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BANASCHEWSKI-MULVEY TYPE COMPACTIFICATION OF PROXIMAL CSÁSZÁR FRAMES

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Abstract: A Császár frame is said to be proximal if it is symmetric, strong and regular. Our aim in this paper is to apply the methods used by Banaschewski and Mulvey in constructing the Stone-Céch compactification of completely regular locale to construct a compactification of a proximal Császár frame.

Keywords and Phrases: (proximal) Császár frame, compactification, dense homomorphism.

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1. Introduction and Preliminaries

We recall that a *frame* is a complete lattice L satisfying the property:

$$x \wedge \bigvee S = \bigvee \{x \wedge s \mid s \in S),$$

for all $x \in L$ and all $S \subseteq L$. The *bottom* (respectively, *top*) element of a frame L will be denoted by 0 (resp., e). A frame homomorphism between two frames is a mapping that preserves finite meets (including e) and arbitrary joins (including 0). The resulting category will be denoted by **Frm**. For general knowledge on frames, we refer to [Johnstone 12 and Picado and Pultr 13].

A frame L is said to be *regular* if every $a \in L$ is expressible as

$$a = \bigvee\{b \in L \mid b \prec a\} = \bigvee\{b \in L \mid b^* \vee a = e\},$$

where $b \prec a$ means that $b^* \vee a = e$ and $b^* = \bigvee\{c \in L \mid c \wedge b = 0\}$. We will denote by **RegFrm** the category of regular frames and frame homomorphisms.

Lemma 1.1. ([Pultr 14, 8.1.1 (5)]) *If $a_i \prec b_i$ for $i = 1,\ 2$ then $a_1 \vee a_2 \prec b_1 \vee b_2$ and $a_1 \wedge a_2 \prec b_1 \wedge b_2$.*

Császár frames were introduced by Chung [6] into the category of frames and frame homomorphisms as pointfree analogues of syntopogenous spaces, themselves owing their origin to [Ákos Császár 8]. See also [Flax 10]. Proximal Császár frames are completely regular, and so, they have compactifications. Classically, compactifications can only be constructed for Hausdorff spaces [Tamano 15]. Since proximal frames are regular (by definition), it makes sense to investigate a compactification of a proximal frame. There have been developments over the past 30 years in the area of frames (locales) that were driven by compactification of frames (locales): see, for example, [Banaschewski and Pultr 4, Ferreira et al 9, Frith and Schauerte 11, and Bezhanishivili and Harding 5]. For background on compactifications, we follow [Banaschewski 2]. See also [Baboolal and Banaschewski 1].

Definition 1.2. ([Chung 6, Definition 2.1.1]) *A Császár order on a frame L is a binary relation $\lhd_i^L$ on L satisfying the following properties:*

$CO_1)$: $0 \lhd_i^L 0$ *and* $e \lhd_i^L e$;

$CO_2)$: $x \lhd_i^L y \Rightarrow x \leq y$; *and*

$CO_3)$: $x \leq u \lhd_i^L v \leq y \Rightarrow x \lhd_i^L y$.

Note that given a collection $\{\lhd_i^L \mid i \in I\}$ of Császár orders on L, it is easily seen that $\mathcal{L} = \cup\{\lhd_i^L \mid i \in I\}$ is a Császár order on L: in this case, the pair $(L, \mathcal{L})$ is called a *Császár frame*.

Definition 1.3. ([Chung 7, Section 2]) *A Császár frame $(L, \mathcal{L})$ is said to be proximal if it symmetric, strong and regular in the sense that:*

(a) A Császár order $\lhd_i^L$ on a Császár frame L is said to be symmetric if whenever $a \lhd_i^L b$ then $b^* \lhd_i^L a^*$. The Császár frame $(L, \mathcal{L})$ is symmetric if each Császár order on L is symmetric.

(b) A Császár frame $(L, \mathcal{L})$ is said to be strong if for each $\lhd_i^L \in \mathcal{L}$ with $u \lhd_i^L v$, there is a $\lhd_0^L \in \mathcal{L}$ and some $w \in L$ such that $u \lhd_0^L w \lhd_0^L v$.

(c) A Császár frame $(L, \mathcal{L})$ is said to be regular if every $\lhd^L \in \mathcal{L}$ is coarser than $\prec$, that is, $\lhd_i^L \subseteq \prec$ if and only if $u \lhd_i^L v$ implies $u \prec v$.

