South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 39-52

DOI: 10.56827/SEAJMMS.2022.1802.4 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

FRACTIONAL INTEGRAL OF WHITTAKER k-FUNCTION AND ITS PROPERTIES

Savita Panwar and Prakriti Rai*

Department of Mathematics,
Amity Institute of Applied Sciences,
Amity University, Noida, Uttar Pradesh - 201313, INDIA

E-mail : savitapanwar1119@gmail.com

*Department of Mathematics,
Siddharth University, Kapilvastu, Uttar Pradesh, INDIA

E-mail : prakritirai.rai@gmail.com

(**Received:** *Dec. 10, 2021* **Accepted:** *Aug. 10, 2022* **Published:** *Aug. 30, 2022)*

Abstract: In this paper, we introduce a generalized form of Whittaker function with the help of generalized confluent k-hypergeometric function. We establish several interesting properties of the Whittaker k-function such as its integral representations, derivative, Laplace transform and Hankel transform. Further, we investigate the Riemann-Liouville fractional integral and k-Riemann-Liouville fractional integral of Whittaker k-function. Some intriguing particular cases of the main results are also mentioned.

Keywords and Phrases: k-Gamma function, k-Beta function, Confluent k-hypergeometric function, Whittaker function, Laplace transform, Hankel transform and Riemann-Liouville fractional integral.

2020 Mathematics Subject Classification: 33B15, 33C15, 26A33.

1. Introduction

Special Functions are remarkable for their significance and utilization in several domains, particularly in mathematical physics such as astronomy, string theory, statistics, and engineering sciences. Numerous authors considered the various generalizations of the most eminent Special functions like Beta function, Gamma function, and hypergeometric functions [1, 8, 20, 22, 25, 28]. Many functions, including the Legendre and Logarithm functions, are particular cases of hypergeometric functions, while some other functions (e.g., Bessel functions and the Exponential functions) are the limiting cases of hypergeometric functions. Whittaker function is also a solution of the modified form of a confluent hypergeometric differential equation. As a result, the generalization of hypergeometric functions has many applications in mathematics as well as in other fields. Whittaker functions have been generalized in a number of intriguing ways by numerous authors [2, 17]. In response to the frequency occurrence of expressions of the form $\alpha(\alpha+k)...(\alpha+(n-1)k)$ in a variety of contexts, particularly in the combinatorics of creation and in the computation of Feynman integrals (see [4, 5]).

Diaz and Pariguan introduced a new form of the Pochhammer symbol as the Pochhammer k-symbol and gamma k-function [6]. Successively, k-analogue of other functions like Beta k-function, hypergeometric k-function, zeta k-function, and Appell k-functions based on k-analogue of Pochhammer symbol was proposed and investigated [13, 15, 16, 27]. Kokologiannki [10] derived many inequalities and properties of beta k-function and gamma k-function. Further, Krasniqi [11] studied limiting behavior for the k-gamma and k-beta functions. Fractional calculus provides numerous prospects for applications across many fields of science and engineering, and it has emerged as an intriguing research topic. In recent years, attention to fractional operators and special functions has developed and a spectacular array of advances and generalizations have been made by many researchers (see [3, 8, 9, 24]). The Riemann-Liouville fractional integral has a significant role in fractional calculus, where the fractional derivatives are defined through fractional integrals [9, 12, 21]. By utilizing the k-gamma function, Mubeen et al. introduced the k-Riemann Liouville fractional integral [14, 18]. This paper is organized as follows. In the next section, we examine the properties of confluent k-hypergeometric functions. In section 3, we introduce the Whittaker k-function and discuss the main properties of the Whittaker k-function. In section 4, we develop the Laplace transforms and Hankel transform of the Whittaker k-function. In section 5, we investigate the fractional integrals of the Whittaker k-function. Section 6 presents a brief conclusion. We begin by recalling some important definitions.
Diaz et al. [6] defined the following k-generalized gamma function Γ_k given by:

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k}) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{k}}\,dt, \tag{1}$$

$$(\Re(\alpha) > 0 \; and \; k > 0).$$

In the same paper, they have also defined the k-beta function B_k as follows:

$$\mathrm{B}_k(\alpha, \beta) = \frac{1}{k} \int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}\, dt, \tag{2}$$

$$(k > 0\, , \Re(\alpha) > 0 \; and \; \Re(\beta) > 0).$$

Let $\alpha \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}^+$, the Pochhammer k-symbol is defined by (see [6])

$$(\alpha)_{n,k} = \alpha(\alpha + k)(\alpha + 2k)\ldots(\alpha + (n-1)k). \tag{3}$$

