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NEW RESULTS OF COMMON FIXED-POINT THEOREMS FOR T-CONTRACTION TYPE MAPPINGS IN CONE METRIC SPACES WITH C-DISTANCE

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Abstract: In this manuscript, we prove the existence and uniqueness of common fixed point of T- contraction type mapping under the concept of c- distance in cone metric spaces. Our results generalize, refinement and improvement the well-known previous result of Dubey et al. [9].

Keywords and Phrases: Fixed point, common fixed point, cone metric space, T- contraction mapping, c- distance.

2020 Mathematics Subject Classification: 47H09, 47H10, 54H25.

1. Introduction

The concept of fixed point was established by S. Banach in 1922 for contraction mapping, which is known as Banach fixed point theorem [2]. Since then, many authors have obtained various extensions and generalizations of [2] by using contraction mapping in different spaces.

In 2007, Huang and Zhang [18] generalized the concept of metric spaces and define a new space, which is called a cone metric space by replacing the set of real numbers by an ordered Banach space. Also they described the convergence of sequences in the cone metric spaces and they proved the following theorems:

Theorem 1.1. *Let (X, d) be a cone metric space, P be a normal cone with normal* *constant K. Suppose a mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq kd(x, y) \tag{1}$$

$\forall x, y \in X$ and where $k \in [0, 1)$. Then T has a unique fixed point in$X$. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.2. *Let (X, d) be a sequentially compact cone metric space, P be a Regular cone . Suppose a mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq d(x, y) \tag{2}$$

$\forall x, y \in X$ and $x \neq y$. Then T has a unique fixed point in X.

Theorem 1.3. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose a mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq k[d(Tx, x) + d(Ty, y)] \tag{3}$$

$\forall x, y \in X$ and where $k \in [0, 1)$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

Theorem 1.4. *Let (X, d) be a cone metric space, P be a normal cone with normal constant K. Suppose a mapping $T : X \to X$ satisfies the contractive condition*

$$d(Tx, Ty) \leq k[d(Tx, y) + d(Ty, x)] \tag{4}$$

$\forall x, y \in X$ and where $k \in [0, 1)$. Then T has a unique fixed point in X. And for any $x \in X$, iterative sequence $\{T^n x\}$ converges to the fixed point.

The above results of [18] were generalized by Sh. Rezapour and R. H. Hagi in [25] omitting the assumption of normality of the cone. Subsequently, many authors have generalized the results of [18] and have studied fixed point theorems for normal and non-normal cone.

In 2009, A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh [3] introduced a new class of contractive mappings: $T-$ Contraction and $T-$ Contractive functions, extending the Banach Contraction Principle and the Edelstein's fixed-point theorem [20], respectively. Subsequently, various authors have generalized and extended in different type of T-contraction mappings on different spaces (see for [21],[22], [23], [24], [6], [7], [8],[27], [18], [29]).

Recently, Cho et al. [4], Wang and Guo [17] defined a concept of c- distance in a cone metric space, which is a cone version of the w-distance Kada et al. [35] and proved some fixed point theorems in ordered cone metric spaces. After that,

several authors studied the existence and uniqueness of the fixed point, common fixed point, coupled fixed point and common coupled fixed point problems using this distance in cone metric spaces and ordered cone metric spaces see for examples [1, 10-16, 26].
Quick recently, in 2017 Fadail et al. [4], studied some fixed point theorems of T-Reich contraction type mappings under the concept of c-distance in complete cone metric spaces depended on another function. Since then, fixed point theorems for T-contraction mapping on cone metric spaces with c-distance have been appeared (see for instance [17],[33], [34],[5],[30] and [9, 31]).

The purpose of this paper is to generalize and improved common fixed point theorems for T-contraction type mappings with c- distance in cone metric spaces. Our results pull out and generalized the results of [9]. Throughout this paper, we do not impose the normality condition for the cones, but the only assumption is that the cone P is solid, that is int $P \neq \phi$. Also, in this paper we assume $\mathbb{R}$ as a set of real numbers and $\mathbb{N}$ as a set of natural numbers.