In [10, 1.4], Flax showed that if $\leq$ is a topogenous order on a set X, then the relation $c(\leq)$ defined on the powerset of X by $A\, c(\leq) B$ if and only if $X - B \leq X - A$ is also a topogenous order on X, for any subsets A and B of X. We prove a pointfree version of this result, even though stronger than in syntopogenous spaces.

Lemma 1.4. *Given a Császár order $\lhd_i^L$ on a frame L, the relation $\lhd_{i(c)}^L$ defined by*

$$u \lhd_{i(c)}^L v \text{ if and only if } v^* \lhd_i^L u^*$$

is also a Császár order on L.

Proof:

$CO_1)$: We have that $0^* \lhd^L 0^*$ since $\lhd_i^L$ is a Császár order, hence $0 \lhd_{i(c)}^L 0$. Similarly, $e^* \lhd_i^L e^*$ implying that $e \lhd_{i(c)}^L e$.

$CO_2)$: If $u \lhd_{i(c)}^L v$, then $v^* \lhd_i^L u^*$. But $\lhd_i^L$ is a Császár order, we must have $v^* \leq u^*$ so that $u \leq v$ as desired.

$CO_3)$: Suppose $u \leq x \lhd_{i(c)}^L y \leq v$. We want show that $u \lhd_{i(c)}^L v$. By definition of $\lhd_{i(c)}^L$, we have $y^* \lhd_i^L x^*$, and so $v^* \leq y^* \lhd_i^L x^* \leq u^*$. Since $\lhd_i^L$ is a Császár order on L, it follows that $v^* \lhd_i^L u^*$ and this shows that $u \lhd_{i(c)}^L v$.

2. Main Results: Banaschewski-Mulvey type Compactification of a Proximal Császár Frame

In this section, we follow the construction of the Stone-Céch compactification of a locale by [Banaschewski and Mulvey 3] to construct the compactification of a proximal Császár frame. In their paper, Banschewski and Mulvey gave a construction which showed that both compact regular locales and compact completely regular locales are reflective in the category **Loc** of locales and localic maps: **Loc** is the opposite category to **Frm**, thus, **Loc** = **Frm**^{op}. For more on the category **Loc**, please refer to [Johnstone 12]. In the following construction, we will show that the category **PCsFrm** of proximal Császár frames and their homomorphisms is reflective in the category **Frm**.

Recall that a frame L is said to be *compact* if for every subset U of L with $\bigvee U = e$ there exists a finite subset T of U with $\bigvee T = e$. By a *compactification* of a frame L we mean a dense onto frame homomorphism $h : M \to L$, where M is a compact regular frame. Ideals are dual to filters in the sense that a non-empty proper subset I of a frame L is said to be an *ideal* of L if:

i) Whenever $u,\ v\ \in I$ then $u\ \vee\ v \in I$, and

ii) Whenever $u\ \in I$ and $v\ \leq\ u$, then $v\ \in I$.

An ideal I of a proximal Császár frame $(M, \mathcal{M})$ is said to be *strongly regular* if whenever $u \in I$ there exists $v \in I$ and some $\lhd_i^M \in \mathcal{M}$ such that $u \lhd_i^M v$. We shall denote by $\mathfrak{R}_{\mathcal{M}}$ the set of all strongly regular ideals of M.

We will need the following result:

Lemma 2.1. (cf. 3, Lemma 1) *Let $(M, \mathcal{M})$ be a Császár frame and let $\lhd_i^M \in \mathcal{M}$.*

i) If $x\ \leq\ y\ \lhd_i^M z$, then $x\ \lhd_i^M z$: Since $x \leq z$, it follows from the definition that $x \lhd_i^M z$.

ii) (Refer also to Lemma 1.1 above) If $x \lhd_i^M y$ and $u \lhd_i^M v$, then $x \wedge u \lhd_i^M y \wedge v$: By hypothesis, we have $x \wedge u \leq x \lhd_i^M y$ and $x \wedge u \leq u \lhd_i^M v$, and then $x \wedge u \lhd_i^M y$ and $x \wedge u \lhd_i^M v$. Since $\lhd^{\mathcal{M}}$ is a meet-sublattice of $M \times M$, we must have $x \wedge u \lhd_i^M y \wedge v$.

