J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 9, No. 2 (2022), pp. 217-234

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

STRESS-STRENGTH RELIABILITY MODELS INVOLVING H-FUNCTION DISTRIBUTIONS

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(**Received:** *Feb. 25, 2022* **Accepted:** *Jun. 01, 2022* **Published:** *Jun. 30, 2022)*

Abstract: This paper deals with four theorems for the stress-strength reliability measure $R = P(Y < X)$, when X and Y are independent H-function random variables with different parameters. Several new particular cases of the general results are given along with known results in the literature. For ready reference, the results are given in tabular form.

Keywords and Phrases: Stress-strength reliability, $P(X < Y)$ and H-function.

2020 Mathematics Subject Classification: 60E05, 62N05.

1. Introduction

The stress-strength probability plays an important role in the reliability theory. If Y is the strength of a system which is subjected to a stress X, then $R = P(X <$

Y) measures the performance of the system. Also, R may be interpreted as the probability of a system failure when the applied stress X is greater than its strength Y.

One can find a comprehensive treatment of different stress-strength models till 2002 in Kotz et al. (2003). In this paper, we have provided most of the results available during the period 2000 to 2020 on the subject which are particular cases of the unified results of this paper. The results are provided in tabular form for ready reference. The reliability for Levy random variables are given recently by Rathie and Ozelim (2017, 2018).

The paper is organized as follows: In Section 2, we define a few distributions and state known results. Section 3 deals with the derivation of R, in the form of the theorems, when X e Y are independent H-function random variables. In Section 4, we derive particular cases of the main result for (a) generalized gamma distributions, (b) generalized F-Shah-Rathie distributions, and (c) Fréchet distributions. The last section deals with conclusions.

2. Distribution Functions and Results

In this section, we give some definitions and results which will be used subsequently.

2.1. Reliability

If $X > 0$ and $Y > 0$ are independent, then

$$R = P(X < Y) = \int_0^\infty \int_0^y f_X(x) g_Y(y) dx dy, \tag{1}$$

where $f_X(x)$ and $g_Y(y)$ are density functions corresponding to the random variables X and Y respectively.

2.2. G-function

The G-function is defined as

$$G^{m,n}_{p,q}\left[x \middle| \begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right] = \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^m \Gamma(b_j - s) \prod_{j=1}^n \Gamma(1 - a_j + s)}{\prod_{j=m+1}^q \Gamma(1 - b_j + s) \prod_{j=n+1}^p \Gamma(a_j - s)} x^s ds, \tag{2}$$

where $x \neq 0$, an empty product is interpreted as unity, $0 \leq m \leq q$ and $0 \leq n \leq p$ (not both m and n zeros simultaneously). The parameters b_j, $j = 1, 2, \ldots, m$ and a_j, $j = 1, 2, \ldots, n$, are such that no pole of $\prod_{j=1}^m \Gamma(b_j - s)$ coincides with any pole of $\prod_{j=1}^n \Gamma(1 - a_j + s)$. See Luke (1969) for details about the contour L and conditions of convergence of the integral.

2.3. The H-function

The H-function is defined by

$$H_{p,q}^{m,n}\left[x\left|\begin{matrix}{}_1(a_j,A_j)_p\\ {}_1(b_j,B_j)_p\end{matrix}\right.\right]=\frac{1}{2\pi i}\int_L\frac{\prod_{j=1}^{m}\Gamma(b_j-B_js)\prod_{j=1}^{n}\Gamma(1-a_j+A_js)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_js)\prod_{j=n+1}^{p}\Gamma(a_j-A_js)}x^s ds, \tag{3}$$

where, $0\leq m\leq q$, $0\leq n\leq p$ (not both m and n zeros simultaneously), $A_j>0$, $j=1,2,\ldots,p$, $B_j>0$, $j=1,2,\ldots,q$, a_j and b_j are complex numbers such that no poles of $\Gamma(b_j-B_js)$, $j=1,2,\ldots,m$ coincide with poles of $\Gamma(1-a_j+A_js)$, $j=1,2,\ldots,n$. L is a suitable contour $w-i\infty$ to $w+i\infty$, $w\in\mathbb{R}$, separating the poles of two types mentioned above. For more details, see Mathai et al. (2010) and Springer (1979).

2.4. The H-function Distribution

Let $X>0$ be a random variable with density function defined as

$$f_X(x)=kH_{p,q}^{m,n}\left[\eta x^{\xi}\left|\begin{matrix}{}_1(a_j,A_j)_p\\ {}_1(b_j,B_j)_q\end{matrix}\right.\right], \tag{4}$$

where k is given by

$$k=\frac{\xi\eta^{\frac{1}{\xi}}\prod_{j=m+1}^{q}\Gamma\left(1-b_j-\frac{B_j}{\xi}\right)\prod_{j=n+1}^{p}\Gamma\left(a_j+\frac{A_j}{\xi}\right)}{\prod_{j=1}^{m}\Gamma\left(b_j+\frac{B_j}{\xi}\right)\prod_{j=1}^{n}\Gamma\left(1-a_j-\frac{A_j}{\xi}\right)}. \tag{5}$$

Notationally, we denote $X\sim HD[x,k,\eta,\xi,{}_1(a_j,A_j)_p,{}_1(b_j,B_j)_q]$. Here the parameters $k,\eta,\xi,a_j,A_j,j=1,2,\ldots,p$, and $b_j,B_j,j=1,2,\ldots,q$ satisfy

a) $f_X(x)\geq 0$ and

b) $\int_0^\infty f_X(x)dx=1$.