In particular, $(\alpha)_{0,k} := 1$. If $\alpha \in \mathbb{C}$ and $p, n \in \mathbb{N}^+$ then for $k \in \mathbb{R}^+$, we have

$$(\alpha)_{\mathrm{n},k} = \frac{\Gamma_k(\alpha + \mathrm{n}k)}{\Gamma_k(\alpha)}, \tag{4}$$

and

$$(\alpha)_{p+\mathrm{n},k} = (\alpha)_{p,k}(\alpha + pk)_{\mathrm{n},k}.$$

Diaz [6] introduced the following form of hypergeometric function:

$$\mathrm{F}(\alpha, k, \beta, s)(z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_{n,k_1}(\alpha_2)_{n,k_2}\cdots(\alpha_p)_{n,k_p}}{(\beta_1)_{n,s_1}(\beta_2)_{n,s_2}\cdots(\beta_q)_{n,s_q}} \frac{z^n}{n!}, \tag{5}$$

where $\alpha = (\alpha_1, \cdots, \alpha_p) \in \mathbb{C}^p$, $\beta = (\beta_1, \cdots, \beta_q) \in \mathbb{C}^q$, $k = (k_1, \cdots, k_p) \in (\mathbb{R}^+)^p$ and $s = (s_1, \cdots, s_q) \in (\mathbb{R}^+)^q$ such that $\beta_i \in \mathbb{C} \setminus s_i\mathbb{Z}^-$.
In 2012, Mubeen et. al [13] defined the confluent k-hypergeometric function ${}_1\mathrm{F}_{1,k}$ as follows:

$${}_1\mathrm{F}_{1,k}(\beta; \gamma; z) = \sum_{n=0}^{\infty} \frac{(\beta)_{n,k}}{(\gamma)_{n,k}} \frac{z^n}{n!}, \tag{6}$$

where $k \in \mathbb{R}^+, n \in \mathbb{N}$; $\beta, \gamma, z \in \mathbb{C}$ and γ is neither zero nor a negative integer and its integral representation is given by

$${}_1\mathrm{F}_{1,k}(\beta; \gamma; z) = \frac{\Gamma_k(\gamma)}{k\Gamma_k(\beta)\Gamma_k(\gamma - \beta)} \int_0^1 t^{\frac{\beta}{k}-1}(1-t)^{\frac{\gamma-\beta}{k}-1} e^{zt}\, dt, \tag{7}$$

$$(k > 0, \Re(\gamma) > \Re(\beta) > 0).$$

In other paper, Mubeen [15] gave the following k-analogue of Kummer's first formula:

$${}_1\mathrm{F}_{1,k}(\beta; \gamma; z) = e^z\, {}_1\mathrm{F}_{1,k}(\gamma - \beta; \gamma; -z). \tag{8}$$

In [23], the Riemann-Liouville fractional integral for a function f of order $-\mu$ is defined as follows:

$$\mathrm{D}_z^{\mu}\{f(z)\} = \frac{1}{\Gamma(-\mu)} \int_0^z f(t)(z-t)^{-\mu-1}\, dt, \tag{9}$$

where $\Re(\mu) < 0$.
In particular, for the case $n-1 < \Re(\mu) < n$ (n=1,2,...), (9) is written in the form:

$$\mathrm{D}_z^{\mu}\{f(z)\} = \frac{d^n}{dz^n}\left\{\frac{1}{\Gamma(-\mu+n)} \int_0^z f(t)(z-t)^{-\mu+n-1}\, dt\right\}. \tag{10}$$

The k-Riemann-Liouville fractional integral of order $-\mu$ is defined by (see [14, 18])

$${}_k\mathrm{D}_z^{\mu}\{f(z)\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^z f(t)(z-t)^{-\frac{\mu}{k}-1}\, dt, \tag{11}$$

where $\Re(\mu) < 0$ and $k \in \mathbb{R}^+$. If we set $k = 1$ in (11) then this k-Riemann-Liouville fractional integral of order $-\mu$ reduces to the Riemann-Liouville fractional integral given in (9). In particular, for the case, $n-1 < \Re(\mu) < n$ where n =1,2,3...