2. Preliminaries and Basic Concept

Definition 2.1. ([18]) *Let E be a real Banach space and θ denote the zero element in E .A cone P is a subset of E such that :*

(i) P is a non – empty, closed, and $P \neq \{\theta\}$;

(ii) If a, b are non – negative real numbers and $x, y \in P$ then $ax + by \in P$;

(iii) $x \in P$ and $-x \in P \Rightarrow x = \theta$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x$. We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x \ll y$ if and only if $y - x \in \text{int } P$, int P denotes the interior of the set P.

Definition 2.2. ([18]) *A cone P is called normal if there is a number $K > 0$ such that for all, $x, y \in E$,$\theta \leq x \leq y$ implies $||x|| \leq K||y||$. The least positive number satisfying the above is called the normal constant of P.*
In the following we always suppose E is a Banach space, P is a cone in E with $intP \neq \phi$and $\leq$ is partial ordering with respect to P.

Definition 2.3. ([18]) *Let X be a non – empty set and E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone P. Suppose that the mapping $d : X \times X \to E$ satisfies the following conditions:*

(d_1). $\theta \preceq d(x, y)$ and $d(x, y) = 0$ if and only if $x = y$, for all $x, y \in X$;

(d_2). *$d(x,y) = d(y,x)$ for all $x, y \in X$;*

(d_3). *$d(x,y) \leq d(x,z) + d(z,y)$ for all $x, y, z \in X$.*

Then, d is called a cone metric on X and the pair (X, d) is called a cone metric space. Notice that the notion of cone metric space is more general than the corresponding of metric space.examples of cone metric space can be found of [18] and [4].

Example 2.4. Let $E = \mathbb{R}^2$, and $P = \{(x,y) \in E : x, y \geq 0\} \subset \mathbb{R}^2$, $X = \mathbb{R}^2$ and suppose that $d : X \times X \to E$ is defined by $d(x,y) = d((x_1,x_2),(y_1,y_2))$=$(|x_1 - y_1|) + |x_2 - y_2|$,$\alpha \max\{|x_1 - y_1|, |x_2 - y_2|\})$, where $\alpha \geq 0$ is a constant.Then $(X, d)$is a cone metric space.It is east to see that d is a cone metric space, and hence (X, d) becomes a cone metric space over(E, P). Also, we have P is a solid and normal cone where the normal constant $K = 1$.

Definition 2.5. ([18]) *Let (X, d) be a cone metric space, let $\{x_n\}$ be a sequence in X and $x \in X$,*

(i) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer $\mathbb{N}$ such that $d(x_n, x) \ll c$ for all $n > N$,then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converges to x.

(ii) for all $c \in E$ with $\theta \ll c$, if there exists a positive integer $\mathbb{N}$ such that for all $n, m \geq N$, $d(x_n, x_m) \ll c$. Then $\{x_n\}$ is called a Cauchy sequence.

(iii) if every Cauchy sequence in X is converge in X then (X, d) is called a complete cone metric space.

The following Lemma is useful to prove our main results.

Lemma 2.6. ([26])
1. If E be a real Banach space with a cone P and $a \leq \lambda a$ wherea $\in P$ and $0 \leq \lambda < 1$,then $a = \theta$.
2. If $c \in intP$, $\theta \leq a_n$ and $a_n \to \theta$, then there exists a positive integer N such that $a_n \ll c$ for all $n \geq N$.
Next, we give the definition of c-distance on a cone metric space (X, d) of Cho et al. in [4].

Definition 2.7. ([4]) *Let (X, d) be a cone metric space. A function $q : X \times X \to X$ is called a c- distance on X, if the following conditions hold:*

(q_1). *$\theta \leq q(x,y)$ for all $x, y \in X$;*

(q_2). *$q(x,z) \leq q(x,y) + q(y,z)$ for all $x, y, z \in X$;*

(q_3). *for each* $x \in X$ *and* $n \geq 1$ *if* $q(x, y_n) \leq u$ *for some* $u = u_x \in P$, *then* $q(x, y) \leq u$ *whenever* $\{y_n\}$ *is sequence in* X *converging to a point* $y \in X$.