2.5. Generalized Gamma Distribution

The generalized gamma density, denoted by $fGG(x,a,b,c)$, is defined as

$$f(x)=\frac{cb^{\frac{a}{c}}}{\Gamma\left(\frac{a}{c}\right)}x^{a-1}e^{-bx^c},\quad x,a,b,c>0. \tag{6}$$

The corresponding distribution function $FGG(x,a,b,c)$ is given by

$$F(x)=\frac{1}{\Gamma(\frac{a}{c})}G_{1,2}^{1,1}\left[bx^c\left|\begin{matrix}1\\ \frac{a}{c},0\end{matrix}\right.\right]. \tag{7}$$

The following well-known distributions are the special cases of density $fGG(x, a, b, c)$: Chi: $fGG(x, n, \frac{1}{2}, 2)$, Chi-square: $fGG(x, \frac{\nu}{2}, \frac{1}{2}, 1)$, Erlang: $fGG(x, p, \frac{1}{a}, 1)$, Exponential: $fGG(x, 1, \frac{1}{\phi}, 1)$, Gamma: $fGG(x, \theta, \frac{1}{\phi}, 1)$, Half-normal: $fGG(x, 1, \frac{1}{2\theta^2}, 2)$, Maxwell: $fGG(x, 3, \frac{1}{\theta^2}, 2)$, Rayleigh : $fGG(x, 1, \frac{2}{2a^2}, 2)$ and Weibull: $fGG(x, \phi, \theta, \phi)$.

2.6. Generalized F-Shah-Rathie Distribution

In 1974, Shah and Rathie defined the generalized F density function, denoted by $fGFSR(x, p, \alpha, h, m)$, as

$$f(x) = \frac{kx^{p-1}}{(1+\alpha x^h)^m}, \quad x, \alpha, m, p, h > 0, \tag{8}$$

where

$$k = \frac{h\alpha^{\frac{p}{h}}}{B\left(\frac{p}{h}, m - \frac{p}{h}\right)}, \quad m > \frac{p}{h}, \tag{9}$$

and $B(\cdot, \cdot)$ is the well-known beta function.

The corresponding distribution function $FGFSR(x, p, \alpha, h, m)$ is given by

$$F(x) = \frac{1}{\Gamma\left(\frac{p}{h}\right)\Gamma\left(m - \frac{p}{h}\right)} G^{1,2}_{2,2}\left[\alpha x^h \middle| \begin{matrix} 1+\frac{p}{h}-m, 1 \\ \frac{p}{h}, 0 \end{matrix}\right] \tag{10}$$

The following well-known distributions are special cases of $fGFSR(x, p, \alpha, h, m)$: Beta-2 Inverted: $fGFSR(x, \alpha, 1, 1, \alpha + \beta)$, Dagum: $fGFSR(x, \sigma\beta, \frac{1}{\lambda}, \sigma, \beta + 1)$, F: $fGFSR(x, \theta_1, \frac{\theta_1}{\theta_2}, 1, \theta_1 + \theta_2)$, Half-Cauchy: $fGFSR(x, 1, \frac{1}{\theta^2}, 2, 1)$, Half-Student-t: $fGFSR(x, 1, 1, 2, \theta + \frac{1}{2})$, Log-logistic: $fGFSR(x, \gamma, \alpha^{-\gamma}, \gamma, 2)$, Singh-Maddala: $fGFSR(x, p, \alpha, p, q)$, Lomax: $fGFSR(x, 1, \frac{1}{\lambda}, 1, \alpha + 1)$.

2.7. Known Results

The following known results will be needed in proving the results of the subsequent sections:

Duplication formula for gamma function

$$\Gamma(2x) = 2^{2x-1}\pi^{-\frac{1}{2}}\Gamma(x)\Gamma\left(x + \frac{1}{2}\right), \tag{11}$$

using $(1 + x)^{-a}$ as

$$(1+x)^{-a} = \frac{1}{\Gamma(a)} H^{1,1}_{1,1}\left[x \middle| \begin{matrix} (1-a, 1) \\ (0, 1) \end{matrix}\right], \tag{12}$$

using e^{-x} as

$$e^{-x} = H^{1,0}_{0,1}\left[x \middle| (0, 1)\right], \tag{13}$$

this equality

$$x^c H_{p,q}^{m,n}\left[x\middle|\begin{matrix} {}_1(a_j,A_j)_p \\ {}_1(b_j,B_j)_q\end{matrix}\right] = H_{p,q}^{m,n}\left[x\middle|\begin{matrix}(a_1+A_1c,A_1),\dots,(a_p+A_pc,A_p)\\(b_1+B_1c,B_1),\dots,(b_q+B_qc,B_q)\end{matrix}\right], \tag{14}$$

finally

$$\int_0^\infty x^{s-1} H_{p_1,q_1}^{m_1,n_1}\left[\eta x\middle|\begin{matrix} {}_1(d_j,D_j)_{p_1} \\ {}_1(e_j,E_j)_{q_1}\end{matrix}\right] H_{p,q}^{m,n}\left[z x^\sigma\middle|\begin{matrix} {}_1(a_j,A_j)_p \\ {}_1(b_j,B_j)_q\end{matrix}\right] dx \;=$$

$$= \eta^{-s} H_{p+q_1,q+p_1}^{m+n_1,n+m_1}\left[z\eta^{-\sigma}\middle|\begin{matrix} {}_1(a_j,A_j)_n, {}_1(1-e_j-sE_j,\sigma E_j)_{q_1}, {}_{n+1}(a_j,A_j)_p \\ {}_1(b_j,B_j)_m, {}_1(1-d_j-sD_j,\sigma D_j)_{p_1}, {}_{m+1}(b_j,B_j)_q\end{matrix}\right], \tag{15}$$

for conditions of existence etc. see Mathai et al. (2010).
When no two b_j's differ by an integer or zero, we have

$$H_{p,q}^{m,n}\left[x\middle|\begin{matrix} {}_1(a_j,A_j)_p \\ {}_1(b_j,B_j)_q\end{matrix}\right] = \sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{\prod_{j=1\neq h}^{m}\Gamma(\vartheta_{j,h})\prod_{j=1}^{n}\Gamma(1-\varphi_{j,h})}{\prod_{j=m+1}^{q}\Gamma(1-\vartheta_{j,h})\prod_{j=n+1}^{p}\Gamma(\varphi_{j,h})}\frac{(-1)^v x^{\frac{b_h+v}{B_h}}}{v!B_h}, \tag{16}$$

where

$$\vartheta_{j,h} = b_j - B_j\frac{(b_h+v)}{B_h},\quad \varphi_{j,h} = a_j - \frac{A_j(b_h+v)}{B_h},$$

for $x \neq 0$ if $\mu > 0$ and for $0 < |x| < \frac{1}{\beta}$ if $\mu = 0$. Here

$$\beta^{-1} = \frac{\prod_{j=1}^{q} B_j^{B_j}}{\prod_{j=1}^{p} A_j^{A_j}},\quad \mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j.$$

