South East Asian J. of Mathematics and Mathematical Sciences
Vol. 18, No. 2 (2022), pp. 13-26

DOI: 10.56827/SEAJMMS.2022.1802.2 **ISSN (Online): 2582-0850**

ISSN (Print): 0972-7752

ON SUMS OF BIVARIATE FIBONACCI POLYNOMIALS AND BIVARIATE LUCAS POLYNOMIALS

Yashwant K. Panwar, Akhlak Mansuri* and Jaya Bhandari**

Department of Mathematics,
Government Model College, Jhabua, INDIA

E-mail : yashwantpanwar@gmail.com

*Department of Mathematics,
Government Girls College, Mandsaur, INDIA

E-mail : akhlaakmansuri@gmail.com

**Department of Mathematics,
Mandsaur University, Mandsaur, INDIA

E-mail : jostwal.222@gmail.com

(**Received:** *Apr. 24, 2021* **Accepted:** *May 25, 2022* **Published:** *Aug. 30, 2022)*

Abstract: In this paper, we present the sum of $s+1$ consecutive member of Bivariate Fibonacci Polynomials and Bivariate Lucas Polynomials and related identities consisting even and odd terms. We present its two cross two matrix and find interesting properties such as nth power of the matrix. Also, we present the identity which generalizes Catlan's, Cassini's and d'Ocagne's identity. Binet's formula will employ to obtain the identities.

Keywords and Phrases: Bivariate Fibonacci Polynomials, Bivariate Lucas Polynomials, Binet's formula and two cross two matrix.

2020 Mathematics Subject Classification: 11B39.

1. Introduction

In [4, 5, 6], Catalani define generalized bivariate polynomials, from which specifying initial conditions the bivariate Fibonacci and Lucas polynomials are obtained

and derived many interesting identities. Also derive a collection of identities for bivariate Fibonacci and Lucas polynomials using essentially a matrix approach as well as properties of such polynomials when the variables x and y are replaced by polynomials.
For $n \geq 2$, the bivariate Fibonacci polynomials sequence is defined by

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y) \tag{1.1}$$

So, the first bivariate Fibonacci polynomials are

$$\{F_n(x,y)\} = \left\{0, 1, x, x^2 + y, x^3 + 2xy, x^4 + 3x^2y + y^2, \ldots\right\}$$

Binet's formula for the bivariate Fibonacci polynomials:

$$F_n(x,y) = \frac{\mathfrak{R}_1^n - \mathfrak{R}_2^n}{\mathfrak{R}_1 - \mathfrak{R}_2} \tag{1.2}$$

For $n \geq 2$, the bivariate Lucas polynomials sequence is defined by

$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y) \tag{1.3}$$

So, the first bivariate Lucas polynomials are

$$\{L_n(x,y)\} = \left\{2, x, x^2 + 2y, x^3 + 3xy, x^4 + 4x^2y + 2y^2, \ldots\right\}$$

Binet's formula for the bivariate Lucas polynomials:

$$L_n(x,y) = \mathfrak{R}_1^n + \Re_2^n \tag{1.4}$$

The characteristic equation of recurrence relation (1.1) and (1.3) is:

$$t^2 - xt - y = 0 \tag{1.5}$$

where $x \neq 0, y \neq 0, x^2 + 4y \neq 0$. This equation has two real roots: $\mathfrak{R}_1 = \frac{x+\sqrt{x^2+4y}}{2}$ and $\mathfrak{R}_2 = \frac{x-\sqrt{x^2+4y}}{2}$.
Note that: $\mathfrak{R}_1 + \mathfrak{R}_2 = x, \mathfrak{R}_1\mathfrak{R}_2 = -y, \mathfrak{R}_1 - \mathfrak{R}_2 = \sqrt{x^2+4y}$. Also, $F_{-n}(x,y) = \frac{-1}{(-y)^n}F_n(x,y)$ and $L_{-n}(x,y) = \frac{1}{(-y)^n}L_n(x,y)$.
Some of relations between bivariate Fibonacci and Lucas polynomials are as follows:

$$L_n(x,y) = F_{n+1}(x,y) + yF_{n-1}(x,y) \tag{1.6}$$

$$\left(x^2 + 4y\right) F_n(x,y) = L_{n+1}(x,y) + yL_{n-1}(x,y) \tag{1.7}$$

Most of the authors introduce and present many properties and identities of Bivariate Fibonacci and Bivariate Lucas polynomials [1, 2, 3, 11, 12, 13, 14, 22, 23, 25, 26]. In [15], Kim et al., present sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials and derive Fourier series expansions of functions associated with them. Also, express those sums of finite products in terms of Bernoulli polynomials and obtain some identities by using those expressions. In [16], Kim et al., present a new approach to the convolved Fibonacci numbers arising from the generating function of them and give some new and explicit identities for the convolved Fibonacci numbers. In [7, 9, 10], Z. Čerin defines many results based on for alternating sums, sums of product, on sums of squares of odd and even terms and properties of odd and even terms for Fibonacci and Lucas numbers. In [8], Z. Čerin defines explicit formulae for sums of products of a fixed number of consecutive generalized Fibonacci and Lucas numbers. In this paper, we present the sum of s+1 consecutive member of Bivariate Fibonacci Polynomials and Bivariate Lucas Polynomials and the same thing for even and for odd and their product and square. Also, we present its two cross two matrix and find interesting properties and we present the identity which generalizes Catlan's, Cassini's and d'Ocagne's identity. Binet's formula will be used to establish identities.

