J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 9, No. 2 (2022), pp. 199-206

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

STUDY OF RL-SEPARATION AXIOMS

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(**Received:** *Mar. 23, 2022* **Accepted:** *Apr. 07, 2022* **Published:** *Jun. 30, 2022)*

Abstract: In this paper, we introduce the notion of RL-T_i spaces for $i = 1, 2, 3, 4$ by using the RL-closed sets. We investigate their characterizations for such notions and established some relations among these spaces.

Keywords and Phrases: RL-open sets, RL-closed sets, RL-T_1 space, RL - Hausdorff space, RL-T_3 space, RL-T_4 space.

2020 Mathematics Subject Classification: 54A05, 54A10, 54D10.

1. Introduction

In topological spaces, the separation axioms are primarily formulated to identify the non-homeomorphic topological spaces. The axioms T_0 or kolmogorov space, T_1 or Frechet space and T_2 or Hausdorff spaces were introduced by Andrey kolmogorov, Frechet and Felix Hausdorff respectively.

Later, several kind of separation axioms and their properties have been studied and investigated by many topologist. For instances, Alias Barakat Khalat and etal. [1] studied new type of separation axioms namely, w-regular and w-normal spaces via w-open sets [6]. Hariwan Z. Ibrahim [7] defined and studied some separation axioms called Bc - T_k for k = 0, 1/2, 1, 2 spaces by using Bc-open sets. Benchalli etal. [2] defined δgb-closed sets and separation axioms namely, $\delta gb - T_1$, $\delta gb - T_2$, δgb - regular and δgb - normal spaces. Mahesh Bhat and Hanif Page [3] used the notion of sgp-open sets [11] in order to study $sgp - T_0$, $sgp - T_1$, and $sgp - T_2$ separation axioms. G. Navalagi and R. G. Charantimath [12] defined and studied gsp-separation axioms called, $gsp - T_0$, $gsp - T_1$, $gsp - T_2$ via gsp-open sets [4].

Ittanagi and Govardhana Reddy [8] introduced and studied new kind of separation axioms namely, $gg-T_0$, $gg-T_1$, $gg-T_2$, gg -regular space and gg -normal space via gg - open sets [9]. Recently, M. Jeyachitra and K. Bageerathi [10] have introduced and investigated separation axioms namely, $p^*gp - T_0$, $p^*gp - T_1$ and $p^*gp - T_2$ axioms.

The RL-closed sets and the RL-open sets are introduced and studied by the authors in [14]. In this paper, we introduced the notion of RL-T_i spaces, $i = 0, 1, 2, 3, 4$, by using the RL-open sets and studied and investigated their properties. Further, RL-homeomorphisms and RL^*-homeomorphisms were introduced and the RL-Hausdorff conditions were established for these two homeomorphisms.

2. Preliminaries

Throughout this paper, we represent X and Y as the topological spaces (X, τ) and (Y, σ) respectively. For a subset A of X, $cl(A)$ denotes the closure of A and $int(A)$ denotes the interior of A.
We recall the following definitions in topological spaces.

Definition 2.1. [13] *A subset A of a topological space X is called a regular open set if $A = int(cl(A))$. The complement of a regular open set is called a regular closed set.*

Definition 2.2. [5] *A subset A of a topological space X is called locally closed set (briefly lc set) if $A = U \cap V$, where U is open and V is closed in X.*
The authors also recall the RL-closed sets defined by the same authors in the paper [14] with some other related definitions.

Definition 2.3. [14] *Let A be a subset of a topological space X.Then,*

(i) A is called a regular locally closed set (briefly RL-closed set) if $A = U \cap V$, where U is a regular open set and V is a closed set in X. The complement of RL-closed set is called RL-open set in X.

(ii) The intersection of all RL-closed supersets of A is called regular locally closure of A and is denoted by $cl_{RL}(A)$.
Let $f : X \to Y$ be a map between the topological spaces X and Y.

(iii) f is called a regular locally closed continuous map (briefly RLC-continuous map) if $f^{-1}(V)$ is a RL-closed set in X for every closed set V in Y.

(iv) f is called regular locally closed irresolute map (briefly RLC-irresolute map) if $f^{-1}(V)$ is a RL-closed set in X for each RL-closed set V in Y.