Similarly to (16), when no two a_j's differ by integer or zero, we have

$$H_{p,q}^{m,n}\left[x\middle|\begin{matrix} {}_1(a_j,A_j)_p \\ {}_1(b_j,B_j)_q\end{matrix}\right] = \sum_{h=1}^{n}\sum_{v=0}^{\infty}\frac{\prod_{j=1}^{m}\Gamma(\vartheta^*_{j,h})\prod_{j=1\neq h}^{n}\Gamma(1-\varphi^*_{j,h})}{\prod_{j=m+1}^{q}\Gamma(1-\vartheta^*_{j,h})\prod_{j=n+1}^{p}\Gamma(\varphi^*_{j,h})}\frac{(-1)^v x^{-\frac{1-a_h+v}{A_h}}}{v!A_h}, \tag{17}$$

where

$$\vartheta^*_{j,h} = b_j + B_j\frac{(1-a_h+v)}{A_h},\quad \varphi^*_{j,h} = a_j + \frac{A_j(1-a_h+v)}{A_h},$$

for $x \neq 0$ if $\mu < 0$ and $|x| > \frac{1}{\beta}$ if $\mu = 0$.

3. Main Result in Reliability

In this section, we prove the main result in reliability. Let X and Y be independent random variables with respectively density functions given by

$$f_X(x) = k_1 H_{p_1,q_1}^{m_1,n_1}\left[\eta_1 x^{\xi_1}\middle|\begin{matrix} {}_1(a_j,A_j)_{p_1} \\ {}_1(b_j,B_j)_{q_1}\end{matrix}\right]\mathbb{1}_{(0,\infty)}(x), \tag{18}$$

and

$$f_Y(y) = k_2 H^{m_2,\, n_2}_{p_2,\, q_2}\left[\eta_2 y^{\xi_2}\middle|\begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right] \mathbb{1}_{(0,\infty)}(y), \tag{19}$$

where

$$k_1 = \frac{\xi_1 \eta_1^{\frac{1}{\xi_1}} \prod_{j=m_1+1}^{q_1} \Gamma\big(1 - b_j - \frac{B_j}{\xi_1}\big) \prod_{j=n_1+1}^{p_1} \Gamma\big(a_j + \frac{A_j}{\xi_1}\big)}{\prod_{j=1}^{m_1} \Gamma\big(b_j + \frac{B_j}{\xi_1}\big) \prod_{j=1}^{n_1} \Gamma\big(1 - a_j - \frac{A_j}{\xi_1}\big)} \tag{20}$$

and

$$k_2 = \frac{\xi_2 \eta_2^{\frac{1}{\xi_2}} \prod_{j=m_2+1}^{q_2} \Gamma\big(1 - d_j - \frac{D_j}{\xi_2}\big) \prod_{j=n_2+1}^{p_2} \Gamma\big(c_j + \frac{C_j}{\xi_2}\big)}{\prod_{j=1}^{m_2} \Gamma\big(d_j + \frac{D_j}{\xi_2}\big) \prod_{j=1}^{n_2} \Gamma\big(1 - c_j - \frac{C_j}{\xi_2}\big)}. \tag{21}$$

Then,

$$\begin{aligned} R = P(X<Y) &= \int_0^\infty \int_0^y k_1 H^{m_1,\, n_1}_{p_1,\, q_1}\left[\eta_1 x^{\xi_1}\middle|\begin{matrix} {}_1(a_j,A_j)_{p_1} \\ {}_1(b_j,B_j)_{q_1}\end{matrix}\right] k_2 H^{m_2,\, n_2}_{p_2,\, q_2}\left[\eta_2 y^{\xi_2}\middle|\begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right] dxdy \\ &= k_1 k_2 \int_0^\infty H^{m_2,\, n_2}_{p_2,\, q_2}\left[\eta_2 y^{\xi_2}\middle|\begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right] \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m_1} \Gamma(\tau_j)}{\prod_{j=m_1+1}^{q_1} \Gamma(1-\tau_j)} \times \\ &\quad \times \frac{\prod_{j=1}^{n_1} \Gamma(1-\psi_j)}{\prod_{j=n_1+1}^{p_1} \Gamma(\psi_j)} \int_0^y \big(\eta_1 x^{\xi_1}\big)^s dxdsdy. \end{aligned} \tag{22}$$

Using (3),

$$\begin{aligned} R &= k_1 k_2 \int_0^\infty H^{m_2,\, n_2}_{p_2,\, q_2}\left[\eta_2 y^{\xi_2}\middle|\begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right] \frac{1}{2\pi i} \int_L \frac{\prod_{j=1}^{m_1} \Gamma(\tau_j)}{\prod_{j=m_1+1}^{q_1} \Gamma(1-\tau_j)} \times \\ &\quad \times \frac{\prod_{j=1}^{n_1} \Gamma(1-\psi_j)}{\prod_{j=n_1+1}^{p_1} \Gamma(\psi_j)} \big(\eta_1 y^{\xi_1}\big)^s y \frac{\Gamma(\xi_1 s + 1)}{\Gamma(\xi_1 s + 2)} dsdy, \end{aligned} \tag{23}$$

where $\Re(\xi_1 s + 1) > 0$. Here $\tau_j = b_j - B_j s$, and $\psi_j = a_j - A_j s$. Thus,

$$R = k_1 k_2 \int_0^\infty y H^{m_2,\, n_2}_{p_2,\, q_2}\left[\eta_2 y^{\xi_2}\middle|\begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right] H^{m_1,\, n_1+1}_{p_1+1,\, q_1+1}\left[\eta_1 y^{\xi_1}\middle|\begin{matrix} (0,\xi_1),\, {}_1(a_j,A_j)_{p_1} \\ {}_1(b_j,B_j)_{q_1},\, (-1,\xi_1)\end{matrix}\right] dy.$$