2. Result and Discussion

In this section, we prove some identities for sums of a finite number of consecutive terms of the Bivariate Fibonacci Polynomials. First, we find the formula for the $\sum_{k=0}^{s} F_{v+k}(x,y)$ and $\sum_{k=0}^{s} L_{v+k}(x,y)$ when $s \geq 0, v \geq 0$.

Proposition 2.1. *For $s \geq 0$ & $v \geq 0$ the following equality holds:*

$$(i) \quad \sum_{k=0}^{s} F_{v+k}(x,y) = \frac{\{L_{v+s+1}(x,y) - L_v(x,y)\} - (x-2)\{L_{v+s+1}(x,y) - L_v(x,y)\}}{2(x+y-1)} \tag{2.1}$$

$$(ii) \quad \sum_{k=0}^{s} L_{v+k}(x,y) = \frac{y\left\{L_{v-1}(x,y) - \sqrt{x^2+4y}F_{v+s}(x,y)\right\} - \{L_{v+s+1}(x,y) - L_v(x,y)\}}{1-(x+y)} \tag{2.2}$$

Proof. (i): By Binet's formula (1.2), we have

$$\sum_{k=0}^{s} F_{v+k}(x,y) = \sum_{k=0}^{s} \left(\frac{\mathfrak{R}_1^{v+k} - \mathfrak{R}_2^{v+k}}{\mathfrak{R}_1 - \mathfrak{R}_2} \right) = \frac{1}{\mathfrak{R}_1 - \mathfrak{R}_2} \left[\frac{\mathfrak{R}_1^{v+s+1} - \mathfrak{R}_1^{v}}{\mathfrak{R}_1 - 1} - \frac{\mathfrak{R}_2^{v+s+1} - \mathfrak{R}_2^{v}}{\mathfrak{R}_2 - 1} \right]$$

$$= \frac{2}{\Re_1 - \Re_2}\left[\frac{\Re_1^{v+s+1} - \Re_1^v}{\sqrt{x^2+4y}+(x-2)} + \frac{\Re_2^{v+s+1} - \Re_2^v}{\sqrt{x^2+4y}-(x-2)}\right]$$

$$= \frac{1}{2(x+y-1)}\left[\left(\Re_1^{v+s+1} + \Re_2^{v+s+1}\right) - (x-2)\left(\frac{\Re_1^{v+s+1} - \Re_2^{v+s+1}}{\Re_1 - \Re_2}\right)\right.$$

$$\left. - \left(\Re_1^v + \Re_2^v\right) + (x-2)\left(\frac{\Re_1^v - \Re_2^v}{\Re_1 - \Re_2}\right)\right]$$

$$= \frac{\{L_{v+s+1}(x,y) - L_v(x,y)\} - (x-2)\{F_{v+s+1}(x,y) - F_v(x,y)\}}{2(x+y-1)}$$

This completes the proof.

Proof. (ii): By Binet's formula (1.4), we have

$$\sum_{k=0}^{s} L_{v+k}(x,y) = \sum_{k=0}^{s}\left(\Re_1^{v+k} + \Re_2^{v+k}\right) = \left[\frac{\Re_1^{v+s+1} - \Re_1^v}{\Re_1 - 1} + \frac{\Re_2^{v+s+1} - \Re_2^v}{\Re_2 - 1}\right]$$

$$= \left[\frac{(\Re_2 - 1)\left(\Re_1^{v+s+1} - \Re_1^v\right) + (\Re_1 - 1)\left(\Re_2^{v+s+1} - \Re_2^v\right)}{(\Re_1 - 1)(\Re_2 - 1)}\right]$$

$$= \left[\frac{\Re_1^{v+m}(\Re_1\Re_2) - \Re_1^v\Re_2 - \Re_1^{v+m+1} + \Re_1^v + \Re_2^{v+m}(\Re_1\Re_2) - \Re_1\Re_2^v - \Re_2^{v+m+1} + \Re_2^v}{\Re_1\Re_2 - \Re_1 - \Re_2 + 1}\right]$$

$$= \frac{y\left\{L_{v-1}(x,y) - \sqrt{x^2+4y}F_{v+s}(x,y)\right\} - \{L_{v+s+1}(x,y) - L_v(x,y)\}}{1-(x+y)}$$

This completes the proof.

