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ON TRANSFORMATION FORMULAS OF ORDINARY HYPERGEOMETRIC SERIES

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Abstract: In this paper, certain transformation formulas have been established by using well-known Bailey's transform and some known summation formulas.

Keywords and Phrases: Bailey transform, ordinary hypergeometric series, summation formula, transformation formula.

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1. Introduction, Notations and Definitions

Hypergeometric series form a particularly powerful class of series, as they appear in a great variety of different scientific contexts while at the same time allowing a rather simple definition. Popularized by the work of Gauss, hypergeometric series have been intensively studied since the 19th century and they are still subject of ongoing research [1, 9]. Nowadays, they are also well understood from an algorithmic point of view, and in this paper, we will see some of the most important algorithms for dealing with them. The hypergeometric function takes a prominent

position amongst the world of standard mathematical functions used in both pure and applied mathematics. Many transformations and summations formulas and results recorded [2, 3, 4, 5, 10, 11] in ordinary as well as basic hypergeometric series with the help of Bailey transformation and certain Rogers-Ramanujan type identities. One can refer [6, 7, 8] and to get more transformation, identities and summation formulas with the help of Bailey transform.
Generalized ordinary hypergeometric function is defined as,

$$ {}_rF_s\left[\begin{array}{l} a_1, a_2, ..., a_r; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r)_n}{(b_1, b_2, ..., b_s)_n} \frac{z^n}{n!}, \tag{1.1} $$

where

$$ \begin{aligned} (a)_n &= a(a+1)...(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}, \\ (a)_0 &= 1. \end{aligned} $$

Also, $(a_1, a_2, ..., a_r)_n = (a_1)_n (a_2)_n ... (a_r)_n$.
For $r \leq s$, series (1.1) converges for all values of z i.e. in the region $|z| < \infty$ and for $r = s+1$, it converges in the unit disc $|z| < 1$. For $r > s+1$, (1.1) does not converge except at $z = 0$. On the other hand the truncated hypergeometric function is defined as,

$$ {}_rF_s\left[\begin{array}{l} a_1, a_2, ..., a_r; z \\ b_1, b_2, ..., b_s \end{array}\right]_n = \sum_{k=0}^{n} \frac{(a_1, a_2, ..., a_r)_k}{(b_1, b_2, ..., b_s)_k} \frac{z^k}{k!}. $$

The Bailey's transform was first stated explicity by W. N. Bailey [1] in 1994. It states as,
If

$$ \beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.2} $$

and

$$ \gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n}, \tag{1.3} $$

where α_r, δ_r, u_r and v_r are any functions of r only, such that the series γ_n exists, then

$$ \sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n. \tag{1.4} $$

We shall make use of following summation formulas in our analysis.

$$ {}_3F_2\left[\begin{matrix} a, b, -n; 1 \\ 1+a-b, 1+a+n \end{matrix}\right] = \frac{(1+a)_n \left(1+\frac{a}{2}-b\right)_n}{\left(1+\frac{a}{2}\right)_n (1+a-b)_n}. \tag{1.5} $$

[6; App. III(III.9), p. 243]

$$ {}_3F_2\left[\begin{matrix} a, 1+\frac{a}{2}, -n; 1 \\ \frac{a}{2}, b \end{matrix}\right] = \frac{(b-a-1-n)_n (b-a)_{n-1}}{(b)_n}. \tag{1.6} $$

[6; App. III(III.15), p. 244]

$$ {}_3F_2\left[\begin{matrix} a, b, -n; 1 \\ 1+a-b, 1+2b-n \end{matrix}\right] = \frac{(a-2b)_n \left(1+\frac{a}{2}-b\right)_n (-b)_n}{(1+a-b)_n \left(\frac{a}{2}-b\right)_n (-2b)_n}. \tag{1.7} $$

[6; App. III(III.16), p. 244]

$$ {}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, -n; 1 \\ \frac{a}{2}, 1+a-b, 1+2b-n \end{matrix}\right] = \frac{(a-2b)_n (-b)_n}{(1+a-b)_n (-2b)_n}. \tag{1.8} $$