Substituting $u = y^{\xi_1}$ and using (15), we get

Theorem 1.

$$\begin{aligned} R &= k_1 k_2 \int_0^\infty u^{\frac{1}{\xi_1}} u^{\frac{1}{\xi_1}-1} H^{m_2,\, n_2}_{p_2,\, q_2}\left[\eta_2 u^{\frac{\xi_2}{\xi_1}}\middle|\begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2}\end{matrix}\right] H^{m_1,\, n_1+1}_{p_1+1,\, q_1+1}\left[\eta_1 u\middle|\begin{matrix} (0,\xi_1),\, {}_1(a_j,A_j)_{p_1} \\ {}_1(b_j,B_j)_{q_1},\, (-1,\xi_1)\end{matrix}\right] \frac{du}{\xi_1} \\ &= \tfrac{k_1 k_2}{\xi_1} \eta_1^{-\frac{2}{\xi_1}} H^{m_2+n_1+1,\, n_2+m_1}_{p_2+q_1+1,\, q_2+p_1+1}\left[\eta_2 \eta_1^{-\frac{\xi_2}{\xi_1}}\middle|\begin{matrix} {}_1(c_j,C_j)_{n_2},\, {}_1\big(1-b_j-\frac{2}{\xi_1}B_j, \frac{\xi_2}{\xi_1}B_j\big)_{q_1}, (0,\xi_2),\, {}_{n_2+1}(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{m_2}, (-1,\xi_2),\, {}_1\big(1-a_j-\frac{2}{\xi_1}A_j, \frac{\xi_2}{\xi_1}A_j\big)_{p_1},\, {}_{m_2+1}(d_j,D_j)_{q_2}\end{matrix}\right] \end{aligned} \tag{24}$$

In (23), writing $\Gamma(\xi_1 s+1)/\Gamma(\xi_1 s+2)$ as $\Gamma(s+\frac{1}{\xi_1})/[\xi_1\Gamma(s+\frac{1}{\xi_1}+1)]$ and following the same procedure, we arrive at the alternative result:

Theorem 2.

$$R = \frac{k_1k_2}{\xi_1^2\eta_1^{2/\xi_1}} H^{m_2+n_1+1,\,n_2+m_1}_{p_2+q_1+1,\,q_2+p_1+1}\left[\eta_2\eta_1^{-\frac{\xi_2}{\xi_1}}\middle|\begin{matrix} {}_1(c_j,C_j)_{n_2},{}_1\left(1-b_j-\frac{2}{\xi_1}B_j,\frac{\xi_2}{\xi_1}B_j\right)_{q_1},(1-\frac{1}{\xi_1},\frac{\xi_2}{\xi_1}),{}_{n_2+1}(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{m_2},(-\frac{1}{\xi_1},\frac{\xi_2}{\xi_1}),{}_1\left(1-a_j-\frac{2}{\xi_1}A_j,\frac{\xi_2}{\xi_1}A_j\right)_{p_1},{}_{m_2+1}(d_j,D_j)_{q_2}\end{matrix}\right]. \quad (25)$$

Let

$$F_X(y) = k_1\int_0^y H^{m_1,\,n_1}_{p_1,\,q_1}\left[\eta_1 x^{\xi_1}\middle|\begin{matrix}{}_1(a_j,A_j)_{p_1}\\ {}_1(b_j,B_j)_{q_1}\end{matrix}\right]dx = k_3 H^{m_3,\,n_3}_{p_3,\,q_3}\left[\eta_3 y^{\xi_3}\middle|\begin{matrix}{}_1(e_j,E_j)_{p_3}\\ {}_1(f_j,F_j)_{q_3}\end{matrix}\right]. \quad (26)$$

Then, similarly we obtain:

Theorem 3.

$$R = \frac{k_2k_3}{\xi_3^2\eta_3^{1/\xi_3}} H^{m_2+n_3,\,n_2+m_3}_{p_2+q_3,\,q_2+p_3}\left[\eta_2\eta_3^{-\frac{\xi_2}{\xi_3}}\middle|\begin{matrix} {}_1(c_j,C_j)_{n_2},{}_1\left(1-f_j-\frac{F_j}{\xi_3},\frac{\xi_2}{\xi_3}F_j\right)_{q_3},{}_{n_2+1}(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{m_2},{}_1\left(1-e_j-\frac{E_j}{\xi_3},\frac{\xi_2}{\xi_3}E_j\right)_{p_3},{}_{m_2+1}(d_j,D_j)_{q_2}\end{matrix}\right]. \quad (27)$$

If we take

$$F_X(y) = 1 - k_4 H^{m_1,\,n_1}_{p_1,\,q_1}\left[\eta_1 y^{\xi_1}\middle|\begin{matrix}{}_1(e_j,E_j)_{p_1}\\ {}_1(f_j,F_j)_{q_1}\end{matrix}\right], \quad (28)$$

then, one obtains:

Theorem 4.

$$R = 1-\frac{k_2k_4}{\xi_1\eta_1^{1/\xi_1}} H^{m_2+n_1,\,n_2+m_1}_{p_2+q_1,\,q_2+p_1}\left[\eta_2\eta_1^{-\frac{\xi_2}{\xi_1}}\middle|\begin{matrix} {}_1(c_j,C_j)_{n_2},{}_1\left(1-f_j-\frac{F_j}{\xi_1},\frac{\xi_2}{\xi_1}F_j\right)_{q_1},{}_{n_2+1}(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{m_2},{}_1\left(1-e_j-\frac{E_j}{\xi_1},\frac{\xi_2}{\xi_1}E_j\right)_{p_1},{}_{m_2+1}(d_j,D_j)_{q_2}\end{matrix}\right]. \quad (29)$$

4. Particular Cases

Three particular cases of (22) are derived in this section.