(q_4). *for all* $c \in E$ *with* $\theta \ll c$, *there exists* $c \in E$ *with* $\theta \ll c$ *such that* $q(z, x) \ll e$ *and* $q(z, y) \ll e$ *imply* $d(x, y) \ll c$.

Example 2.8. ([4]) Let $E = \mathbb{R}$, and $P = \{x \in E : x \geq 0\}, X = [0, \infty)$ and define a mapping $d : X \times X \to E$ such that $d(x, y) = |x - y|$,for all $x, y, \in X$.Then $(X, d)$is a cone metric space.And define a mapping $q : X \times X \to E$ by $q(x, y) = y$ for all $x, y \in X$.Then q is a c-distance on X.

The following Lemma is very important to prove our results.

Lemma 2.9. ([4]) *Let* $X, d)$ *be a cone metric space and* q *is* c- *distance on* X. *Let* $\{x_n\}$ *and* $\{y_n\}$ *be a sequence in* X *and* $x, y, z \in X$.*Suppose that* $\{u_n\}$ *is a sequence in* P *converging to* θ. *Then the following hold:*

(i) If $q(x_n, y) \leq u_n$ *and* $q(x_n, z) \leq u_n$, *then* $y = z$.

(ii) If $q(x_n, y_n) \leq u_n$ *and* $q(x_n, z) \leq u_n$, *then* $\{y_n\}$ *converges tobz.*

(iii) If $q(x_n, x_m) \leq u_n$ *for* $m > n$, *then* $\{x_n\}$ *is Cauchy sequence in* X.

(iv) If $(y, x_n) \leq u_n$, *then* $\{x_n\}$ *is a Cauchy sequence in* X.

Remark 2.10. ([4])
(1). $q(x, y) = q(y, x)$ *does not necessarily for all* $x, y, \in X$.
(2). $q(x, y) = \theta$ *is not necessarily equivalent to* $x = y$ *for all* $x, y \in X$.
Next definition is taken from [3]:

Definition 2.11. *Let* (X, d) *be a cone metric space,* P *a solid cone and* $d : X \to X$. *Then*

(i) T *is said to be continuous if* $\lim\limits_{n\to\infty} x_n = x^*$ *implies that* $\lim\limits_{n\to\infty} Tx_n = Tx^*$ *for all* $\{x_n\}$ *in* X.

(ii) T *is said to be subsequentially convergent, if we have, for every sequence* $\{x_n\}$,*that* $\{Tx_n\}$ *is convergent,implies* $\{x_n\}$ *has a convergent subsequence.*

(iii) T *is said to be sequentially convergent if we have, for every sequence* $\{x_n\}$, *if* $\{Tx_n\}$ *is convergent, then* $\{x_n\}$ *is also convergent.*

3. Main Results

In this section, we generalize and extend the results of [9].

Theorem 3.1. *Let (X, d) be a complete cone metric space, P be solid cone and q be c-distance on X. and $T : X \to X$ be an one to one, continuous function and subsequentilly convergent and $P, Q : X \to X$ be a pair of mappings. In addition, suppose that there exists mapping $r_1, r_2 : X \to [0, 1)$ such that the following conditions hold:*

(a) $r_1(Px) \leq r_1(x), r_2(Px) \leq r_2(x)$, for all $x \in X$;

(b) $(r_1 + 2r_2)(x) < 1$ for all $x \in X$;

(c) $q(TPx, TQy) \leq r_1(x)q(Tx, Ty) + r_2(x)[q(TPx, Ty) + q(TQy, Tx)]$

forall $x, y \in X$ Then the mapping P and Q have a unique common fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{P^{2k}x\}$and $\{Q^{2k+1}x\}$ converges to the common fixed point. If $u = Pu = Qu$. Then $q(Tu, Tu) = \theta$.