(v) *f is called regular locally open map (briefly RL-open map) if the image of each open set in X is a RL-open set in Y.*

Proposition 2.1. [14] *If both A and B are RL-closed sets in a topological space (X, τ), then $A \cap B$ is RL-closed set.*

Proposition 2.2. [14] *If both A and B are RL-open sets, then $A \cup B$ is a RL-open set in a topological space X.*

3. RL- Separation Axioms

Definition 3.1. *A topological space X is said to be a regular locally-T_1 space (briefly RL-T_1 space) if for any pair of distinct points x_1, x_2 in X, each belongs to a RL-open set which does not contain the other.*

Theorem 3.1. *A topological space X is said to be RL-T_1 space if and only if every singleton subset of X is a RL-closed set.*

Proof. Suppose that X is a RL-T_1 space and $p \in X$. We shall prove that $\{p\}^C$ is a RL-open set. Let $x \in \{p\}^C$. Then, $x \neq p$, so $\exists$ a RL-open set U_x in X such that $x \in U_x$ but $p \notin U_x$. That is, $x \in U_x \subseteq \{p\}^C$. Thus, $\bigcup_{x \in \{p\}^C} \{x\} \subseteq \bigcup \{U_x : x \in \{p\}^C\} \subseteq \{p\}^C$. That is, $\{p\}^C = \bigcup \{U_x : x \in \{p\}^C\}$. Since $\bigcup \{U_x : x \in \{p\}^C\}$ is RL-open, $\{p\}^C$ is RL-open.

Conversely suppose that $\{p\}$ is RL-closed for any $p \in X$. Let a, $b \in X$ with $a \neq b$. Then, $b \in \{a\}^C$ where $\{a\}^C$ is RL-open and $a \notin \{a\}^C$. Similarly, $a \in \{b\}^C$, where $\{b\}^C$ is RL-open and $b \notin \{b\}^C$. Thus, X is RL-T_1 space.

Definition 3.2. *A topological space X is called a regular locally Hausdorff (briefly RL-Hausdorff) space if for each pair x_1, x_2 of distinct points of X, there are disjoint RL-open sets U_1 and U_2 such that x_1 belongs to U_1 and x_2 belongs to U_2.*

This is clear that every Hausdorff space is a RL-Hausdorff space.

Lemma 3.2. *Every RL-Hausdorff space is a RL-T_1 space.*

Proof. Suppose that the topological space X is a RL-Hausdorff space and let x_1, $x_2 \in X$ such that $x_1 \neq x_2$. Since X is a RL-Hausdorff space, there exist disjoint RL-open sets U_1 and U_2 such that $x_1 \in U_1$, $x_2 \in U_2$. That is, there exist RL-open sets U_1 and U_2 such that $x_1 \in U_1$, $x_1 \notin U_2$ and $x_2 \in U_2$, $x_2 \notin U_1$. Thus, X is a RL-T_1 space.

Definition 3.3. *The RL-subspace Y of a topological space X is a subset of X determined by all sets of the form $Y \cap U$ where U is a RL-open set in X.*

Proposition 3.1. *Every RL-subspace of a RL-Hausdorff space is a RL-Hausdorff* *space.*

Proof. Let Y be a RL-subspace of a RL-Hausdorff space X and let x_1, $x_2 \in X$ such that $x_1 \neq x_2$. Since X is RL-Hausdorff, there exist disjoint RL-open sets U_1, U_2 in X such that $x_1 \in U_1$, $x_2 \in U_2$. Then, $x_1 \in Y \cap U_1$ and $x_2 \in Y \cap U_2$ and $(Y \cap U_1) \cap (Y \cap U_2) = \emptyset$. Since Y is RL-subspace, $Y \cap U_1$ and $Y \cap U_2$ are RL-open sets in Y. Therefore, Y is a RL-Hausdorff space.

Proposition 3.2. *All singleton sets are RL-closed sets in the RL-Hausdorff space.*

Proof. Suppose that X is a RL-Hausdorff space and let $x \in X$. We shall prove that $\{x\}^C$ is a RL-open set. Let $y \in \{x\}^C$, then, $y \neq x$. Since X is RL-Hausdorff, $y \in \{x\}^C$ has a RL-open set U_y which does not intersect with some RL-open set of x. Then, we get $\{x\}^C = \bigcup\{U_y : y \in \{x\}^C\}$. Since $\bigcup\{U_y : y \in \{x\}^C\}$ is RL-open, $\{x\}^C$ is RL-open.

Proposition 3.3. *Let $f : X \to Y$ be a bijective RLC-continuous map between the topological space X and Y. If Y is a Hausdorff space, then X is a RL-Hausdorff space.*

Proof. Let x_1, x_2 be two distinct points in X. Then, $f(x_1)$, $f(x_2)$ are distinct points in Y. Since Y is Hausdorff, there exist disjoint open sets U_1 and U_2 in Y such that $f(x_1) \in U_1$, $f(x_2) \in U_2$. This implies that $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$ where $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are RL-open sets in X by the RL-continuity of f. Now, $f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2) = \emptyset$, we get, X is a RL-Hausdorff space.