Proposition 2.2. *For $s \geq 0 \& v \geq 0$ the following equality holds:*

$$(i)\quad \sum_{k=0}^{s} F_{2v+2k}(x,y)$$

$$= \frac{xy\{F_{2v-1}(x,y) - F_{2v+2s+1}(x,y)\} + (y-1)\{F_{2v+2s+2}(x,y) - F_{2v}(x,y)\}}{y^2 - (x^2+2y) + 1} \quad (2.3)$$

$$(ii)\quad \sum_{k=0}^{s} L_{2v+2k}(x,y)$$

$$= \frac{-xy\{L_{2v+2s+1}(x,y) - L_{2v-1}(x,y)\} + (y-1)\{L_{2v+2s+2}(x,y) - L_{2v}(x,y)\}}{y^2 - (x^2+2y) + 1} \quad (2.4)$$

Proof. Using the Binet's formula (1.2) and (1.4), the proof is clear.

Proposition 2.3. *For $s \geq 0 \& v \geq 0$ the following equality holds:*

$$(i) \quad \sum_{k=0}^{s} \{F_{v+k+1}(x,y) + yF_{v+k-1}(x,y)\}$$

$$= \frac{y\left\{L_{v-1}(x,y) - \sqrt{x^2+4y}F_{v+s}(x,y)\right\} - \{L_{v+s+1}(x,y) - L_v(x,y)\}}{1-(x+y)} \quad (2.5)$$

$$(ii) \quad \sum_{k=0}^{s} \{L_{v+k+1}(x,y) + yL_{v+k-1}(x,y)\}$$

$$= \frac{(x^2+4y)}{1-(x+y)}\{L_{v+s+1}(x,y) - L_v(x,y)\} - (x-2)\{L_{v+s+1}(x,y) - L_v(x,y)\} \quad (2.6)$$

Proof. Using the Binet's formula (1.2) and (1.4), the proof is clear.

3. Alternating sums of Bivariate Fibonacci Polynomials and Bivariate Lucas Polynomials

Proposition 3.1. *For $s \geq 0$ & $v \geq 0$ the following equality holds:*

$$(i) \quad \sum_{k=0}^{s} (-1)^k F_{v+k}(x,y)$$

$$= \frac{(x+2)\{F_v(x,y) + (-1)^s F_{v+s+1}(x,y)\} - \{L_v(x,y) + (-1)^s L_{v+s+1}(x,y)\}}{2(1+x-y)} \quad (2.7)$$

$$(ii) \quad \sum_{k=0}^{s} (-1)^k L_{v+k}(x,y) = 2\left[(x+2)\{L_v(x,y)\right.$$

$$\left. + (-1)^s L_{v+s+1}(x,y)\} - \left(x^2+4y\right)\{F_v(x,y) + (-1)^s F_{v+s+1}(x,y)\}\right] \quad (2.8)$$

Proof. (i): By Binet's formula (1.2), we have

$$\sum_{k=0}^{s} (-1)^k F_{v+k}(x,y) = \sum_{k=0}^{s} (-1)^k \left(\frac{\Re_1^{v+k} - \Re_2^{v+k}}{\Re_1 - \Re_2}\right)$$

$$= \frac{1}{\Re_1 - \Re_2}\left[\frac{\Re_1^v + (-1)^s \Re_1^{v+s+1}}{\Re_1 + 1} - \frac{\Re_2^v + (-1)^s \Re_2^{v+s+1}}{\Re_2 + 1}\right]$$

$$= \frac{2}{\Re_1 - \Re_2}\left[\frac{\Re_1^v + (-1)^s \Re_1^{v+s+1}}{(x+2) + \sqrt{x^2+4y}} - \frac{\Re_2^v + (-1)^s \Re_2^{v+s+1}}{(x+2) - \sqrt{x^2+4y}}\right]$$

$$= \frac{2}{\Re_1 - \Re_2} \left[\frac{\begin{matrix} (x+2)\left(\Re_1^v - \Re_2^v\right) + (-1)^{s+1}(x+2)\left(\Re_1^{v+s+1} - \Re_2^{v+s+1}\right) \\ -\sqrt{x^2+4y}\left(\Re_1^v + \Re_2^v\right) - (-1)^s\sqrt{x^2+4y}\left(\Re_1^{v+s+1} + \Re_2^{v+s+1}\right) \end{matrix}}{4(1+x-y)} \right]$$

$$= \frac{(x+2)\left\{F_v(x,y) + (-1)^s F_{v+s+1}(x,y)\right\} - \left\{L_v(x,y) + (-1)^s L_{v+s+1}(x,y)\right\}}{2(1+x-y)}$$

This completes the proof.