[6; App. III(III.17), p. 244]

$$ {}_4F_3\left[\begin{matrix} a, 1+\frac{a}{2}, b, -n; 1 \\ \frac{a}{2}, 1+a-b, 2+2b-n \end{matrix}\right] = \frac{(a-2b-1)_n \left(\frac{1}{2}+\frac{a}{2}-b\right)_n (-b-1)_n}{(1+a-b)_n \left(\frac{a}{2}-\frac{1}{2}-b\right)_n (-2b-1)_n}. \tag{1.9} $$

[6; App. III(III.18), p. 244]

$$ {}_7F_6\left[\begin{matrix} a, 1+\frac{a}{2}, \frac{d}{2}, \frac{1}{2}+\frac{d}{2}, a-d, 1+2a-d+n, -n; 1 \\ \frac{a}{2}, 1+a-\frac{d}{2}, \frac{1}{2}+a-\frac{d}{2}, 1+d, d-a-n, 1+a+n \end{matrix}\right] = \frac{(1+a)_n (1+2a-2d)_n}{(1+a-d)_n (1+2a-d)_n}. \tag{1.10} $$

[6; App. III(III.19), p. 244]

$$ {}_4F_3\left[\begin{matrix} \frac{a}{2}, \frac{1}{2}+\frac{a}{2}, b+n, -n; 1 \\ \frac{b}{2}, \frac{1}{2}+\frac{b}{2}, 1+a \end{matrix}\right] = \frac{(b-a)_n}{(b)_n}. \tag{1.11} $$

[6; App. III(III.20), p. 245]

2. Main Results

In this section we establish following transformation formulas

(i)

$$\frac{\Gamma(1+a)\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\Gamma(1+a-\beta)}\ {}_4F_3\left[\begin{array}{l} a,b,\alpha,\beta;-1 \\ 1+a-b,1+a-\alpha,1+a-\beta \end{array}\right]$$

$$= {}_3F_2\left[\begin{array}{l} \alpha,\beta,1+\frac{a}{2}-\beta;1 \\ 1+\frac{a}{2},1+a-b \end{array}\right], \tag{2.1}$$

provided $Re(1+a-\alpha-\beta)>0$.

(ii)

$$e^z\ {}_2F_2\left[\begin{array}{l} a,1+\frac{a}{2};-z \\ \frac{a}{2},b \end{array}\right] = {}_2F_2\left[\begin{array}{l} b-a-1,2+a-b;z \\ b,1+a-b \end{array}\right]. \tag{2.2}$$

(iii)

$$(1-z)^{2b}\ {}_2F_1\left[\begin{array}{l} a,b;z \\ 1+a-b \end{array}\right] = {}_3F_2\left[\begin{array}{l} a-2b,1+\frac{a}{2}-b;z \\ 1+a-b,\frac{a}{2}-b \end{array}\right], \tag{2.3}$$

provided $|z|<1$.

(iv)

$$(1-z)^{2b}\ {}_3F_2\left[\begin{array}{l} a,1+\frac{a}{2},b;z \\ \frac{a}{2},1+a-b \end{array}\right] = {}_2F_1\left[\begin{array}{l} a-2b,-b;z \\ 1+a-b \end{array}\right], \tag{2.4}$$

where $|z|<1$.

(v)

$$(1-z)^{1+2b}\ {}_3F_2\left[\begin{array}{l} a,1+\frac{a}{2},b;z \\ \frac{a}{2},1+a-b \end{array}\right] = {}_3F_2\left[\begin{array}{l} a-2b-1,\frac{1}{2}+\frac{a}{2}-b,-b-1;z \\ 1+a-b,\frac{a}{2}-b-\frac{1}{2} \end{array}\right], \tag{2.5}$$

provided $|z|<1$.

