamma_i)f\|^2 \leq \lambda\sum_{i\in I}\|\Lambda_i f\|^2 \quad \forall f \in \mathscr{H}$. For any $f \in \mathscr{H}$, we have

$$\begin{aligned}\sum_{i\in I}\|\Gamma_i f\|^2 &= \sum_{i\in I}\|(\Lambda_i+\Gamma_i)f-\Lambda_i f\|^2\\ &\le 2\left(\sum_{i\in I}\|(\Lambda_i+\Gamma_i)f\|^2+\sum_{i\in I}\|\Lambda_i f\|^2\right)\\ &\le 2\left(\lambda\sum_{i\in I}\|\Lambda_i f\|^2+\sum_{i\in I}\|\Lambda_i f\|^2\right)\\ &\le 2(\lambda+1)\sum_{i\in I}\|\Lambda_i f\|^2\\ &\le 2B(\lambda+1)\|f\|^2\end{aligned}$$

This implies that $\{\Gamma_i : i \in I\}$ is a g-Bessel sequence for $\mathscr{H}$.
Conversely, let D be the bound of the g-Bessel sequence $\{\Gamma_i : i \in I\}$.

Since $A\|f\|^2 \le \sum_{i\in I}\|\Lambda_i f\|^2 \Rightarrow \|f\|^2 \le \dfrac{\sum_{i\in I}\|\Lambda_i f\|^2}{A}, \quad \forall f \in \mathscr{H}.$

Then for any $f \in \mathscr{H}$, we have
$\sum_{i\in I}\|\Gamma_i f\|^2 \le D\|f\|^2 \le \dfrac{D}{A}\sum_{i\in I}\|\Lambda_i f\|^2, \quad \forall f \in \mathscr{H}$. Therefore,

$$\begin{aligned}\sum_{i\in I}\|(\Lambda_i+\Gamma_i)f\|^2 &\le 2\Big(\sum_{i\in I}\|\Lambda_i f\|^2+\sum_{i\in I}\|\Gamma_i f\|^2\Big)\\ &\le 2\Big(1+\frac{D}{A}\Big)\sum_{i\in I}\|\Lambda_i f\|^2\\ &\le \lambda\sum_{i\in I}\|\Lambda_i f\|^2 \quad \text{where} \quad \lambda = 2\Big(1+\frac{D}{A}\Big).\end{aligned}$$

Moreover, let T_Λ be the synthesis operator of $\{\Lambda_i : i \in I\}$. Since T_Λ is onto, by Lemma (2.9), there exists an operator $T_\Lambda^\dagger$ such that $T_\Lambda T_\Lambda^\dagger = I_{\mathscr{H}}$ with
$B^{-1} \le \|T_\Lambda^\dagger\| \le A^{-1}$. Then for all $f \in \mathscr{H}$,

$$\begin{aligned}\|f\|^2 &= \|T_\Lambda T_\Lambda^\dagger f\|^2 \le \|T_\Lambda^\dagger\|^2\|T_\Lambda\|^2\|f\|^2\\ &= \|T_\Lambda^\dagger\|^2\sum_{i\in I}\|\Lambda_i f\|^2,\end{aligned}$$

thus $\sum_{i\in I}\|\Lambda_i f\|^2 \ge \|T_\Lambda^\dagger\|^{-2}\|f\|^2 \ge \dfrac{\|T_\Lambda^\dagger\|^{-2}}{B}\sum_{i\in I}\|\Lambda_i f\|^2$. Since

$$\begin{aligned}\sum_{i\in I}\|\Lambda_i f\|^2 &= \sum_{i\in I}\|\Lambda_i f+\Gamma_i f-\Gamma_i f\|^2\\ &\le 2\big(\sum_{i\in I}\|(\Lambda_i+\Gamma_i)f\|^2+\sum_{i\in I}\|\Gamma_i f\|^2\big)\end{aligned}$$

by hypothesis, we also have

$$\sum_{i\in I}\|\Gamma_i f\|^2 \ge \frac{1}{2}\Big(\sum_{i\in I}\|\Lambda_i f\|^2 - 2\sum_{i\in I}\|(\Lambda_i+\Gamma_i)f\|^2\Big)$$

$$\geq \frac{1}{2}\Big(\frac{\|T_\Lambda^\dagger\|^{-2}}{B} - 2\lambda\Big)\sum_{i\in I}\|\Lambda_i f\|^2 > 0.$$

Hence $\{\Gamma_i : i \in I\}$ is a g- frame for $\mathscr{H}$.

References

[1] Arefijamaal A. A., Ghasemi S., On characterization and stability of alternate duals of g-frames, Turkish J Math., 37 (1) (2013), 71-79.

[2] Christensen O. and Jensen T. K., An Introduction to the Theory of Bases, Frames, and Wavelets, Technical University of Denmark, 1999.

[3] Chugh Renu and Goel Shashank, On sums of G-frames in Hilbert spaces, Jordan Journal of Mathematics and Statistics (JJMS), 5 (2) (2012), 115-124.

[4] Daubechies I., Grossmann A., Meyer Y., Painless nonorthogonal expansions, J. Math Phys., (27) (1986), 1271-1283.

[5] Duffin R. J., Schaeffer A. C., A class of nonharmonic Fourier series, Trans Amer Math Soc., (72) (1952), 341-366.

[6] Enayati F. and Asgari M. S., Duality properties for Generalized frames, Banach J. Math. Anal., 11(4) (2017), 880-898.

[7] Guo X., Decompositions of g-Frames and Duals and Pseudoduals of g-Frames in Hilbert spaces, Journal of Function Spaces, Vol. 2015, Article ID 305961, 7 pages.

[8] Guo X., Perturbations of invertible operators and stability of g-frames in Hilbert spaces, Res Math., 64 (3-4) (2013), 405-421.

[9] Najati A., Faroughi M. H., Rahimi A., G-frames and stability of g-frames in Hilbert spaces, Methods of Functional Analysis and Topology, 14 (3) (2008), 271-286.

[10] Rajeswari K. N., George Neelam, On Alternate Duals of Generalized Frames, JJMS, 12 (4) (2019), 473-483.

[11] Sun W., G-frames and g-riesz bases, J. Math Anal Appl., 322 (1) (2006), 437-452.

[12] Sun W., Stability of g-frames, J. Math. Anal. Appl., 326 (2) (2007), 858-868.