Proof. Choose $x_0 \in X$. We define the iterative sequences $\{x_{2k}\}$ and $\{x_{2n+1}\}$ by

$$\begin{aligned} x_{2k+1} &= Px_{2K} = P^{2K}x_0, \\ x_{2K+2} &= Qx_{2K+1} = Q^{2K+1}x_0, . \end{aligned} \tag{5}$$

Then we have

$$\begin{aligned} &q(Tx_{2k}, Tx_{2k+1})) = q(TPx_{2k-1}, TQx_{2k})) \\ &\leq r_1(x_{2k-1})q(Tx_{2k-1}, Tx_{2k}) + r_2(x_{2k-1})[q(TPx_{2k-1}, Tx_{2k}) + q(TQx_{2k}, Tx_{2k-1})] \\ &= r_1(Px_{2k-1})q(Tx_{2k-1}, Tx_{2k}) + r_2(Px_{2k-1})[q(Tx_{2k}, Tx_{2k}) + q(Tx_{2k+1}, Tx_{2k-1})] \\ &\leq r_1(x_{2k-2})q(Tx_{2k-1}, Tx_{2k}) + r_2(x_{2k-2})[q(Tx_{2k-1}, Tx_{2k}) + q(Tx_{2k}, Tx_{2k+1})] \end{aligned}$$

Continuing in this manner, we get

$$\begin{aligned} q(Tx_{2k}, Tx_{2k+1})) \leq\ & r_1(x_0)q(Tx_{2k-1}, Tx_{2k}) + r_2(x_0)q(Tx_{2k-1}, Tx_{2k}) \\ &+ r_2(x_0)q(Tx_{2k}, Tx_{2k+1}) \end{aligned}$$

and hence

$$\begin{aligned} q(Tx_{2k}, Tx_{2k+1}) &\leq \frac{[r_1(x_0) + r_2(x_0)]}{1 - r_2(x_0)} q(Tx_{2k-1}, Tx_{2k}) \\ &= \xi q(Tx_{2k-1}, Tx_{2k}) \\ &\leq \xi^2 q(Tx_{2k-2}, Tx_{2k-1}) \\ &\leq \xi^n q(Tx_0, Tx_1) \end{aligned}$$

where $\xi = \frac{[r_1(x_0)+r_2(x_0)]}{1-r_2(x_0)} < 1$. Note that

$$\begin{aligned} q(TPx_{2k-1}, TQx_{2k}) &= q(Tx_{2k}, Tx_{2k-1}) \\ &= \xi q(Tx_{2k-1}, Tx_{2k}) \end{aligned} \tag{6}$$

Let $k > n \geq 1$. Then it follows that,

$$\begin{aligned} q(Tx_{2k}, Tx_{2n})) &\leq q(d(x_{2k}, x_{2k+1}) + q(Tx_{2k+1}, Tx_{2n+2} + \cdots + q(Tx_{2k-1}, x_{2k}] \\ &\leq \xi^{2k} + \xi^{2k+1} + \xi^{2k+2} + \cdots + \xi^{2n})(Tx_0, Tx_1) \\ &\leq \frac{\xi^{2k}}{1-\xi} q(Tx_0, Tx_1) \to \infty, \ as\ n \to \infty. \end{aligned} \tag{7}$$

Thus, Lemma 2.9(iii) shows that $\{Tx_{2k}\}$ is Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $x_{2k} \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_{2k}\}$ has convergent sequence. So, there are $x^* \in X$ and $\{x_{2ki}\}$ such that $x_{2ki} \to x^*$ as $i \to \infty$. Since T is continuous. So, we obtain $\lim_{n\to\infty} Tx_{2ki} \to Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_{2k}, Tx^*) \leq \frac{\lambda^k}{1-\lambda} q(Tx_0, Tx_1) \tag{8}$$