$${}_k\mathrm{D}_z^{\mu}\{f(z)\} = \frac{d^n}{dz^n}\left\{\frac{1}{k\Gamma_k(-\mu+nk)} \int_0^z f(t)(z-\mathrm{t})^{-\frac{\mu}{k}+n-1}\, dt\right\}. \tag{12}$$

The classical Whittaker function is defined as (see [25])

$$\mathrm{M}_{\lambda,\rho}(z) = z^{\rho+\frac{1}{2}} e^{-\frac{z}{2}} \phi(\rho-\lambda+\frac{1}{2}; 2\rho+1; z), \tag{13}$$

$$(\Re(\rho) > \frac{-1}{2}\ \mathit{and}\ \Re(\rho\pm\lambda) > \frac{-1}{2}),$$

where ϕ is the confluent hypergeometric function which is defined as (see [19]),

$${}_1\mathrm{F}_1(\beta;\gamma;z) = \phi(\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{(\beta)_n}{(\gamma)_n} \frac{z^n}{n!}, \tag{14}$$

$$(\beta \in \mathbb{C}, \gamma \in \mathbb{C} \setminus \mathbb{Z}_0^-; z \in \mathbb{C}).$$

2. Some Properties of Confluent k-hypergeometric Function

For the new confluent k-hypergeometric function, we have

$${}_1\mathrm{F}_{1,k}(\beta;\gamma;z) = \sum_{n=0}^{\infty} \frac{B_k(\beta+nk, \gamma-\beta)}{B_k(\beta, \gamma-\beta)} \frac{z^n}{n!}. \tag{15}$$

Theorem 2.1. *For confluent k-hypergeometric function, we have the following differential formula:*

$$\frac{d^m}{dz^m}({}_1\mathrm{F}_{1,k}(\beta;\gamma;z)) = \frac{(\beta)_{m,k}}{(\gamma)_{m,k}}\ {}_1\mathrm{F}_{1,k}(\beta+mk;\gamma+mk;z), \tag{16}$$

where $m \in \mathbb{N}$ and $k > 0$.

Proof. Using mathematical induction, for m=1 and from (15)

$$\frac{d}{dz}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \frac{d}{dz}\left\{\sum_{n=0}^{\infty}\frac{B_k(\beta+nk,\gamma-\beta)}{B_k(\beta,\gamma-\beta)}\frac{z^n}{n!}\right\},$$

replacing n by n+1 and after simplification, we get

$$\frac{d}{dz}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \sum_{n=0}^{\infty}\frac{B_k(\beta+(n+1)k,\gamma-\beta)}{B_k(\beta,\gamma-\beta)}\frac{z^n}{n!}.$$

Since

$$B_k(\beta,\gamma-\beta) = \frac{\gamma}{\beta}B_k(\beta+k,\gamma-\beta).$$

Using this, we get

$$\frac{d}{dz}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \sum_{n=0}^{\infty}\frac{\beta}{\gamma}\frac{B_k(\beta+nk+k,\gamma-\beta)}{B_k(\beta+k,\gamma-\beta)}\frac{z^n}{n!},$$

$$\frac{d}{dz}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \frac{\beta}{\gamma}{}_1\mathrm{F}_{1,k}(\beta+k;\gamma+k;z), \tag{17}$$

let it is true for m-1, then we have

$$\frac{d^{m-1}}{dz^{m-1}}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \frac{(\beta)_{m-1,k}}{(\gamma)_{m-1,k}}{}_1F_{1,k}(\beta+(m-1)k;\gamma+(m-1)k;z).$$

Now,

$$\frac{d^m}{dz^m}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \frac{(\beta)_{m-1,k}}{(\gamma)_{m-1,k}}\frac{d}{dz}\{{}_1\mathrm{F}_{1,k}(\beta+(m-1)k;\gamma+(m-1)k;z)\}.$$

Using (17), we get

$$\frac{d^m}{dz^m}\{{}_1\mathrm{F}_{1,k}(\beta;\gamma;z)\} = \frac{(\beta)_{m-1,k}(\beta+(m-1)k)}{(\gamma)_{m-1,k}(\gamma+(m-1)k)}\ {}_1\mathrm{F}_{1,k}(\beta+mk;\gamma+mk;z).$$

After simplification, we get the desired result.