Lemma 2.2. *$\mathfrak{R}_{\mathcal{M}}$ is a compact frame.*

Proof.

a) The *bottom* and *top* elements of $\mathfrak{R}_{\mathcal{M}}$ are respectively $\{0_M\}$ and $\{M\}$. Recall that for ideals (even strongly regular ones) $I, J \in \mathfrak{R}_{\mathcal{M}}$, it holds that $I \wedge J = \{x \wedge y \mid x \in I, y \in J\}$. Now, if $u \in I \wedge J$ then $u = x \wedge y$, so that there exists $u_x \in I$ and $u_y \in J$ such that $u = x \wedge y \lhd_i^M u_x \wedge u_y \in I \wedge J$ for some $\lhd_i^M \in \mathcal{M}$, so strongly regular ideals are closed under finite meets. It is also easily seen that $I \vee J \in \mathfrak{R}_{\mathcal{M}}$ for any $I, J \in \mathfrak{R}_{\mathcal{M}}$: For, if $x \in I \vee J$ there exist $u \in I$ and $v \in J$, and then $u_x \in I, v_x \in J$ such that $u \lhd_i^M u_x$ and $v \lhd_i^M v_x$, for some $\lhd_i^M \in \mathcal{M}$. Then $x = u \vee v \lhd_i^M u_x \vee v_x \in I \vee J$, which establishes that $I \vee J$ is a strongly regular ideal. We then consider $\bigcup_i I_i$, where each $I_i \in \mathfrak{R}_{\mathcal{M}}$. It is easily shown that for any $x \in \bigcup_i I_i$, there is a $y \in \bigcup_i I_i$ such that $x\ \lhd_i^M y$, for some $\lhd_i^M \in \mathcal{M}$. (For the construction that follows, we are indebted to [Pultr 10].) Given a strongly regular ideal I and a collection $\{J_i\}_i$ of strongly regular ideals in $\mathfrak{R}_{\mathcal{M}}$, let us consider

$$\bigvee_i J_i = \{\bigvee E \mid E \subseteq \bigcup_i J_i \text{ finite}\}.$$

We claim that

$$I \cap (\bigvee_i J_i) \ = \ \bigvee_i (I \cap J_i):$$

Note that $J_i\ \leq\ \bigvee_i J_i$ and $J_i \cap I\ \subseteq\ I$, so it easily follows that

$$\bigvee_i (J_i \cap I)\ \leq\ (\bigvee_i J_i) \cap I.$$

For the opposite implication, we take $u \in I \cap (\bigvee_i J_i)$, say $u = u_1 \vee u_2 \vee \ \dots \ \vee u_n$.

Since $u_j \ \leq \ u$, the definition of an ideal implies that $u \ = \ u_1 \vee u_2 \vee \ \dots \ \vee u_n \in \bigvee_i (I \cap J_i)$, and so

$$(\bigvee_i J_i) \cap I \ \leq \ \bigvee_i (J_i \cap I),$$

showing that

$$(\bigvee_i J_i) \cap I \ = \ \bigvee_i (J_i \cap I).$$

Consequently, $\mathfrak{R}_{\mathcal{M}}$ is a frame.

b) Finally, for compactness of $\mathfrak{R}_{\mathcal{M}}$, suppose $\bigvee_i \{J_i\} = \{M\}$. Remember, $\{M\}$ is the top element of $\mathfrak{R}_{\mathcal{M}}$. Since $e_M \in M$, it follows that there are finitely many $u_j \in J_{i_j}$ such that

$$u_1 \vee u_2 \vee \ \dots \ \vee u_n \ = \ e_M \in \bigvee_{i_j}^{n} J_{i_j}.$$

But then we must have $\{M\} = \bigvee_{i_j}^{n} J_{i_j}$, making $\mathfrak{R}_{\mathcal{M}}$ a compact frame.

Lemma 2.3. $\mathfrak{R}_{\mathcal{M}}$ *is a proximal Császár frame.*

Proof. We start with the Császár part: we consider strongly regular ideals of the form:

$$I \ \lhd_{i\mathcal{C}}^{M} \ J \text{ if and only if for every } x \in I \text{ there exists a } y \in J \text{ such that } x \lhd_i^M y, \text{ for some } \lhd_i^M \in \mathcal{M}.$$

We claim that: $\lhd_{i\mathcal{C}}^{M}$ is a Császár order on $\mathfrak{R}_{\mathcal{M}}$:

(CO_1): For any $\lhd_i^M \in \mathcal{C}$, we have $0_M \lhd_i^M 0_M$, so $\{0_M\} \lhd_i^M \{0_M\}$. Similarly, $\{M\} \lhd_i^M \{M\}$.

(CO_2): Suppose that $I \lhd_{i\mathcal{C}}^{M} \ J$. To see that $I \leq J$, we take $x \in I$ and $y \in J$ such that $x \ \lhd_i^M y$, for some $\lhd_i^M \in \mathcal{C}$; this implies that $x \leq y$, thus $I \leq \ J$.