Proposition 3.4. *Let $f : X \to Y$ be a bijective RL-open map between the topological spaces X and Y. If X is a Hausdorff space, then Y is a RL-Hausdorff space.*

Proof. Let y_1, y_2 be two distinct points in Y. Then, there exists two distinct points x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Now X being a Hausdorff space, there exist disjoint open sets U_1, U_2 in X, such that $x_1 \in U_1$ and $x_2 \in U_2$. Since f is a RL-open map, there exists disjoint RL-open sets $f(U_1), f(U_2)$ in Y such that $f(x_1) \in f(U_1)$ and $f(x_2) \in f(U_2)$. Therefore Y is a RL-Hausdorff space.

4. Regular Locally Homeomorphism

Definition 4.1. *A bijection map $f : X \to Y$ between the topological spaces X and Y is called a regular locally homeomorphism (briefly RL-homeomorphism) if both f and f^{-1} are RLC-continuous maps.*

It is clear that every homeomorphism is a RL-homeomorphism but, in general, the converse need not be true. This can be seen in the following example:

Example 4.1. Let $X = \{a, b, c\}$ and $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and let $Y = \{a, b, c\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a, c\}\}$. Define $f : X \to Y$ by $f(a) = b$, $f(b) = a$ and

$f(c) = c$. Then, f is a RL-homeomorphism but not a homeomorphism.

Definition 4.2. *A bijection map $f : X \to Y$ between the topological spaces X and Y is called a RL^*-homeomorphism if both f and f^{-1} are RLC-irresolute maps.*
We note that every RL^*-homeomorphism is a RL-homeomorphism. But not conversely. That is for any space X, $RL - h(X) \subseteq RL^* - h(X)$.

Theorem 4.2. *Let $f : X \to Y$ be a bijective RL-homeomorphism between the topological spaces X and Y. Then,*

1. *if Y is a Hausdorff space, then X is a RL-Hausdorff space,*
2. *if X is a Hausdorff space, then Y is a RL-Hausdorff space.*

Proof. Let $f : X \to Y$ be a RL-homeomorphism. Then, by definition, both f and f^{-1} are RLC-continuous maps and so f is both RLC-continuous and RLC-open map. Hence by the above 3.3 and 3.4 Propositions, we get the result.

Theorem 4.3. *Let $f : X \to Y$ be a bijective map and let both f and f^{-1} are RLC-irresolute maps. Then, X is a RL-Hausdorff space if and only if Y is a RL-Hausdorff space.*
Proof. Suppose that X is a RL-Hausdorff space. Let y_1, y_2 be two distinct points in Y. Then, there exists two distinct points x_1, x_2 in X such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X being RL-Hausdorff , there exist two disjoint RL-open sets U_1, U_2 in X, such that $x_1 \in U_1$ and $x_2 \in U_2$. Since f^{-1} is a RLC-irresolute map, the inverse image of U_1, U_2 under f^{-1}, $f(U_1), f(U_2)$ are disjoint RL-open sets in Y such that $y_1 \in f(U_1)$ and $y_2 \in f(U_2)$. Therefore, Y is a RL-Hausdorff space.
Conversely, suppose that Y is a RL-Hausdorff space. Let x_1, x_2 be two distinct points in X. Then, $f(x_1)$, $f(x_2)$ are the distinct points in Y. Since Y being RL-Hausdorff, there exist disjoint RL-open sets U_1 and U_2 in Y such that $f(x_1) \in U_1$, $f(x_2) \in U_2$. This implies that $x_1 \in f^{-1}(U_1)$ and $x_2 \in f^{-1}(U_2)$ and also we get, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are RL-open sets in X. Now, $f^{-1}(U_1) \cap f^{-1}(U_2) = f^{-1}(U_1 \cap U_2) = \emptyset$, we get, X is a RL-Hausdorff space.

Theorem 4.4. *Let the bijective map be a $f : X \to Y$ RL^*-homeomorphism. Then, X is a RL-Hausdorff space if and only if Y is a RL-Hausdorff space.*
Proof. Let $f : X \to Y$ be a RL^*-homeomorphism. Then, by definition, both f and f^{-1} are RLC-irresolute maps. Hence by theorem, we get the result.

Theorem 4.5. *Every convergent sequence in a RL-Hausdorff space has a unique limit.*
Proof. Suppose that X is a RL-Hausdorff space and let (x_n) be a sequence in X

which converges to the points x_1 and x_2 where $x_1 \neq x_2$. Since X is RL-Hausdorff, there exist disjoint RL-open sets U_1 and U_2 in X such that $x_1 \in U_1$, $x_2 \in U_2$. Then, there exist N_1, $N_2 \in \mathbb{N}$ such that if $n > \sup\{N_1, N_2\}$, then, $x_n \in U_1$ for $n > N_1$ and $x_n \in U_2$ for $n > N_2$. This is a contradiction that U_1 and U_2 are disjoint. Therefore, (x_n) has a unique limit.