Proof. (ii): By Binet's formula (1.4), we have

$$\sum_{k=0}^{s}(-1)^k L_{v+k}(x,y) = \sum_{k=0}^{s}(-1)^k \left(\Re_1^{v+k} + \Re_2^{v+k}\right)$$
$$= \left[\frac{\Re_1^v + (-1)^s \Re_1^{v+s+1}}{\Re_1 + 1} + \frac{\Re_2^v + (-1)^s \Re_2^{v+s+1}}{\Re_2 + 1}\right]$$
$$= 2\left[(x+2)\left(\Re_1^v + \Re_2^v\right) + (x+2)(-1)^s\left(\Re_1^{v+s+1} + \Re_2^{v+s+1}\right)\right.$$
$$= 2\left[(x+2)\left\{L_v(x,y) + (-1)^s L_{v+s+1}(x,y)\right\}\right.$$
$$- \left(x^2 + 4y\left\{\left(\Re_1^v - \Re_2^v\right) - (-1)^s\left(\Re_1^{v+s+1} - \Re_2^{v+s+1}\right)\right\}\right]$$
$$= \left[F_v(x,y) + (-1)^s F_{v+s+1}(x,y)\right\}\right]$$

This completes the proof.

Proposition 3.2. *For $s \geq 0 \& v \geq 0$ the following equality holds:*

$$(i) \quad \sum_{k=0}^{s}(-1)^k F_{2v+2k}(x,y)$$
$$= \frac{xy\left\{F_{2v+2s+1}(x,y) - F_{2v-1}(x,y)\right\} + (y+1)\left\{F_{2v}(x,y) - F_{2v+2s+2}(x,y)\right\}}{x^2+y^2+2y+1} \tag{2.9}$$

$$(ii) \quad \sum_{k=0}^{s}(-1)^k L_{2v+2k}(x,y)$$
$$= \frac{xy\left\{L_{2v+2s+1}(x,y) - L_{2v-1}(x,y)\right\} + (y+1)\left\{F_{2v}(x,y) - F_{2v+2s+2}(x,y)\right\}}{x^2+y^2+2y+1} \tag{2.10}$$

Proof. Using the Binet's formula (1.2) and (1.4), the proof is clear.

4. Product and Square of adjacent Bivariate Fibonacci polynomials and Bivariate Lucas Polynomials

Proposition 4.1. *For $s \geq 0 \& v \geq 0$ the following equality holds:*

$$(i)\quad \sum_{k=0}^{s} F_{v+k}(x,y)F_{v+k+1}(x,y) = \frac{1}{x^2+4y}$$
$$\left[\frac{xy\left\{L_{2v}(x,y) - L_{2v+2s+2}(x,y)\right\} + (y-1)\left\{L_{2v+2s+3}(x,y) - L_{2v+1}(x,y)\right\}}{y^2-x^2-2y+1}\right.$$
$$\left. -x\left\{\frac{(-y)^{v+s+1} - (-y)^v}{y+1}\right\}\right] \tag{2.11}$$

$$(ii)\quad \sum_{k=0}^{s} L_{v+k}(x,y)L_{v+k+1}(x,y)$$
$$= \left[\frac{xy\{L_{2v}(x,y)-L_{2v+2s+2}(x,y)\}+(y-1)\{L_{2v+2s+3}(x,y)-L_{2v+1}(x,y)\}}{y^2-x^2-2y+1} - x\left\{\frac{(-y)^{v+s+1}-(-y)^v}{y+1}\right\}\right] \tag{2.12}$$

Proof. By Binet's formula (1.2) and (1.4), the proof is clear.

Proposition 4.2. *For $s \geq 0$ & $v \geq 0$ the following equality holds:*

$$(i)\quad \sum_{k=0}^{s} F^2_{v+k}(x,y) = \frac{1}{x^2+4y}$$
$$\left[\frac{xy\left\{L_{2v-1}(x,y) - L_{2v+2s+1}(x,y)\right\} + (y-1)\left\{L_{2v+2s+2}(x,y) - L_{2v}(x,y)\right\}}{y^2-x^2-2y+1}\right]$$
$$\frac{1}{y+2}\left[\left[\frac{(-y)^{v+s+1} - (-y)^v}{y+1}\right\}\right] \tag{2.13}$$

$$(ii)\quad \sum_{k=0}^{s} L^2_{v+k}(x,y)$$
$$= \frac{xy\left\{L_{2v-1}(x,y) - L_{2v+2s+1}(x,y)\right\} + (y-1)\left\{L_{2v+2s+2}(x,y) - L_{2v}(x,y)\right\}}{y^2-x^2-2y+1}$$
$$-2\left\{\frac{(-y)^{v+s+1} - (-y)^v}{y+1}\right\} \tag{2.14}$$

Proof. By Binet's formula (1.2) and (1.4), the proof is clear.