3. Whittaker k-Function

The Whittaker k-function, denoted by $M_{\lambda,\rho,k}(z)$, for $k>0$ is defined as

$$M_{\lambda,\rho,k}(z) = z^{\rho+\frac{1}{2}}\, e^{\frac{-z}{2}}\ {}_1F_{1,k}(\rho-\lambda+\frac{1}{2};2\rho+1;z), \tag{18}$$

where $\Re(\rho) > -1/2$, $\Re(\rho\pm\lambda) > \frac{-1}{2}$ and ${}_1F_{1,k}$ is the confluent k-hypergeometric function given in (6).

Remark. *If we take k=1, then the Whittaker k-function given in (18) reduces to classical Whittaker function given in (13).*

3.1. Integral representations of Whittaker k-Function

Theorem 3.1. *Each of the following integral representations holds true:*

$$\begin{aligned} M_{\lambda,\rho,k}(z) &= \frac{z^{\rho+1/2}\, e^{\frac{-z}{2}}\ \Gamma_k(2\rho+1)}{k\Gamma_k(\rho-\lambda+1/2)\Gamma_k(\rho+\lambda+1/2)} \\ &\quad\times \int_0^1 t^{\frac{\rho-\lambda+1/2}{k}-1}(1-t)^{\frac{\rho+\lambda+1/2}{k}-1} e^{zt}\, dt, \end{aligned} \tag{19}$$

$$(k>0,\ \beta>\alpha, \Re(\rho) > \frac{-1}{2}\ \textit{and}\ \Re(\rho\pm\lambda) > \frac{-1}{2}),$$

$$\begin{aligned} M_{\lambda,\rho,k}(z) &= \frac{z^{\rho+1/2}\, e^{\frac{z}{2}}\ \Gamma_k(2\rho+1)}{k\Gamma_k(\rho-\lambda+1/2)\Gamma_k(\rho+\lambda+1/2)} \\ &\quad\times \int_0^1 u^{\frac{\rho+\lambda+1/2}{k}-1}(1-u)^{\frac{\rho-\lambda+1/2}{k}-1} e^{-zu}\, du, \end{aligned} \tag{20}$$

$$\begin{aligned} M_{\lambda,\rho,k}(z) &= \frac{(\beta-\alpha)^{1-\frac{2\rho+1}{k}} z^{\rho+1/2}\, e^{\frac{-z}{2}}\ \Gamma_k\,(2\rho+1)}{k\Gamma_k(\rho-\lambda+1/2)\Gamma_k(\rho+\lambda+1/2)} \\ &\times \int_\alpha^\beta (u-\alpha)^{\frac{\rho-\lambda+1/2}{k}-1}(\beta-u)^{\frac{\rho+\lambda+1/2}{k}-1} e^{z(\frac{u-\alpha}{\beta-\alpha})}\, du. \end{aligned} \tag{21}$$

Proof. Using the integral representation of ${}_1F_{1,k}$ given in (7) in the definition (18), we get (19). Now,if we put t= 1-u, t= $\frac{u-\alpha}{\beta-\alpha}$ in (19), we get (20) and (21) respectively.

Corollary. *For k=1, in equations (19), (20) and (21) respectively, we get the integral representations of Classical Whittaker functions as follows:*

$$M_{\lambda,\rho}(z) = \frac{z^{\rho+1/2}\, e^{\frac{-z}{2}}\, \Gamma(2\rho+1)}{\Gamma(\rho-\lambda+1/2)\Gamma(\rho+\lambda+1/2)} \times \int_0^1 t^{\rho-\lambda-1/2}(1-t)^{\rho+\lambda-1/2} e^{zt}\, dt, \tag{22}$$

$$M_{\lambda,\rho}(z) = \frac{z^{\rho+1/2}\, e^{\frac{z}{2}}\, \Gamma(2\rho+1)}{\Gamma(\rho-\lambda+1/2)\Gamma(\rho+\lambda+1/2)} \times \int_0^1 u^{\rho+\lambda-1/2}(1-u)^{\rho-\lambda-1/2} e^{-zu}\, du, \tag{23}$$

$$M_{\lambda,\rho}(z) = \frac{(\beta-\alpha)^{-2\rho} z^{\rho+1/2}\, e^{\frac{-z}{2}}\, \Gamma(2\rho+1)}{\Gamma(\rho-\lambda+1/2)\Gamma(\rho+\lambda+1/2)} \times \int_\alpha^\beta (u-\alpha)^{\rho-\lambda-1/2}(\beta-u)^{\rho+\lambda-1/2} e^{z(\frac{u-\alpha}{\beta-\alpha})}\, du, \tag{24}$$

Remark. *Using equation (7) in the above equation (20), we get*

$$M_{\lambda,\rho,k}(z) = z^{\rho+\frac{1}{2}}\, e^{\frac{z}{2}}\, {}_1F_{1,k}(\rho+\lambda+\frac{1}{2}; 2\rho+1; -z), \tag{25}$$

Thus, it is seen that generalized Whittaker k-function can also be expressed by equation (25).