Definition 4.3. *A topological space X is said to be a regular locally-regular space (briefly RL-regular space) if V is a RL-closed subset of X and $p \in X$ does not belong to V, then there exist disjoint RL-open sets U_1 and U_2 such that $V \subset U_1$ and $p \in U_2$.*

Definition 4.4. *A topological space X is said to be a regular locally-regular T_1 space (briefly RL-regular T_1 space) if the RL-regular space X is also a separation axiom of RL-T_1 space. A RL-regular T_1 space is called a RL-T_3 space.*

Theorem 4.6. *A topological space X is RL-regular if and only if for a given point x of X and a RL-open set U of x, there exists a RL-open set V of x such that $cl_{RL}(V) \subset U$.*

Proof. Suppose that X is a RL-regular space and let x be a point of X and U be a RL-open set of X. Let $A = X - U$. Then, by hypothesis, there exist disjoint RL-open sets V and W containing x and A respectively. Now, the set $cl_{RL}(V)$ is disjoint from A, since if $y \in A$, then the set W is a RL-open set of y disjoint from V. Therefore, $cl_{RL}(V) \subset U$.

Conversely, suppose that the point x and the RL-closed set A which is not containing x are given. Let $U = X - A$, then by hypothesis, there is a RL-open set V of x such that $cl_{RL}(V) \subset U$. The RL-open sets V and $X - cl_{RL}(V)$ are disjoint RL-open sets containing x and A respectively. Thus X is a RL-regular space.

Theorem 4.7. *If X is a RL-T_3 space, then X is a RL-Hausdorff space.*

Proof. Suppose that X is a RL-T_3 space and let x_1, $x_2 \in X$, where $x_1 \neq x_2$. Since X being a RL-T_1 space, every singleton set is RL-closed; that is, $\{x_1\}$ and $\{x_2\}$ are both RL-closed sets in X. Since X being a RL-regular space, there exist RL-open sets U $\{x_1\}$ and V $\{x_2\}$ such that $U \cap V = \emptyset$. Therefore, X is a RL-Hausdorff space.

Definition 4.5. *A topological space X is said to be regular locally-normal space (briefly RL-normal space) if V_1 and V_2 are disjoint RL-closed sets of X, then there exist disjoint RL-open sets U_1 and U_2 such that $V_1 \subset U_1$ and $V_2 \subset U_2$.*

Definition 4.6. *A topological space X is said to be a regular locally-normal T_1 space (briefly RL-normal T_1 space) if the RL-normal space X is also a RL-T_1 space. A RL-normal T_1 space is called a RL-T_4 space.*

Theorem 4.8. *A topological space X is RL-normal if and only if for a given RL-closed set A of X and a RL-open set U_1 containing A, there exists a RL-open set U_2 containing A such that $cl_{RL}(U_2) \subset U_1$.*

Proof. Suppose that X is a RL-normal space and the RL-closed set A and the RL-open set U_1 containing A are given. Let $B = X - U_1$, then by hypothesis, there exist disjoint RL-open sets U_2 and U_3 containing A and B respectively. Then, the set $cl_{RL}(U_2)$ is disjoint from B, since if $y \in B$, the set U_3 is a RL-open set of y disjoint from U_2. Therefore, $cl_{RL}(U_2) \subset U_1$.

Conversely, suppose that the RL-closed sets A and B are given. Let $U_1 = X - A$. By hypothesis, there is a RL-open set U_2 of A such that $cl_{RL}(U_2) \subset U_1$. Then, U_2 and $X - cl_{RL}(U_2)$ are disjoint RL-open sets containing A and B respectively. Thus X is a RL-normal space.

Theorem 4.9. *If X is a RL-T_4 space, then X is a RL-regular space.*

Proof. Suppose that X is a RL-T_4 space and let $x \in X$ and A be a RL-closed set in X that does not contain x. Since X is a RL-T_1 space, $\{x\}$ is RL-closed. Since X is RL-normal space, there exist RL-open sets U and V with $\{x\} \subset U$ and $A \subset V$ such that $U \cap V = \emptyset$. Therefore, X is RL-regular space.

5. Conclusion

In this paper, we introduced the notions of RL-T_1 space, RL-Hausdorff space, RL-T_3 space, and RL-T_4 space by using the RL-closed sets. We investigated their properties and found the relationship between them. We also introduced the notion of RL-homeomorphisms and RL^*-homeomorphisms and investigated their relationships with the RL-Hausdorff space.

Acknowledgments

The authors acknowledge the reviewers for their valuable comments.

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