5. Matrix Representation of Bivariate Fibonacci polynomials and Bivariate Lucas Polynomials

In [17], Özkan and Altun, find elements of the Lucas polynomials by using two matrices and extend the study to the n-step Lucas polynomials. In [18] Özkan and Taştan, define the new families of Gauss k-Jacobsthal numbers and Gauss k-Jacobsthal-Lucas numbers and obtain some exciting properties of the families. Also, find the new generalizations of these families and the polynomials in matrix representation. In [19], Özkan and Taştan, define a new family of Gauss k-Lucas numbers and find new generalizations of these families and the polynomials in matrix representation. In [20], Özkan and Taştan, define the Gauss Fibonacci polynomials and define the matrices of the Gauss Fibonacci polynomials and the Gauss Lucas polynomials. Also, examine properties of the matrices. In [21], Taştan, Özkan and Shannon define new families of Generalized Fibonacci polynomials and Generalized Lucas polynomials and develop some elegant properties of these families and also, find new generalizations of these families and the polynomials in matrix representation. In this section, we present two cross two matrix for bivariate Fibonacci polynomials and bivariate Lucas Polynomials is given by $\mathrm{A} = \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix}$.

Theorem 5.1. *For $n \in \mathrm{N}$ we have*

$$\begin{bmatrix} F_{n+1}(x,y) \\ yF_n(x,y) \end{bmatrix} = \mathrm{A} \begin{bmatrix} F_n(x,y) \\ yF_{n-1}(x,y) \end{bmatrix} \tag{3.1}$$

Proof. To prove the result we will use induction on n. (3.1) is true for $n = 1$. Suppose (3.1) is true for n, we get

$$\begin{aligned}
\begin{bmatrix} F_{n+2}(x,y) \\ yF_{n+1}(x,y) \end{bmatrix} &= \begin{bmatrix} xF_{n+1}(x,y) + yF_n(x,y) \\ yF_{n+1}(x,y) \end{bmatrix} \\
&= \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} F_{n+1}(x,y) \\ yF_n(x,y) \end{bmatrix} \\
&= \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} F_n(x,y) \\ yF_{n-1}(x,y) \end{bmatrix} \\
&= \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} xF_n(x,y) + yF_{n-1}(x,y) \\ yF_n(x,y) \end{bmatrix} \\
&= \begin{bmatrix} x & 1 \\ y & 0 \end{bmatrix} \begin{bmatrix} F_{n+1}(x,y) \\ yF_n(x,y) \end{bmatrix} \\
&= \mathrm{A} \begin{bmatrix} F_{n+1}(x,y) \\ yF_n(x,y) \end{bmatrix}
\end{aligned}$$

Theorem 5.2. *For $n \in \mathrm{N}$ we have*

$$\begin{bmatrix} L_{n+1}(x,y) \\ yL_n(x,y) \end{bmatrix} = \mathrm{A} \begin{bmatrix} L_n(x,y) \\ yL_{n-1}(x,y) \end{bmatrix} \tag{3.2}$$

Theorem 5.3. *For $n \in N$ we have*

$$\begin{bmatrix} F_{n+1}(x,y) \\ yF_n(x,y) \end{bmatrix} = A^n \begin{bmatrix} F_1(x,y) \\ yF_0(x,y) \end{bmatrix} \tag{3.3}$$

Theorem 5.4. *For $n \in \mathrm{N}$ we have*

$$\begin{bmatrix} L_{n+1}(x,y) \\ yL_n(x,y) \end{bmatrix} = \mathrm{A}^n \begin{bmatrix} L_1(x,y) \\ yL_0(x,y) \end{bmatrix} \tag{3.4}$$

6. Generalized identity of Bivariate Fibonacci polynomials and Bivariate Lucas Polynomials

In this section, we present some generalized identities for bivariate Fibonacci polynomials and bivariate Lucas Polynomials, from which we obtain Catlan's identity, Cassini's identity and d'Ocagne's identity.

Proposition 6.1. (Generalized identity) For $n > m \geq k \geq 1$,

$$F_m(x,y)F_n(x,y) - F_{m-k}(x,y)F_{n+k}(x,y) = (-y)^{m-k}F_k(x,y)F_{n-m+k}(x,y) \tag{4.1}$$

Proof. By Binet's formula (1.2), we have

$$\begin{aligned}
&F_m(x,y)F_n(x,y) - F_{m-k}(x,y)F_{n+k}(x,y) \\
&= \left(\frac{\mathfrak{R}_1^m - \mathfrak{R}_2^m}{\mathfrak{R}_1 - \mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^n - \mathfrak{R}_2^n}{\mathfrak{R}_1 - \mathfrak{R}_2}\right) - \left(\frac{\mathfrak{R}_1^{m-k} - \mathfrak{R}_2^{m-k}}{\mathfrak{R}_1 - \mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{n+k} -}{\mathfrak{R}_1 -}\mathfrak{R}_1\right) \\
&= \frac{\mathfrak{R}_1^m\mathfrak{R}_2^n\left(\mathfrak{R}_1^{-k}\mathfrak{R}_2^k - 1\right) + \mathfrak{R}_1^n\mathfrak{R}_2^m\left(\mathfrak{R}_1^k\mathfrak{R}_2^{-k} - 1\right)}{\left(\mathfrak{R}_1 - \mathfrak{R}_2\right)^2} \\
&= \frac{(-y)^m\left(\mathfrak{R}_1^k - \mathfrak{R}_2^k\right)}{\left(\mathfrak{R}_1 - \mathfrak{R}_2\right)^2}\left(\frac{\mathfrak{R}_1^{n-m}}{\mathfrak{R}_2^k} - \frac{\mathfrak{R}_2^{n-m}}{\mathfrak{R}_1^k}\right) \\
&= (-y)^{m-k}\left(\frac{\mathfrak{R}_1^k - \mathfrak{R}_2^k}{\mathfrak{R}_1 - \mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{n-m+k} - \mathfrak{R}_2^{n-m+k}}{\mathfrak{R}_1 - \mathfrak{R}_2}\right) \\
&= (-y)^{m-k}F_k(x,y)F_{n-m+k}(x,y)
\end{aligned}$$