Theorem 3.2. *For $k > 0$, the following relation holds true:*

$$M_{\lambda,\rho,k}(-z) = (-1)^{\rho+\frac{1}{2}} M_{-\lambda,\rho,k}(z), \tag{26}$$

$$(\Re(\rho) > \frac{-1}{2} \textit{and } \Re(\rho\pm\lambda) > \frac{-1}{2}).$$

Proof. Replacing z by -z in (18), we get

$$M_{\lambda,\rho,k}(-z) = (-z)^{\rho+\frac{1}{2}} e^{\frac{z}{2}}\, {}_1F_{1,k}(\rho-\lambda+\frac{1}{2}; 2\rho+1; -z),$$

using k-analogue of Kummer's first formula (8), we get

$$M_{\lambda,\rho,k}(-z) = (-1)^{\rho+\frac{1}{2}} (z)^{\rho+\frac{1}{2}}\, e^{-\frac{z}{2}}\, {}_1F_{1,k}(\rho+\lambda+\frac{1}{2}; 2\rho+1; z).$$

Again using (18), we get the desired result.

Corollary. *When k=1, equation (26) reduces to the standard transformation formula for* $M_{\lambda,\rho}(z)$ *(see* [26]*).*

$$M_{\lambda,\rho}(-z) = (-1)^{\rho+\frac{1}{2}} M_{-\lambda,\rho}(z), \tag{27}$$

3.2. Derivative of Whittaker k-Function

Theorem 3.3. *The following differential formula for Whittaker k-function holds true:*

$$\frac{d^n}{dz^n}[e^{\frac{z}{2}} z^{-\rho-\frac{1}{2}} \ M_{\lambda,\rho,k}(z)] = \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} e^{\frac{z}{2}} \ z^{-(\rho+\frac{nk}{2}+\frac{1}{2})} \ M_{\lambda-\frac{nk}{2},\rho+\frac{nk}{2},k}(z), \tag{28}$$

where $n \in \mathbb{N}$.

Proof. Using (18), we have

$$\frac{d^n}{dz^n}[e^{\frac{z}{2}} z^{-\rho-\frac{1}{2}} M_{\lambda,\rho,k}(z)] = \frac{d^n}{dz^n}[{}_1F_{1,k}(\rho-\lambda+\frac{1}{2}; 2\rho+1; z)],$$

using Theorem (2.1), we get

$$\frac{d^n}{dz^n}[e^{\frac{z}{2}} z^{-\rho-\frac{1}{2}} \ M_{\lambda,\rho,k}(z)] = \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} \ {}_1F_{1,k}(\rho-\lambda+nk+\frac{1}{2}; 2\rho+1+nk; z).$$

Again using (18), we get the desired result.

4. Integral transforms of Whittaker k-Function

4.1. Laplace transform of Whittaker k-Function

Theorem 4.1. *For* $k > 0$ *and* $\Re(\lambda \pm \rho) > \frac{-1}{2}$*, the following Laplace transform holds true:*

$$\mathcal{L}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{1}{s^{\rho+\frac{3}{2}}} \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)} \times \frac{\Gamma(\rho+n+3/2)}{s^n n!}. \tag{29}$$

Proof. By using the definition of Laplace transform and equation (18), we have

$$\mathcal{L}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \int_0^{\infty} e^{-sz} z^{\rho+1/2} \ {}_1F_{1,k}(\rho-\lambda+1/2; 2\rho+1; z) dz,$$

using (15), we get

$$= \int_0^\infty e^{-sz} z^{\rho+1/2} \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)} \frac{z^n}{n!}\, dz,$$

changing the order of integration and summation, we get

$$= \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)n!} \int_0^\infty e^{-sz} z^{\rho+n+1/2}\, dz,$$

using the definition of classical gamma function, we get

$$= \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)n!} \frac{\Gamma(\rho+n+3/2)}{s^{\rho+n+3/2}}.$$

After simplification, we get the desired result.