4.1. Generalized Gamma Distributions

Let $X \sim GG(x, a_1, b_1, c_1)$ and $Y \sim GG(y, a_2, b_2, c_2)$ be independent random variables having generalized gamma density functions. Then, using (6), (13) and (14), we have the following density functions respectively:

$$f_X(x) = \frac{c_1 b_1^{\frac{1}{c_1}}}{\Gamma\left(\frac{a_1}{c_1}\right)} H^{1,0}_{0,1}\left[b_1x^{c_1}\middle|((a_1-1)/c_1,1)\right], \quad x, a_1, b_1, c_1 > 0$$

and

$$f_Y(y) = \frac{c_2 b_2^{\frac{1}{c_2}}}{\Gamma\left(\frac{a_2}{c_2}\right)} H_{0,1}^{1,0}\left[b_2 x^{c_2}\middle|((a_2-1)/c_2, 1)\right], \quad y, a_2, b_2, c_2 > 0.$$

Using (22) and (24), we get

$$R = P(X < Y) = \frac{c_2 b_2^{\frac{1}{c_2}} b_1^{-\frac{1}{c_1}}}{\Gamma\left(\frac{a_1}{c_1}\right)\Gamma\left(\frac{a_2}{c_2}\right)} H_{2,2}^{2,1}\left[b_2 b_1^{-\frac{c_2}{c_1}}\middle|\begin{matrix}\left(1-\frac{a_1+1}{c_1}, \frac{c_2}{c_1}\right), (0, c_2)\\ \left(\frac{a_2-1}{c_2}, 1\right), (-1, c_2)\end{matrix}\right]. \tag{30}$$

Alternatively, (30) may be written as

$$R = P(X < Y) = \frac{c_2 b_2^{\frac{1}{c_2}}}{c_1 b_1^{\frac{1}{c_1}} \Gamma\left(\frac{a_1}{c_1}\right)\Gamma\left(\frac{a_2}{c_2}\right)} H_{2,2}^{2,1}\left[\frac{b_2}{b_1^{\frac{c_2}{c_1}}}\middle|\begin{matrix}\left(1-\frac{a_1+1}{c_1}, \frac{c_2}{c_1}\right), \left(1-\frac{1}{c_1}, \frac{c_2}{c_1}\right)\\ \left(\frac{a_2-1}{c_2}, 1\right), \left(-\frac{1}{c_1}, \frac{c_2}{c_1}\right)\end{matrix}\right]. \tag{31}$$

For $c_1 = c_2$, (31) reduces to the result recently obtained by Nojosa and Rathie (2020). For $c_1 < c_2$, (24) and (16) yield the following expression for computational purposes:

$$R = K \sum_{r=0}^{\infty} \frac{\Gamma\left(\frac{a_1+a_2+c_1 r}{c_2}\right)\Gamma(a_1 + c_1 r)}{r!\,\Gamma(a_1 + 1 + c_1 r)} (-1)^r \left(b_2 b_1^{-\frac{c_2}{c_1}}\right)^{\frac{a_2+1+c_1 r}{c_2}}, \tag{32}$$

where

$$K = \frac{c_1 b_2^{\frac{1}{c_2}} b_1^{-\frac{1}{c_1}}}{\Gamma\left(\frac{a_1}{c_1}\right)\Gamma\left(\frac{a_2}{c_2}\right)}.$$

Using the well-known distributions indicated after Eq. (7), which are special cases of $fGG(x, a, b, c)$, the corresponding results for reliability $P(X < Y)$ may be derived as particular cases of (30) or (31).

4.2. Generalized F-Shah-Rathie Distributions

Let $X \sim GFRS(x, p_1, \alpha_1, h_1, m_1)$ and $Y \sim GFRS(y, p_2, \alpha_2, h_2, m_2)$ be independent random variables having generalized F Shah-Rathie density functions. Then, using (8), (12) and (14), we have the following density functions respectively:

$$f_X(x) = \frac{h_1 \alpha_1^{\frac{1}{h_1}}}{\Gamma\left(\frac{p_1}{h_1}\right)\Gamma\left(m_1 - \frac{p_1}{h_1}\right)} H_{1,1}^{1,1}\left[\alpha_1 x^{h_1}\middle|\begin{matrix}\left(1-m_1+\frac{p_1-1}{h_1}, 1\right)\\ \left(\frac{p_1-1}{h_1}, 1\right)\end{matrix}\right], \quad x, p_1, \alpha_1, h_1, m_1 > 0$$

and

$$f_Y(y) = \frac{h_2 \alpha_2^{\frac{1}{h_2}}}{\Gamma\big(\frac{p_2}{h_2}\big)\Gamma\big(m_2 - \frac{p_2}{h_2}\big)} H_{1,1}^{1,1}\left[\alpha_2 y^{h_2}\middle|\begin{matrix}\big(1-m_2+\frac{p_2-1}{h_2},1\big)\\ \big(\frac{p_2-1}{h_2},1\big)\end{matrix}\right], \quad y, p_2, \alpha_2, h_2, m_2 > 0.$$

Applying (22) and (24), we get

$$R = P(X < Y) = K^* H_{3,3}^{3,2}\left[\alpha_2 \alpha_1^{-\frac{h_2}{h_1}}\middle|\begin{matrix}\big(1-m_2+\frac{p_2-1}{h_2},1\big),\big(1-\frac{p_1+1}{h_1},\frac{h_2}{h_1}\big),(0,h_2)\\ \big(\frac{p_2-1}{h_2},1\big),(-1,h_2),\big(m_1-\frac{p_1+1}{h_1},\frac{h_2}{h_1}\big)\end{matrix}\right], \tag{33}$$

where

$$K^* = \frac{h_2 \alpha_1^{-\frac{1}{h_1}} \alpha_2^{\frac{1}{h_2}}}{\Gamma\big(\frac{p_1}{h_1}\big)\Gamma\big(m_1 - \frac{p_1}{h_1}\big)\Gamma\big(\frac{p_2}{h_2}\big)\Gamma\big(m_2 - \frac{p_2}{h_2}\big)}.$$