This completes the proof.

Corollary 6.2. (Catlan's identity). *If $m = n$ in the generalized identity (4.1), we obtain,*

$$F_n^2(x,y) - F_{n-k}(x,y)F_{n+k}(x,y) = (-y)^{n-k}F_k^2(x,y) \tag{4.2}$$

Corollary 6.3. (Cassini's identity). *If $m = n$ and $k = 1$ in the generalized identity (4.1), we obtain,*

$$F_n^2(x,y) - F_{n-1}(x,y)F_{n+1}(x,y) = (-y)^{n-1} \tag{4.3}$$

Corollary 6.4. (d'Ocagne's identity). *If $n = m, m = n+1$ and $k = 1$ in the generalized identity (4.1), we obtain,*

$$F_m(x,y)F_{n+1}(x,y) - F_n(x,y)F_{m+1}(x,y) = (-y)^n F_{m-n}(x,y) \tag{4.4}$$

Proposition 6.5. *(Generalized identity) For $n > m \geq k \geq 1$,*

$$L_m(x,y)L_n(x,y) - L_{m-k}(x,y)L_{n+k}(x,y) = -\left(x^2+4y\right)(-y)^{m-k}F_k(x,y)F_{n-m+k}(x,y) \tag{4.5}$$

Proof. By Binet's formula (1.4), we have

$$\begin{aligned}
&L_m(x,y)L_n(x,y) - L_{m-k}(x,y)L_{n+k}(x,y)\\
&= \left(\Re_1^m + \Re_2^m\right)\left(\Re_1^n + \Re_2^n\right) - \left(\Re_1^{m-k} + \Re_2^{m-k}\right)\left(\Re_1^{n+k} + \Re_2^{n+k}\right)\\
&= \Re_1^m\Re_2^n\left(1 - \Re_1^{-k}\Re_2^k\right) + \Re_1^n\Re_2^m\left(1 - \Re_1^k\Re_2^{-k}\right)\\
&= -(-y)^{m-k}\left(\Re_1^k - \Re_2^k\right)\left(\Re_1^{n-m+k} - \Re_2^{n-m+k}\right)\\
&= -\left(x^2+4y\right)(-y)^{m-k}\left(\frac{\Re_1^k - \Re_2^k}{\Re_1 - \Re_2}\right)\left(\frac{\Re_1^{n-m+k} - \Re_2^{n-m+k}}{\Re_1 - \Re_2}\right)\\
&= -\left(x^2+4y\right)(-y)^{m-k}F_k(x,y)F_{n-m+k}(x,y)
\end{aligned}$$

This completes the proof.

Corollary 6.6. (Catlan's identity). *If $m = n$ in the generalized identity (4.5), we obtain,*

$$L_n^2(x,y) - L_{n-k}(x,y)L_{n+k}(x,y) = -\left(x^2+4y\right)(-y)^{n-k}F_k^2(x,y) \tag{4.6}$$

Corollary 6.7. (Cassini's identity). *If $m = n$ and $k = 1$ in the generalized identity (4.5), we obtain,*

$$L_n^2(x,y) - L_{n-1}(x,y)L_{n+1}(x,y) = -\left(x^2+4y\right)(-y)^{n-1} \tag{4.7}$$

Corollary 6.8. (d'Ocagne's identity). *If $n = m, m = n + 1$ and $k = 1$ in the generalized identity (4.5), we obtain,*

$$L_m(x,y)L_{n+1}(x,y) - L_n(x,y)L_{m+1}(x,y) = -\left(x^2+4y\right)(-y)^n F_{m-n}(x,y) \tag{4.8}$$

Proposition 6.9.

$$F_{m+n}(x,y)F_{m+t}(x,y) - F_m(x,y)F_{m+n+t}(x,y) = (-y)^m F_n(x,y)F_t(x,y) \tag{4.9}$$

Proof.