(vi)

$${}_3F_2\left[\begin{array}{l} a,1+\frac{a}{2},a-d;1 \\ \frac{a}{2},1+d \end{array}\right] = 0 \tag{2.6}$$

(vii)

$${}_2F_1\left[\begin{array}{l} \frac{a}{2},\frac{1}{2}+\frac{a}{2};\frac{-4z}{(1-z)^2} \\ 1+a \end{array}\right] = (1-z)^a, \tag{2.7}$$

provided $|z|<1$.

Proof of (2.1).

(a) In order to prove (2.1) let us take $u_r = \dfrac{1}{(1)_r}$, $v_r = \dfrac{1}{(1+a)_r}$ and $\alpha_r = \dfrac{(a)_r(b)_r(-1)^r}{(1+a-b)_r r!}$ in (1.2) we get,

$$\beta_n = \frac{1}{n!(1+a)_n} \; {}_3F_2\left[\begin{array}{l} a, b, -n; 1 \\ 1+a-b, 1+a+n \end{array}\right],$$

which by a appeal of (1.5) yields

$$\beta_n = \frac{\left(1+\frac{a}{2}-b\right)_n}{n!\left(1+\frac{a}{2}\right)_n (1+a-b)_n}. \tag{2.8}$$

Again, taking $\delta_r = (\alpha)_r(\beta)_r$ in (1.3) we get,

$$\gamma_n = \sum_{r=0}^{\infty} \frac{(\alpha)_{r+n}(\beta)_n}{r!(1+a)_{r+2n}} = \frac{(\alpha)_n(\beta)_n}{(1+a)_{2n}} \; {}_2F_1\left[\begin{array}{l} \alpha+n, \beta+n; 1 \\ 1+a+2n \end{array}\right]. \tag{2.9}$$

Now, making use of the summation formula [6; App.III (III.3), p. 243] in (2.9) we have

$$\gamma_n = \frac{\Gamma(1+a)\Gamma(1+a-\alpha-\beta)}{\Gamma(1+a-\alpha)\Gamma(1+a-\beta)} \frac{(\alpha)_n(\beta)_n}{(1+a-\alpha)_n(1+a-\beta)_n}, \tag{2.10}$$

provided $Re(1+a-\alpha-\beta) > 0$.
Putting theses values in (1.4) we get (2.1) after some simplifications.

(b) In order to prove (2.2) let us choose $u_r = \dfrac{1}{(1)_r}$, $v_r = 1$ and $\alpha_r = \dfrac{(a)_r\left(1+\frac{a}{2}\right)_r(-1)^r}{\left(\frac{a}{2}\right)_r (b)_r r!}$ in (1.2) we find,

$$\beta_n = \frac{1}{n!} \; {}_3F_2\left[\begin{array}{l} a, 1+\frac{a}{2}, -n; 1 \\ \frac{a}{2}, b \end{array}\right]. \tag{2.11}$$

Summing the ${}_3F_2$ series in (2.11) by using (1.6) we get,

$$\beta_n = \frac{(2+a-b)_n (b-a-1)_n}{n!(b)_n (1+a-b)_n}. \tag{2.12}$$

Now, taking $\delta_r = z^r$ in (1.3) we get,

$$\gamma_n = z^n \sum_{r=0}^{\infty} \frac{z^r}{r!} = z^n e^z. \tag{2.13}$$

Putting these values in (1.4) we get (2.2) after some simplifications.

(c) In order to prove (2.3) let us take $u_r = \frac{(-2b)_r}{(1)_r}$, $v_r = 1$ and $\alpha_r = \frac{(a)_r\,(b)_r}{(1+a-b)_r\, r!}$ in (1.2) we get,

$$\beta_n = \frac{(-2b)_n}{n!}\ {}_3F_2\left[\begin{array}{l} a, b, -n; 1 \\ 1+a-b, 1+2b-n \end{array}\right]. \tag{2.14}$$

Now, summing the ${}_3F_2$ series by using (1.7) we get,

$$\beta_n = \frac{(a-2b)_n \left(1+\frac{a}{2}-b\right)_n (-b)_n}{(1+a-b)_n \left(\frac{a}{2}-b\right)_n n!}. \tag{2.15}$$

Again, choosing $\delta_r = z^r$ in (1.3) we get,

$$\gamma_n = \sum_{r=0}^{\infty} \frac{(-2b)_r z^{r+n}}{r!} = z^n (1-z)^{2b}. \tag{2.16}$$

Putting these values in (1.4) we get (2.3).