(CO_3): We start with $I \leq A \lhd_{i\mathcal{C}}^{M} B \leq J$. For any $x \in I$, there exist $y \in A, z \in B$ and $t \in J$ such that

$$x \leq y \ \lhd_i^M z \leq \ t.$$

Since $\lhd_i^M$ is a Császár order, we must have $x \lhd_i^M t$, showing that $I \lhd_{i\mathcal{C}}^{M} J$. By construction, $\mathfrak{R}_{\mathcal{M}}$ is regular, being a collection of strongly regular ideals.

Now we denote by $\mathcal{M}_{\mathcal{C}}$ the Császár order on $\mathfrak{R}_{\mathcal{M}}$, that is,

$$\mathcal{M}_{\mathcal{C}} = \cup\{\lhd_{i\mathcal{C}}^{M} \mid i \in I\}.$$

It remains to show that $\mathfrak{R}_{\mathcal{M}}$ is strong and symmetric. Well, for symmetry, we assume that $x \ \lhd^M_{i\mathcal{C}} \ y$, for some $\lhd^M_{i\mathcal{C}} \in \mathcal{M}_{\mathcal{C}}$, and $x \in I$ and $y \in J$. Then, by definition, for some symmetric $\lhd^M_i \in \mathcal{M}$ (since M is proximal),

$$x \lhd^M_i y \text{ and so } y^* \lhd^M_i x^*; \text{thus } y^* \lhd^M_{i\mathcal{C}} x^*,$$

which shows that $J^* \lhd^M_{i\mathcal{C}} I^*$; thus $\mathfrak{R}_{\mathcal{M}}$ is symmetric. Finally, for strongness of $\mathfrak{R}_{\mathcal{M}}$, suppose that $\lhd^M_{i\mathcal{C}} \in \mathcal{M}_{\mathcal{C}}$ and let $I \ \lhd^M_{i\mathcal{C}} \ J$, and let $\lhd^M_i$ be the associated Császár order in $\mathcal{M}$. Since M is strong (being proximal), we find a Császár order $\lhd^M_\circ \in \mathcal{M}$ such that

$$u \lhd^M_i v \Longrightarrow u \lhd^M_\circ w \lhd^M_\circ v,$$

for some $w \in M$. There is a $\lhd^M_{\circ\mathcal{C}}$ generating the $\lhd^M_\circ$ so that $\lhd^M_{i\mathcal{C}} \subseteq (\lhd^M_{\circ\mathcal{C}})^2$, so $\lhd^M_{\circ\mathcal{C}}$ is strong as desired. And we then have a proximal Csśzár frame $(\mathfrak{R}_{\mathcal{M}}, \mathcal{M}_{\mathcal{C}})$.

Remark 2.4. *The category of proximal Császár frames and proximal frame homomorphisms is denoted by* ***PCsFrm****. Now, recalling that a frame homomorphism $h : M \to L$ is dense if whenever $h(a) = e$, then $a = e$. For the following result, we need:*

Definition 2.5. *A proximal homomorphism $h : (M, \mathcal{M}) \to (L, \mathcal{L})$ between proximal Császár frames is one that satisfies:*

i) $h(0) = 0$ *and* $h(e) = e$,

ii) $h(a \wedge b) = h(a) \wedge h(b)$,

iii) If $a_1 \lhd^M b_1$ *and* $a_2 \lhd^M b_2$ *for some* $\lhd^M \in \mathcal{M}$, *then* $f(a_1 \vee b_1) h(\lhd^M) h(a_2) \vee h(b_2)$, *and*

iv) $h(b) = \bigvee\{h(a) \mid a \lhd^M_i b,$ *for some* $\lhd^M_i \in \mathcal{M}\}$.

Lemma 2.6. *The map $\nu_M : \mathfrak{R}_{\mathcal{M}} \to (M, \mathcal{M})$, where $I \mapsto \bigvee I$, is a dense onto proximal map.*

Proof.