$$\begin{aligned}
&F_{m+n}(x,y)F_{m+t}(x,y) - F_m(x,y)F_{m+n+t}(x,y)\\
&= \left(\frac{\mathfrak{R}_1^{m+n}-\mathfrak{R}_2^{m+n}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{m+t}-\mathfrak{R}_2^{m+t}}{\mathfrak{R}_1-\mathfrak{R}_2}\right) - \left(\frac{\mathfrak{R}_1^{m}-\mathfrak{R}_2^{m}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{m+n+t}-\mathfrak{R}_2^{m+n+t}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\\
&\qquad = \frac{\left(\mathfrak{R}_1^k-\mathfrak{R}_2^k\right)}{\left(\mathfrak{R}_1-\mathfrak{R}_2\right)^2}\left\{\mathfrak{R}_1^{m+t}\left(\frac{-y}{\mathfrak{R}_1}\right)^m - \mathfrak{R}_2^{m+t}\left(\frac{-y}{\mathfrak{R}_2}\right)^m\right.\\
&\qquad = (-y)^m\left(\frac{\mathfrak{R}_1^{n}-\mathfrak{R}_2^{n}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{t}-\mathfrak{R}_2^{t}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\\
&\qquad = (-y)^m F_n(x,y)F_t(x,y)
\end{aligned}$$

This completes the proof.

Proposition 6.10.

$$\begin{aligned}
&F_m(x,y)F_{n+1}(x,y) + F_{m-1}(x,y)F_n(x,y)\\
&\quad = \frac{\{L_{m+n+1}(x,y)+L_{m+n-1}(x,y)\} + (y-1)(-y)^n L_{m-n-1}(x,y)}{(x^2+4y)}
\end{aligned} \tag{4.10}$$

Proof. By Binet's formula (1.2), we have

$$\begin{aligned}
&F_m(x,y)F_{n+1}(x,y) + F_{m-1}(x,y)F_n(x,y)\\
&= \left(\frac{\mathfrak{R}_1^{m}-\mathfrak{R}_2^{m}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{n+1}-\mathfrak{R}_2^{n+1}}{\mathfrak{R}_1-\mathfrak{R}_2}\right) - \left(\frac{\mathfrak{R}_1^{m-1}-\mathfrak{R}_2^{m-1}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\left(\frac{\mathfrak{R}_1^{n}-\mathfrak{R}_2^{n}}{\mathfrak{R}_1-\mathfrak{R}_2}\right)\\
&= \frac{\mathfrak{R}_1^{m+n}\left(\mathfrak{R}_1+\mathfrak{R}_1^{-1}\right) + \mathfrak{R}_2^{m+n}\left(\mathfrak{R}_2+\mathfrak{R}_2^{-1}\right) - \mathfrak{R}_1^{m}\mathfrak{R}_2^{n}\left(\mathfrak{R}_2+\mathfrak{R}_1^{-1}\right) - \mathfrak{R}_2^{m}\mathfrak{R}_1^{n}\left(\mathfrak{R}_1+\mathfrak{R}_2^{-1}\right)}{\left(\mathfrak{R}_1-\mathfrak{R}_2\right)^2}\\
&= \frac{\mathfrak{R}_1^{m+n-1}\left(\mathfrak{R}_1^2+1\right) + \mathfrak{R}_2^{m+n-1}\left(\mathfrak{R}_2^2+1\right) - \mathfrak{R}_1^{m-1}\mathfrak{R}_2^{n}\left(\mathfrak{R}_1\mathfrak{R}_2+1\right) - \mathfrak{R}_2^{m-1}\mathfrak{R}_1^{n}\left(\mathfrak{R}_1\mathfrak{R}_2+\right.}{\left(\mathfrak{R}_1-\mathfrak{R}_2\right)^2}\Bigg)\\
&= \frac{\mathfrak{R}_1^{m+n+1}+\mathfrak{R}_2^{m+n+1}+\mathfrak{R}_1^{m+n-1}+\mathfrak{R}_2^{m+n-1} + (y-1)(-y)^n\left(\mathfrak{R}_1^{m-n-1}+\mathfrak{R}_2^{m-n-1}\right)}{\left(\mathfrak{R}_1-\mathfrak{R}_2\right)^2}\\
&= \frac{\{L_{m+n+1}(x,y)+L_{m+n-1}(x,y)\} + (y-1)(-y)^n L_{m-n-1}(x,y)}{(x^2+4y)}
\end{aligned}$$

This completes the proof.

7. Conclusion

In this paper, we have stated and derived many identities. We define the sum of $s+1$ consecutive members of Bivariate Fibonacci polynomials and Bivariate Lucas polynomials and the same thing for even and for odd and for their product and square. Also, we present their two cross two matrix representation and the generalized identities of bivariate Fibonacci polynomials and bivariate Lucas polynomial.

Acknowledgments

The authors are grateful to the editor and anonymous referees for helpful suggestions and comments.

References

[1] Alves, F. R. V., and Catarino, P. M. M. C., The Bivariate (Complex) Fibonacci and Lucas Polynomials: An Historical Investigation with the Maple's Help, Acta Didactica Napocensia, 9 (4) (2016), 71-95.