Alternatively, (33) may be written as

$$R = P(X < Y) = \frac{K^*}{h_1} H_{3,3}^{3,2}\left[\alpha_2 \alpha_1^{-\frac{h_2}{h_1}}\middle|\begin{matrix}\big(1-m_2+\frac{p_2-1}{h_2},1\big),\big(1-\frac{p_1+1}{h_1},\frac{h_2}{h_1}\big),\big(1-\frac{1}{h_1},\frac{h_2}{h_1}\big)\\ \big(\frac{p_2-1}{h_2},1\big),\big(-\frac{1}{h_1},\frac{h_2}{h_1}\big),\big(m_1-\frac{p_1+1}{h_1},\frac{h_2}{h_1}\big)\end{matrix}\right]. \tag{34}$$

Using the well-known distributions indicated after Eq. (10), which are special cases of $fGFRS(x, p, \alpha, h, m)$, the corresponding results for reliability $P(X < Y)$ may be derived as particular cases of (33) or (34).

4.3. Fréchet Distributions

The Fréchet distribution of random variable X has density function as

$$f_X(x) = \mu\alpha x^{-\alpha-1} \exp\left(-\mu x^{-\alpha}\right), \quad x, \mu, \alpha > 0$$

and the distribution function is

$$F_X(x) = \exp\left(-\mu x^{-\alpha}\right).$$

The reliability, when $X \sim$ Fréchet(x, μ, α) and $Y \sim$ Fréchet(y, λ, θ) are independent, is given by

$$R = P(X < Y) = \lambda\theta \int_0^\infty G_{0,1}^{1,0}\left[\mu z^\alpha \middle|_{0}\right] G_{0,1}^{1,0}\left[\lambda z^\theta \middle|_{0}\right] dz.$$

Using (15), we have

$$R = H_{1,1}^{1,1}\left[\mu\lambda^{-\frac{\alpha}{\theta}}\middle|\begin{matrix}(0,\frac{\alpha}{\theta})\\(0,1)\end{matrix}\right].$$

Alternate forms for R may be obtained by using other results of this paper.

5. Conclusions

Four theorems are established for the Stress-Strength reliability $R = P(X < Y)$, when X and Y have independent H-function distributions with different parameters. Three particular cases for generalized gamma, generalized F-Shah-Rathie and Fréchet distributions are derived. Several new and known results are given in tabular forms, for computational purposes for real data analysis. Several new particular cases for other statistical distributions may be obtained from the four general theorems proved in this paper.

Attachments

Table 1: Special cases of Results (30) or (31).

Distributions of X(Stress) and Y(Strength)	$R = P(X < Y)$	Articles
(T1.1) Exponential $GG(x, 1, 1/\theta, 1)$ $GG(y, 1, 1/\gamma, 1)$	$\frac{\theta}{\gamma+\theta}$	Aminzadeh (1997), Chao (1982) and Díaz-Francés and Montoya (2013).
(T1.2) Weibull $GG(x, c, \frac{1}{b_1^c}, c)$ $GG(y, c, \frac{1}{b_2^c}, c)$	$\frac{b_2^c}{b_2^c+b_1^c}$	Neal et al. (1991), Krishnamoorthy and Lin (2010) and Ali and Kannan (2011).
(T1.3) Weibull $GG(x, \frac{\theta}{2}, k_1, \frac{\theta}{2})$ $GG(y, \theta, k_2, \theta)$	$1 - K\left\{1 - \Phi\left(\frac{k_1}{\sqrt{2k_2}}\right)\right\}$ where $K = k_1\sqrt{\frac{\pi}{k_2}}e^{\frac{k_1^2}{4k_2}}$	Ali and Kannan (2011) and Kundu and Gupta (2006).
(T1.4) Weibull $GG(x, \theta, k_1, \theta)$ $GG(y, \frac{\theta}{2}, k_2, \frac{\theta}{2})$	$1 - e^{\frac{k_2^2}{8k_1}} D_{-2}\left[k_2(2\theta)^{-\frac{1}{2}}\right]$ where $D_n(z)$ is cylindric parabolic function.	Ali and Kannan (2011) and Kundu and Gupta (2006).
(T1.5) Gamma $GG(x, \alpha, \frac{1}{\beta}, 1)$ Exponential $GG(y, 1, \frac{1}{\lambda}, 1)$	$\left(\frac{\lambda}{\lambda+\beta}\right)^\alpha$	Jovanovic and Rajic (2014), Ismail et al. (1986), Constantine et al. (1986), Constantine et al. (1990), and Nadarajah (2005).
(T1.6) Power hazard (Weibull) $GG(x, k+1, \frac{a_1}{k+1}, k+1)$ $GG(y, k+1, \frac{a_2}{k+1}, k+1)$	$1 - \frac{a_1}{a_1+a_2}$	Kinaci (2014)

Alternative results:

$$\text{(T1.3)}\ R = \theta\left(\frac{k_2}{k_1^2}\right)^{1/\theta} H_{2,2}^{2,1}\left[\frac{k_2}{k_1^2}\left|\begin{matrix}\left(-\frac{2}{\theta}, 2\right), (0, \theta)\\ \left(1-\frac{1}{\theta}, 1\right), (-1, \theta)\end{matrix}\right.\right]$$

$$\text{(T1.4)}\ R = \frac{\theta}{2}\left(\frac{k_2^2}{k_1}\right)^{1/\theta} H_{2,2}^{2,1}\left[\frac{k_2}{k_1^{1/2}}\left|\begin{matrix}\left(-\frac{1}{\theta}, \frac{1}{2}, \left(0, \frac{\theta}{2}\right)\right)\\ \left(1-\frac{2}{\theta}, 1\right), (-1, \frac{\theta}{2})\end{matrix}\right.\right]$$

Table 1 : Continuation.