We first note that for the map $\rho_M : M \to \mathfrak{R}_{\mathcal{M}}, \ u \mapsto \downarrow u$, the map ν_M is onto because for each $u \in M$, it holds that $\nu_M \circ (\rho_M(u)) = \nu_M(\downarrow u) = u$. In addition, for any $I \in \mathfrak{R}_{\mathcal{M}}$, we have that

$$(\rho_M \circ \nu_M)(I) \ = \ \rho_M(\bigvee I) \ = \downarrow (\bigvee I) \ \geqq I,$$

which means that ν_M is a left adjoint of ρ_M; consequently, it must preserve all updirected joins. So, if $I_1 \lhd^M_{i\mathcal{C}} J_1$ and $I_2 \lhd^M_{i\mathcal{C}} J_2$, then (see Definition 2.5)

$$\nu_M(I_1 \vee I_2) = \nu_M(I_1) \vee \nu_M(I_2) \ \nu_M(\lhd^M_{i\mathcal{C}}) \ \nu_M(J_1) \vee \nu_M(J_2).$$

In addition, if

$$J = \bigvee\{I \in \mathfrak{R}_{\mathcal{M}} \mid I \lhd^{M}_{i\mathcal{C}} J, \text{ for some } J \in \mathfrak{R}_{\mathcal{M}}, \text{and for some } \lhd^{M}_{i\mathcal{C}} \in \mathcal{M}\},$$

this left adjoint satisfies

$$\nu_M(J) = \bigvee\{\nu_M(I) \mid I \lhd^{M}_{i\mathcal{C}} J, \text{for some } J \in \mathfrak{R}_{\mathcal{M}}, \text{ and for some } \lhd^{M}_{i\mathcal{C}} \in \mathcal{M}\}.$$

To show that $\rho_M : \mathfrak{R}_{\mathcal{M}} \to M$ preserves finite meets, we take $I, J \in \mathfrak{R}_{\mathcal{M}}$, and note that

$$\begin{aligned} \nu_M(I \wedge J) &= \bigvee I \wedge \bigvee J \\ &= \bigvee\{u \wedge v \mid u \in I, v \in J\} \\ &\leq \bigvee\{w \mid w \in I \cap J\} \\ &= \nu_M(I \cap J) \\ &\leq \nu_M(I) \wedge \nu_M(J). \end{aligned}$$

On the other hand, we also have $\nu_M(I) \wedge \nu_M(J) = (\bigvee I) \wedge (\bigvee J) \leq \bigvee(I \cap J) = \nu_M(I \wedge J)$, whence $\nu_M(I \wedge J) = \nu_M(I) \wedge \nu_M(J)$. For denseness, we proceed thus: let $\nu_M(I) = 0_M$ and note that then $\bigvee I = 0_M$ can only be true if $I = \{0_M\}$, the bottom element of $\mathfrak{R}_{\mathcal{M}}$. It is also true that $\nu_M(\{L\}) = \bigvee\{L\} = e_M$. We have therefore shown that ν_M is a dense onto proximal map.

Putting these results together, noting also that $\bigvee\{L\} = e_M$, we have proved that.

Proposition 2.7. *The pair* $(\mathfrak{R}_{\mathcal{M}}, \nu_M)$ *is the compactification of a proximal Császár frame* $(M, \mathcal{M})$.
By definition, we have **PCsFrm** $\subseteq$ **RegFrm**, so we derive the following:

Corollary 2.8. *The proximal frame homomorphism* $\nu_M : \mathfrak{R}_{\mathcal{M}} \to M$ *is a monomorphism.*
Proof. Since $\mathfrak{R}_{\mathcal{M}}$ ia proximal, it is regular. Since ν_M is dense (onto), the result follows from a standard result of dense homomorphisms on regular frames. See, for example, [Pultr 14, 8.3.2].

The following result, due to Pultr [10, Proposition 10.3.1], is needed in our last result which is analogous to a result of [Banaschewski and Mulvey 3, Lemma 5].

Proposition 2.9. *Let* L *be regular and let* M *be compact. Then each dense homomorphism* $h : L \to M$ *is one-to-one.*

The proof of the following result requires a adaptation on the original proof of the result of Banaschewski and Mulvey.

Proposition 2.10. *In our construction, if M is compact, then for any strongly regular ideal $I \in \mathfrak{R}_{\mathcal{M}}$, it holds that*

$$x \in I \text{ if and only if } x \; \mathcal{M}_{\mathcal{C}} \bigvee I,$$

for all $x \in M$. Moreover, $\nu_M : \mathfrak{R}_{\mathcal{M}} \to M$ is an isomorphism in $\mathbb{PC}\boldsymbol{s}\mathbb{F}\boldsymbol{rm}$.

Proof. By the above result (Proposition 2.9), ν_M is one-to-one; it is an isomorphism with inverse $\rho_M : (M, \mathcal{M}) \to (\mathfrak{R}_{\mathcal{M}}, \mathcal{M}_{\mathcal{C}})$.

3. Concluding Remarks.

If the underlying frame is completely regular, our construction reduces to that of Banaschewski and Mulvey.

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