[2] Belbachir, H., and Bencherif, F., On some properties on bivariate Fibonacci and Lucas polynomials, (2007). arXiv:0710.1451v1.

[3] Cakmak, T., and Karaduman, E., On the derivatives of bivariate Fibonacci polynomials, Notes on Number Theory and Discrete Mathematics, 24 (3) (2018), 37-46.

[4] Catalani, M., Generalized Bivariate Fibonacci Polynomials, Version 2 (2004). http://front.math.ucdavis.edu/math.CO/0211366

[5] Catalani, M., Identities for Fibonacci and Lucas Polynomials derived from a book of Gould, (2004). http://front.math.ucdavis.edu/math.CO/0407105

[6] Catalani, M., Some Formulae for Bivariate Fibonacci and Lucas Polynomials, (2004). http://front.math.ucdavis.edu/math.CO/0406323

[7] Čerin, Z., Alternating sums of Lucas numbers, Central European Journal of Mathematics, 3 (1) (2005), 1-13.

[8] Čerin, Z., Sums of Product of Generalized Fibonacci and Lucas Numbers, Demonstratio Mathematica, 42 (2) (2009), 247-258.

[9] Čerin, Z., On sums of squares of odd and even terms of the Lucas sequence, Proceedings of the 11th Fibonacci Conference, (to appear).

[10] Čerin, Z., Properties of odd and even terms of the Fibonacci sequence, Demonstratio Mathematica, 39 (1) (2006), 55–60.

[11] Hoggatt, V. E. Jr. and Bicknell, M., Generalized Fibonacci Polynomials, The Fibonacci Quarterly, 11 (5) (1973), 457-465.

[12] Inoue, K., and Aki, S., Bivariate Fibonacci polynomials of order k with statistical applications, Annals of the Institute of Statistical Mathematics, 63 (1) (2011), 197-210.

[13] Jacob, G., Reutenauer, C., and Sakarovitch, J., On a divisibility property of Fibonacci polynomials, (2006). http://perso.telecom-paristech.fr/ jsaka/PUB/Files/DPFP.pdf.

[14] Kaygisiz, K., and Sahin, A., Determinantal and permanental representation of generalized bivariate Fibonacci p-polynomials, (2011). arXiv:1111.4071v1.

[15] Kim, T., Kim, D. S., Dolgy, D. V. and Park, J. -W., Sums of finite products of Chebyshev polynomials of the second kind and of Fibonacci polynomials, J. Inequal. Appl. 2018, 148, (2018).

[16] Kim, D., Dolgy, D. V., Kim, D. S. and Seo, J. J., convolved Fibonacci numbers and their applications, Ars Combin., 135 (2017), 119–131.

[17] Özkan, E. and Altun, I., Generalized Lucas polynomials and relationships between the Fibonacci polynomials and Lucas polynomials, Communications in Algebra, 47 (10) (2019), 4020-4030.

[18] Özkan, E. and Taştan, M., A new families of Gauss k-Jacobsthal numbers and Gauss k-Jacobsthal-Lucas numbers and Their Polynomials, Journal of Science and Arts, 53 (4) (2020), 893-908.

[19] Özkan, E., and Taştan, M., On a new family of gauss k-Lucas numbers and their polynomials, Asian-European Journal of Mathematics, 14 (6) (2021), 215010.

[20] Özkan, E., and Taştan, M., On Gauss Fibonacci polynomials, on Gauss Lucas polynomials and their applications, Communications in Algebra, 48 (3) (2020), 952-960.

[21] Taştan, M., Özkan, E. and Shannon, A. G., The generalized k-Fibonacci polynomials and generalized k-Lucas polynomials, Notes on Number Theory and Discrete Mathematics, 27 (2) (2021), 148-158.

[22] Panwar, Y. K., and Singh, M., Generalized bivariate Fibonacci-Like polynomials, International journal of Pure Mathematics, 1 (2014), 8-13.

[23] Panwar, Y. K., Gupta, V. K. and Bhandari, J., Generalized Identities of Bivariate Fibonacci Polynomials and Bivariate Lucas Polynomials, Journal of Amasya University the Institute of Sciences and Technology, 1 (2) (2020), 146-154.

[24] Saba, N. and Boussayoud, A., Generating functions of binary products of (p,q)-modified Pell numbers with Gaussian numbers and polynomials, 28 (4) (2021), 1311-1327.

[25] Tasci, D., Cetin Firengiz, M. and Tuglu, N., Incomplete Bivariate Fibonacci and Lucas $p-$ Polynomials, Discrete Dynamics in Nature and Society, doi:10.1155/2012/840345.

[26] Tuglu, N., Kocer, E. G. and Stakhov, A., Bivariate fibonacci like p-polynomials, Applied Mathematics and Computation, 217 (24) (2011), 10239-10246.