(d) In order to prove (2.4) let us proceed by taking $u_r = \frac{(-2b)_r}{(1)_r}$, $v_r = 1$ and $\alpha_r = \frac{(a)_r \left(1+\frac{a}{2}\right)_r (b)_r}{\left(\frac{a}{2}\right)_r (1+a-b)_r r!}$ in (1.2) we get,

$$\beta_n = \frac{(-2b)_n}{n!}\ {}_4F_3\left[\begin{array}{l} a, 1+\frac{a}{2}, b, -n; 1 \\ \frac{a}{2}, 1+a-b, 1+2b-n \end{array}\right],$$

which by an appeal of the summing formula (1.8) we get,

$$\beta_n = \frac{(a-2b)_n (-b)_n}{(1+a-b)_n n!}. \tag{2.17}$$

Again, choosing $\delta_r = z^r$ in (1.3) we get,

$$\gamma_n = z^n \sum_{r=0}^{\infty} \frac{(-2b)_r z^r}{r!} = z^n (1-z)^{2b}, \quad |z| < 1. \tag{2.18}$$

Putting these values in (1.4) we get (2.4).

(e) In order to prove (2.5) let us take $u_r = \dfrac{(-1-2b)_r}{(1)_r}$, $v_r = 1$ and $\alpha_r = \dfrac{(a)_r\left(1+\frac{a}{2}\right)_r(b)_r}{\left(\frac{a}{2}\right)_r(1+a-b)_r r!}$ in (1.2) we get,

$$\beta_n = \frac{(-1-2b)_n}{n!}\;{}_4F_3\left[\begin{array}{c} a, 1+\frac{a}{2}, b, -n; 1 \\ \frac{a}{2}, 1+a-b, 2+2b-n \end{array}\right]. \tag{2.19}$$

Now, using the summing formula (1.9) in (2.19) we obtain,

$$\beta_n = \frac{(a-2b-1)_n\left(\frac{1}{2}+\frac{a}{2}-b\right)_n(-1-b)_n}{(1+a-b)_n\left(\frac{a}{2}-\frac{1}{2}-b\right)_n n!}. \tag{2.20}$$

Again, taking $\delta_r = z^r$ in (1.3) we get,

$$\gamma_n = \sum_{r=0}^{\infty}\frac{(-1-2b)_r z^{n+r}}{r!} = z^n(1-z)^{1+2b}, \quad |z|<1. \tag{2.21}$$

Putting these values in (1.4) we get (2.5).

(f) In order to prove (2.6) let us choose $u_r = \dfrac{(1-d+a)_r}{(1)_r}$, $v_r = \dfrac{(1+2a-d)_r}{(1+a)_r}$ and $\alpha_r = \dfrac{(a)_r\left(1+\frac{a}{2}\right)_r\left(\frac{d}{2}\right)_r\left(\frac{1}{2}+\frac{d}{2}\right)_r(a-d)_r}{\left(\frac{a}{2}\right)_r\left(1+a-\frac{d}{2}\right)_r\left(\frac{1}{2}+a-\frac{d}{2}\right)_r(1+d)_r r!}$ in (1.2) we get,

$$\beta_n = \frac{(1+a-d)_n(1+2a-d)_n}{n!(1+a)_n}\times$$

$${}_7F_6\left[\begin{array}{c} a, 1+\frac{a}{2}, \frac{d}{2}, \frac{1}{2}+\frac{d}{2}, a-d, 1+2a-d+n, -n; 1 \\ \frac{a}{2}, 1+a-\frac{d}{2}, \frac{1}{2}+a-\frac{d}{2}, 1+d, 1-a-n, 1+a+n \end{array}\right]. \tag{2.22}$$