4.2. Hankel transform of Whittaker k-Function

Theorem 4.2. *For* $k > 0$, $\Re(\rho \pm \lambda) > \frac{-1}{2}$ *and* $\Re(\rho+\nu) > \frac{-5}{2}$, *the following Hankel transformation holds true:*

$$\begin{aligned} \int_0^\infty z\, M_{\lambda,\rho,k}(z)\, J_\nu(az)\, dz = &\frac{\Gamma(\rho+\nu+5/2)}{(a^2+1/4)^{\rho/2+5/4}} \\ &\times \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)n!} \\ &\times \frac{(\rho+\nu+5/2)_n}{(a^2+1/4)^{n/2}}\, P^{-\nu}_{\rho+n+3/2}\left\{\frac{1}{\sqrt{4a^2+1}}\right\}, \end{aligned} \tag{30}$$

where $P^{-\nu}_{\rho+n+3/2}(z)$ *Legendre function of first kind* [23].

Proof. By using the definition of Hankel transform and equation (18), we have

$$\begin{aligned} \int_0^\infty z\, M_{\lambda,\rho,k}(z)\, J_\nu(az) dz = &\int_0^\infty z^{\rho+3/2} e^{-z/2} \\ &\times {}_1F_{1,k}(\rho-\lambda+1/2; 2\rho+1; z) J_\nu(az)\, dz, \end{aligned}$$

using (15), we have

$$= \int_0^\infty z^{\rho+3/2} e^{-z/2} \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)} \frac{z^n}{n!} J_\nu(az)\, dz.$$

Changing the order of integration and summation and using the well known formula [7]

$$\int_0^\infty e^{-pt}t^\mu J_\nu(at)dt = \Gamma(\mu+\nu+1)\, r^{-\mu-1}P_\mu^{-\nu}(\frac{p}{r}),$$

$$(\Re(\mu+\nu) > -1 \textit{ and } r = \sqrt{p^2+a^2}),$$

we get

$$= \sum_{n=0}^{\infty} \frac{B_k(\rho-\lambda+1/2+nk, \rho+\lambda+1/2)}{B_k(\rho-\lambda+1/2, \rho+\lambda+1/2)n!} \frac{\Gamma(\rho+n+\nu+5/2)}{(a^2+1/4)^{\rho/2+n/2+5/4}} \times P_{\rho+n+3/2}^{-\nu}\left(\frac{1}{\sqrt{4a^2+1}}\right).$$

After simplification, we get the desired result.

5. Fractional Integral of Whittaker k-Function

5.1. Riemann-Liouville fractional integral of order $-\mu$ of Whittaker k-Function

Theorem 5.1. *For* $\Re(\mu) < 0, k \in \mathbb{R}^+$, $\Re(\rho \pm \lambda) > \frac{-1}{2}$, $\Re(\rho) > \frac{-1}{2}$ *and* $\Re(\rho+n) > \frac{-3}{2}$, *we have*

$$\mathrm{D}_z^\mu\{e^{\frac{z}{2}}M_{\lambda,\rho,k}(z)\} = \frac{z^{\rho-\mu+1/2}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+1/2)_{n,k}}{(2\rho+1)_{n,k}} \frac{z^n}{n!} B(\rho+n+3/2, -\mu) \tag{31}$$

Proof. Using the definition of Riemann-Liouville fractional integral given in (9)

$$\mathrm{D}_z^\mu\{e^{\frac{z}{2}}M_{\lambda,\rho,k}(z)\} = \frac{1}{\Gamma(-\mu)} \int_0^z t^{\rho+\frac{1}{2}}\, {}_1F_{1,k}(\rho-\lambda+\frac{1}{2}; 2\rho+1; t)(z-t)^{-\mu-1}\, dt,$$

using (6), we have

$$\mathrm{D}_z^\mu\{e^{\frac{z}{2}}M_{\lambda,\rho,k}(z)\} = \frac{1}{\Gamma(-\mu)} \int_0^z t^{\rho+\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} \frac{t^n}{n!}(z-t)^{-\mu-1}\, dt,$$

Changing the order of integration and summation (which is permissible under the conditions stated along with the theorem), we have

$$\mathrm{D}_z^\mu\{e^{\frac{z}{2}}M_{\lambda,\rho,k}(z)\} = \frac{1}{\Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}n!} \int_0^z t^{\rho+n+\frac{1}{2}}(z-t)^{-\mu-1}\, dt,$$

putting t=uz, we get

$$\mathrm{D}_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{z^{\rho-\mu+\frac{1}{2}}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} \frac{z^n}{n!} \int_0^1 u^{\rho+n+\frac{1}{2}} (1-u)^{-\mu-1}\, du.$$

Using the definition of classical beta function, we get the desired result.