Distributions of X(Stress) and Y(Strength)	$R = P(X < Y)$	Articles
(T1.7) Rayleigh $GG(x, 2, \frac{1}{2a_1^2}, 2)$ $GG(y, 2, \frac{1}{2a_2^2}, 2)$	$\frac{a_2^2}{a_1^2 + a_2^2}$	-
(T1.8) Chi $GG(x, n_1, 1/2, 2)$ $GG(y, n_2, 1/2, 2)$	$\frac{1}{\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})} G_{2,2}^{2,1}\left[1 \middle\vert \begin{matrix} \frac{1-n_1}{2}, \frac{1}{2} \\ \frac{n_2-1}{2}, -\frac{1}{2} \end{matrix}\right]$	-
(T1.9) Chi-square $GG(x, \gamma_1/2, 1/2, 1)$ $GG(y, \gamma_2/2, 1/2, 1)$	$\frac{1}{\Gamma(\frac{\gamma_1}{2})\Gamma(\frac{\gamma_2}{2})} G_{2,2}^{2,1}\left[1 \middle\vert \begin{matrix} -\frac{\gamma_1}{2}, 0 \\ \frac{\gamma_2}{2} - 1, -1 \end{matrix}\right]$	-
(T1.10) Erlang $GG(x, p_1, 1/a_1, 1)$ $GG(y, p_2, 1/a_2, 1)$	$\frac{a_2^{-1}a_1}{\Gamma(p_1)\Gamma(p_2)} G_{2,2}^{2,1}\left[\frac{a_1}{a_2} \middle\vert \begin{matrix} -p_1, 0 \\ p_2 - 1, -1 \end{matrix}\right]$	-
(T1.11) Half-Normal $GG(x, 1, 1/\theta_1^2, 2)$ $GG(y, 1, 1/\theta_2^2, 2)$	$\frac{\theta_1}{\pi\theta_2} G_{2,2}^{2,1}\left[\left(\frac{\theta_1}{\theta_2}\right)^2 \middle\vert \begin{matrix} 0, \frac{1}{2} \\ 0, -\frac{1}{2} \end{matrix}\right]$	-
(T1.12) Maxwell $GG(x, 3, 1/\theta_1^2, 2)$ $GG(y, 3, 1/\theta_2^2, 2)$	$\frac{4\theta_1}{\pi\theta_2} G_{2,2}^{2,1}\left[\left(\frac{\theta_1}{\theta_2}\right)^2 \middle\vert \begin{matrix} -1, \frac{1}{2} \\ 1, -\frac{1}{2} \end{matrix}\right]$	-

Table 2: Particular cases of Results (33) or (34).

Distributions of X(Stress) and Y(Strength)	$R = P(X < Y)$	Articles
(T2.1) Lomax $GFSR(x, 1, \frac{1}{\lambda}, 1, \alpha_1 + 1)$ $GFSR(y, 1, \frac{1}{\lambda}, 1, \alpha_2 + 1)$	$\frac{\alpha_1}{\alpha_1+\alpha_2}$	Ashour et al. (2015)
(T2.2) Lomax truncated at zero $GFSR(x, 1, \frac{1}{\sigma_1}, 1, 2)$ $GFSR(y, 1, \frac{1}{\sigma_2}, 1, 2)$	$\frac{1}{\sigma_1}\int_0^\infty \left[1 + \frac{z}{\sigma_1}\right]^{-2}\left[1 + \frac{z}{\sigma_2}\right]^{-1} dz$	Punathumparambath (2011)
(T2.3) Log-logistic $GFSR(x, \beta, \frac{1}{\sigma_1^\beta}, \beta, 2)$ $GFSR(y, \beta, \frac{1}{\sigma_2^\beta}, \beta, 2)$	$1 - \sum_{i=s}^{k} \binom{k}{i} \int_0^\infty \frac{z^{k-i}\lambda^{\beta(k-i)}}{\left(1+z\lambda^\beta\right)^k (1+z)^2} dz$ where $z = \left(\frac{y}{\sigma_1}\right)^\beta$ and $\lambda = \frac{\sigma_1}{\sigma_2}$.	Rao and Kantam (2010)
(T2.4) Half T-Student $GFSR(x, 1, \frac{1}{2\theta_1}, 2, \theta_1 + 1/2)$ $GFSR(y, 1, \frac{1}{2\theta_2}, 2, \theta_2 + 1/2)$	$\frac{4(\theta_1\theta_2)^{\frac{1}{2}}}{\pi\Gamma(\theta_1+\frac{1}{2})\Gamma(\theta_2+\frac{1}{2})} G_{3,3}^{3,2}\left[\frac{\theta_1}{\theta_2}\middle\vert \begin{matrix} -\theta_2, 0, \frac{1}{2} \\ 0, -\frac{1}{2}, \theta_1 - \frac{1}{2}\end{matrix}\right]$	-
(T2.5) F of Snedecor $GFSR(x, \frac{m_1}{2}, \frac{m_1}{n_1}, 1, \frac{m_1+n_1}{2})$ $GFSR(y, \frac{m_2}{2}, \frac{m_2}{n_2}, 1, \frac{m_2+n_2}{2})$	$KG_{3,3}^{3,2}\left[\frac{n_1 m_2}{n_2 m_1}\middle\vert \begin{matrix} -\frac{n_2}{2}, -\frac{m_2}{2}, 0 \\ \frac{m_2}{2}-1, -1, \frac{n_1}{2}-1\end{matrix}\right]$ where $K = \frac{n_1 n_2\left[\Gamma(\frac{n_1}{2})\Gamma(\frac{n_2}{2})\right]^{-1}}{m_1 m_2 \Gamma(\frac{m_1}{2})\Gamma(\frac{m_2}{2})}$	-

Alternative results

(T2.2) $R = G_{2,2}^{2,2}\left[\frac{\sigma_1}{\sigma_2}\middle\vert \begin{matrix} 0,0 \\ 0,0 \end{matrix}\right]$

(T2.3) $R = H_{2,2}^{2,2}\left[\left(\frac{\sigma_1}{\sigma_2}\right)^\beta \middle\vert \begin{matrix} (0,1),(0,\beta) \\ (0,1),(0,\beta) \end{matrix}\right]$, for $k = 1 = s$.