Making use of the summation formula (1.10) we get,

$$\beta_n = \frac{(1+2a-2d)_n}{n!}. \tag{2.23}$$

Now, taking $\delta_r = 1$ in (1.3) we have

$$\gamma_n = \sum_{r=0}^{\infty} \frac{(1+a-d)_r(1+2a-d)_{2n}(1+2a-d+2n)_r}{r!(1+a)_{2n}(1+a+2n)_r}$$

$$= \frac{(1+2a-d)_{2n}}{(1+a)_{2n}} \, {}_2F_1\left[\begin{array}{c} 1+a-d, 1+2a-d+2n; 1 \\ 1+a+2n \end{array}\right]$$

$$= \frac{(1+2a-d)_{2n}}{(1+a)_{2n}} \frac{\Gamma(1+a+2n)\Gamma(2d-2a-1)}{\Gamma(d+2n)\Gamma(d-a)}$$

$$= \frac{\Gamma(1+a)\Gamma(2d-2a-1)}{\Gamma(d)\Gamma(d-a)} \frac{(1+2a-d)_{2n}}{(d)_{2n}}$$

$$= \frac{\Gamma(1+a)\Gamma(2d-2a-1)}{\Gamma(d)\Gamma(d-a)} \frac{\left(\frac{1}{2}+a-\frac{d}{2}\right)_n \left(1+a-\frac{d}{2}\right)_n}{\left(\frac{d}{2}\right)_n \left(\frac{1}{2}+\frac{d}{2}\right)_n} \tag{2.24}$$

provided $Re(d-a) > \frac{1}{2}$.
Putting these values in (1.4) we get,

$$\frac{\Gamma(1+a)\Gamma(2d-2a-1)}{\Gamma(d)\Gamma(d-a)} \sum_{n=0}^{\infty} \frac{(a)_n \left(1+\frac{a}{2}\right)_n (a-d)_n}{\left(\frac{a}{2}\right)_n (1+d)_n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(1+2a-2d)_n}{n!} = 0.$$

Thus we get

$${}_3F_2\left[\begin{array}{c} a, 1+\frac{a}{2}, a-d; 1 \\ \frac{a}{2}, 1+d \end{array}\right] = 0, \tag{2.25}$$

provided $Re(d-a) > \frac{1}{2}$.

(g) In order to prove (2.7) let us take $u_r = \dfrac{1}{(1)_r}$, $v_r = (b)_r$ and

$\alpha_r = \dfrac{\left(\frac{a}{2}\right)_r \left(\frac{1}{2}+\frac{a}{2}\right)_r (-)^r}{\left(\frac{b}{2}\right)_r \left(\frac{1}{2}+\frac{b}{2}\right)_r (1+a)_r}$ in (1.2) we get,

$$\beta_n = \frac{(b)_n}{n!} \, {}_4F_3\left[\begin{array}{c} \frac{a}{2}, \frac{1}{2}+\frac{a}{2}, b+n, -n; 1 \\ \frac{b}{2}, \frac{1}{2}+\frac{b}{2}, 1+a \end{array}\right]. \tag{2.26}$$

Now, useing of the summation formula (1.11) we get,

$$\beta_n = \frac{(b-a)_n}{n!}. \tag{2.27}$$

Again, choosing $\delta_r = z^r$ in (1.3) we have

$$\gamma_n = z^n (b)_r \sum_{r=0}^{\infty} \frac{(b+2n)_r}{r!} z^r$$

$$= (b)_{2n} z^n (1-z)^{-b-2n}. \tag{2.28}$$

Putting these values in (1.4) we get (2.7) after some simplifications.
Making use of Bailey's transform and certain known summation formulas, many transformations formulas have been established for ordinary as well as basic hypergeometric series.

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