5.2. Riemann-Liouville k-fractional integral of order $-\mu$ of Whittaker k-Function

Theorem 5.2. *For* $\Re(\mu) < 0, k \in \mathbb{R}^+$, $\Re(\rho \pm \lambda) > \frac{-1}{2}$, $\Re(\rho) > \frac{-1}{2}$ *and* $\Re(\rho + n) > \frac{-3}{2}$, *we have*

$$_k\mathrm{D}_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{z^{\rho+\frac{1}{2}-\frac{\mu}{k}}}{k\Gamma_k(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} \frac{z^n}{n!}\, B(\rho+n+\frac{3}{2}, \frac{-\mu}{k}). \tag{32}$$

Proof. Using the definition of k-fractional integral given in (11)

$$_k\mathrm{D}_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^{\rho+\frac{1}{2}}\, {}_1F_{1,k}(\rho-\lambda+\frac{1}{2}; 2\rho+1; t) \times (z-t)^{-\frac{\mu}{k}-1}\, dt,$$

using (6), we have

$$_k\mathrm{D}_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{1}{k\Gamma_k(-\mu)} \int_0^z t^{\rho+\frac{1}{2}} (z-t)^{-\frac{\mu}{k}-1} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} \frac{t^n}{n!}\, dt.$$

Changing the order of summation and integration (which is permissible under the conditions stated along with the theorem), we get

$$_k\mathrm{D}_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{1}{k\Gamma_k(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k} n!} \int_0^z t^{\rho+n+\frac{1}{2}} (z-t)^{-\frac{\mu}{k}-1}\, dt,$$

putting t=uz, we get

$$_k\mathrm{D}_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho,k}(z)\} = \frac{1}{k\Gamma_k(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_{n,k}}{(2\rho+1)_{n,k}} \frac{z^{-\frac{\mu}{k}+\rho+n+\frac{1}{2}}}{n!} \int_0^1 u^{\rho+n+\frac{1}{2}} (1-u)^{-\frac{\mu}{k}-1}\, du,$$

using the definition of classical beta function, we get the desired result.

Corollary. *For k=1 in Theorem (5.1) and (5.2), we get*

$$D_z^{\mu}\{e^{\frac{z}{2}} M_{\lambda,\rho}(z)\} = \frac{z^{\rho+\frac{1}{2}-\mu}}{\Gamma(-\mu)} \sum_{n=0}^{\infty} \frac{(\rho-\lambda+\frac{1}{2})_n}{(2\rho+1)_n} \frac{z^n}{n!} B(\rho+n+\frac{3}{2}, -\mu). \tag{33}$$

$$(\Re(\mu) < 0, \Re(\rho \pm \lambda) > \frac{-1}{2}, \Re(\rho) > \frac{-1}{2}; \Re(\rho+n) > \frac{-3}{2})$$

6. Conclusion

Whittaker functions are utilized in numerous different fields, including mathematical physics, holonomic systems, modelling hydrogen atoms, and more. Therefore, the generalization of Whittaker functions has applicability in some pertinent areas of engineering sciences and mathematical physics. In this paper, we define the k-analogue of Whittaker function by using the confluent k-hypergeometric function. We then discussed its some important properties, like integral representations, derivative and Integral transforms. By applying the theory of the Riemann-Liouville fractional integral to the Whittaker k-function, we obtain some results in terms of the Beta function.

Acknowledgement

The authors are grateful to the reviewers for their valuable comments and suggestions for improving the manuscript.

References

[1] Ali M., Ghayasuddin M., Khan W. A., Nisar K. S., A novel Kind of the multi-index Beta, Gauss, and confluent hypergeometric functions, J. Math. Computer Sci., 23 (2021), 145–154.

[2] Choi J., Ghayasuddin M., Khan N. U., Generalized Extended Whittaker function and its properties, Appl. Math. Sci. 9 (131) (2015), 6529-6541.

[3] Choi J., Agarwal P., Jain S., Certain Fractional Integral Operators and Extended Generalized Gauss Hypergeometric Functions, Kyungpook Math. J., 55 (2015), 695–703.