Table 2: Continuation.

Distributions of X(Stress) and Y(Strength)	$R = P(X < Y)$	Articles
(T2.6) Beta-2 inverted $GFSR(x, \alpha_1, 1, 1, \alpha_1 + \beta_1)$ $GFSR(y, \alpha_2, 1, 1, \alpha_2 + \beta_2)$	$KG^{3,2}_{3,3}\left[1\,\middle\vert\, \begin{matrix} -\beta_2, -\alpha_1, 0 \\ \alpha_2-1, -1, \alpha_1+\beta_1-2 \end{matrix}\right]$, where $K = \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\beta_1)\Gamma(\beta_2)}$	-
(T2.7) Dagum $GFSR(x, \sigma_1\beta_1, \frac{1}{\lambda_1}, \sigma_1, \beta_1+1)$ $GFSR(y, \sigma_2\beta_2, \frac{1}{\lambda_2}, \sigma_2, \beta_2+1)$	$KH^{2,2}_{2,2}\left[\frac{\lambda_2^{\frac{\sigma_1}{\sigma_2}}}{\lambda_1}\,\middle\vert\, \begin{matrix} (1-\beta_2+\frac{1}{\sigma_2}, \frac{\sigma_1}{\sigma_2}), (1+\frac{1}{\sigma_1}, 1) \\ (1+\frac{1}{\sigma_2}, \frac{\sigma_1}{\sigma_2}), (\beta_1, +\frac{1}{\sigma_1}, 1) \end{matrix}\right]$, where $K = \frac{\lambda_1^{1/\sigma_1}}{\lambda_2^{1/\sigma_2}\Gamma(\beta_1)\Gamma(\beta_2)}$.	-
(T2.8) Half-Cauchy $GFSR(x, 1, 1/\theta_1^2, 2, 1)$ $GFSR(y, 1, 1/\theta_2^2, 2, 1)$	$\frac{\theta_2\theta_1}{\pi^2}G^{3,2}_{3,3}\left[\left(\frac{\theta_1}{\theta_2}\right)^2\,\middle\vert\, \begin{matrix} 0, 0, \frac{1}{2} \\ 0, -\frac{1}{2}, 0 \end{matrix}\right]$	-
(T2.9) Singh-Maddala $GFSR(x, \alpha_1, \beta_1^{\alpha_1}, \alpha_1, \lambda_1+1)$ $GFSR(y, \alpha_2, \beta_2^{\alpha_2}, \alpha_2, \lambda_2+1)$	$1-KH^{2,2}_{2,2}\left[\left(\frac{\beta_2}{\beta_1}\right)^{\alpha_2}\,\middle\vert\, \begin{matrix} (1-\lambda_2-\frac{1}{\alpha_2}, 1), (1-\frac{1}{\alpha_1}, \frac{\alpha_2}{\alpha_1}) \\ (1-\frac{1}{\alpha_2}, 1), (\lambda_1-\frac{1}{\alpha_1}, \frac{\alpha_2}{\alpha_1}) \end{matrix}\right]$, where $K = \frac{\alpha_2\beta_2}{\alpha_1\beta_1\Gamma(\lambda_1)\Gamma(\lambda_2)}$.	-
(T2.10) Dagum (ν_1, ν_2, ν_3) $GFSR(x, \nu_1\nu_3, \frac{1}{\nu_2}, \nu_3, \nu_1+1)$ $GFSR(y, p, \alpha, h, m)$	$1-KH^{2,2}_{2,2}\left[\frac{1}{\nu_2\alpha^{\frac{\nu_3}{h}}}\,\middle\vert\, \begin{matrix} (1-\frac{p}{h}, \frac{\nu_3}{h}), (1, 1) \\ (m-\frac{p}{h}, \frac{\nu_3}{h}), (\nu_1, 1) \end{matrix}\right]$, where $K = \frac{1}{\Gamma(\frac{p}{h})\Gamma(m-\frac{p}{h})\Gamma(\nu_1)}$.	Nojosa et al. (2018)
(T2.11) Singh-Maddala $GFSR(x, p, \alpha, p, m)$ $GFSR(y, p_2, \alpha_2, h_2, m_2)$	$1-KH^{2,2}_{2,2}\left[\alpha_2\alpha^{\frac{h_2}{p}}\,\middle\vert\, \begin{matrix} (1-m_2+\frac{p_2-1}{h_2}, 1), (1-\frac{1}{p}, \frac{h_2}{p}) \\ (\frac{p_2-1}{h_2}, 1), (-1+m-\frac{1}{p}, \frac{h_2}{p}) \end{matrix}\right]$, where $K = \frac{h_2\alpha_2^{\frac{1}{h_2}}}{p\alpha^{\frac{1}{p}}\Gamma(m-1)\Gamma(\frac{p_2}{m_2})\Gamma(m_2-\frac{p_2}{h_2})}$.	-

Table 3

Distributions of X(Stress) and Y(Strength)	$R = P(X < Y)$	Articles
(T3.1) Fréchet Fréchet(x, μ, α) Fréchet(y, λ, θ)	$H_{1,1}^{1,1}\left[\mu\lambda^{-\frac{\alpha}{\theta}}\middle\vert\begin{matrix}(0,\frac{\alpha}{\theta})\\(0,1)\end{matrix}\right]$	-
(T3.2) Fréchet Fréchet$(x, \mu, 1)$ Fréchet$(y, \lambda, 1)$	$\frac{\lambda}{\lambda+\mu}$	Abid (2014)
(T3.3) Rayleigh inverse Fréchet$(x, \mu, 2)$ Fréchet$(y, \lambda, 2)$	$\frac{\lambda}{\lambda+\mu}$	Tarvirdizade and Garehchobogh (2014)

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