[4] D´ıaz R., Pariguan E., Quantum symmetric functions, Communications in Algebra, 6 (33) (2005), 1947–1978.

[5] Deligne P., Etingof P., Freed D., Jeffrey L., Kazhdan D., Morgan J., Morrison D., Witten E., Quantum fields and strings, A course for mathematicians, American Mathematical Society, 1999.

[6] Diaz R., Pariguan E., On hypergeometric functions and Pochhammer k-symbol, Divulg. Math., (15) (2007), 179-192.

[7] Erd'elyi A., Magnus W., Oberhettinger F. and Tricomi F.G., Table of Integral Transform, McGraw-Hill, New York, 1954.

[8] Khan O., Khan N., Choi J., Nisar K. S., A Type of Fractional Kinetic Equations Associated with the (p, q)-Extended τ-Hypergeometric and Confluent Hypergeometric Functions, Nonlinear Functional Analysis and Applications, 26 (2) (2021), 381-92.

[9] Kilbas A. A., Srivastava H. M. and Trujillo J. J., Theory and Applications of Fractional Differential Equations, Elsevier B. V., Amsterdam, Netherlands, 2006.

[10] Kokologiannaki C. G., Properties and inequalities of generalized k-gamma, beta and zeta functions, Int. Contemp. Math. Sci., 5 (2010), 653–660.

[11] Krasniqi V., A limit for the k-gamma and k-beta function, Int. Math. Forum., 5 (2010), 1613–1617.

[12] Miller K. S. and Ross B., An introduction to the fractional Calculus and Fractional and Fractional Differential Equations, Wiley, New York, 1993.

[13] Mubeen S., Habibullah G. M., An integral representation of some k-hypergeometric functions, Int. Math. Forum, 7 (4) (2012), 203-207.

[14] Mubeen S., Habibullah G. M., k-fractional integrals and application, Int. J. Contemp. Math. Sci., 7 (2012), 89-94.

[15] Mubeen S., k-analogue of Kummer's first formula, J. Inequal. Spec. Funct., 3 (3) (2012), 41-44.

[16] Mubeen S., Rehman A., A note on k-gamma function and pochhammer k-symbol, J. Informatic Math. Sci., (6) (2014), 93-107.

[17] Nagar D. K., Mor´an-V´asquez R. A., Gupta A. K., Properties of the extended Whittaker function, Progr. Appl. Math., 6 (2) (2013), 70-80.

[18] Rahman G., Nisar K. S., Mubeen S., On generalized k-fractional derivative operator, AIMS Math., (5) (2020), 1936-1945.

[19] Rainville, E. D., Special functions, Macmillan Company, New York, 1960, Reprinted by Chelsea Publishing Company, Bronx, New York, 1971

[20] Saboor A., Rahman G., Ali H., Nisar K. S., Abdeljawad T., Properties and Applications of a New Extended Gamma Function Involving Confluent Hypergeometric Function, Journal of Mathematics, 2021.

[21] Samko S. G., Kilbas A. A. and Marichev, Fractional Integrals and Derivatives, Theory and Applications, Gordan and Breach, Yverdon et alibi, 1993.

[22] Srivastava H. M., Tassaddiq A., Rahman G., Nisar K. S., Khan I., A new extension of the τ-Gauss hypergeometric function and its associated properties, Mathematics, 7(10) (2019), 996.

[23] Srivastava, H. M. and Manocha, H. L., A treatise on generating functions, Ellis Harwood Limited, Halsted Press: a divison of John Wiley and sons, New York, 1984.

[24] Srivastava, H. M.; Parmar, R. K., Chopra, P., A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms, 1 (2012), 238–258.

[25] Whittaker, E. T., An expression of certain known functions as generalized hypergeometric functions, Bull. Amer. Math. Soc., 10 (3) (1903), 125-134.

[26] Whittaker E. T. and Watson, G. N., A Course of Modern Analysis, Cambridge University Press, Cambridge, 1952.

[27] Yilmaz G. O., Aktas R., Tasdelen F., On some formulas for the k-analogue of Apell functions and generating relations via k-fractional derivative, Fractal Fract., 4 (2020), 48.

[28] Younis J., Verma A., Aydi H., Nisar K. S., Alsamir H., Recursion formulas for certain quadruple hypergeometric functions, Advances in Difference Equations, (1) (2021), 1-4.