Now using (6),we have

$$\begin{aligned} q(Tx_k, TPx^*) &= q(TPx_{2k-1}, TPx^*) \\ &\leq hq(Tx_{2k-1}, Tx^*) \\ &\leq \xi \frac{\lambda^{2k-1}}{1-\lambda} q(Tx_0, Tx_1) \\ &\leq \frac{\xi^{2k}}{1-\xi} q(Tx_0, Tx_1). \end{aligned} \tag{9}$$

By Lemma 2.9(i), (8) and (9),we have $Tx^* = TPx^*$. Since T is one to one. So, then $x^* = Px^*$. Thus x^* is fixed point of P. Similarly, we can prove that x^* is fixed point of Q. Therefore, x^* is common fixed point of P and Q.
Moreover, suppose that $u = Pu = Qu$, then we have

$$\begin{aligned} q(Tu, Tu) &\leq q(TPu, TQu) \\ &\leq r_1(u)q(Tu, Tu) + r_2(u)[q(TPu, Tu) + q(TQu, Tu)] \\ &= r_1(u)q(Tu, Tu) + r_2(u)[q(Tu, Tu) + q(Tu, Tu)] \\ &= [(r_1 + 2r_2)](x_0)q(Tu, Tu). \end{aligned}$$

Since $[(r_1+2r_2)](x_0) < 1$, Lemma 2.6(1) shows that $q(Tu, Tu) = \theta$. Next, we prove that the uniqueness of the common fixed point. Suppose that, there is another common fixed point of y^* of P and Q. Then we have

$$\begin{aligned} q(Tx^*, Ty^*) &\leq q(TPx^*, TQy^*) \\ &\leq r_1(x^*)q(Tx^*, Ty^*) + r_2(x^*)[q(TPx^*, Ty^*) + q(TQy^*, Ty^*)] \\ &= r_1(x^*)q(Tx^*, Ty^*) + r_2(x^*)[q(Tx^*, Ty^*) + q(Ty^*, Ty^*)] \\ &= (r_1 + 2r_2)(x^*)q(Tx^*, Ty^*). \end{aligned}$$

Since $(r_1 + 2r_2)(x^*) < 1$, then by Lemma 2.6(1), we have $q(Tx^*, Ty^*) = \theta$. Also, we have $q(Tx^*, Tx^*) = \theta$. Thus 2.9(i), $Tx^* = Ty^*$. Since T is one to one.So, then $x^* = Px^* = Qx^*$. Therefore, the common fixed point is unique. This completes the proof of the theorem.

Corollary 3.2. *Let (X, d) be a complete cone metric space, P be solid cone and q be c-distance on X. and $T : X \to X$ be an one to one, continuous function and subsequentilly convergent and $P, Q : X \to X$ be a pair of mappings. In addition, suppose that there exists mapping $r_1, r_2 : X \to [0, 1)$ such that the following conditions hold:*

(a) $r_1(Px) \leq r_1(x), r_2(Px) \leq r_2(x)$, for all $x \in X$;

(b) $(r_1 + 2r_2)(x) < 1$ for all $x \in X$;

(c) $q(TPx, TQy) \leq r_1(x)q(Tx, Ty) + r_2(x)[q(TPx, Tx) + q(TQy, Ty)]$

for all $x, y \in X$ Then the mapping P and Q have a unique common fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{P^{2k}x\}$and $\{Q^{2k+1}x\}$ converges to the common fixed point. If $u = Pu = Qu$. Then $q(Tu, Tu) = \theta$.

Theorem 3.3. *Let (X, d) be a complete cone metric space, P be solid cone and q be c-distance on X and $T : X \to X$ be an one to one, continuous function and subsequentilly convergent and $P, Q : X \to X$ be a pair of mappings. In addition, suppose that there exists mapping $r_1, r_2 : X \to [0, 1)$ such that the following conditions hold:*

(a) $r_1(Px) \leq r_1(x), r_2(Px) \leq r_2(x)$, for all $x \in X$;

(b) $(r_1 + 2r_2)(x) < 1$ for all $x \in X$;

(c) $q(TPx, TQy) \leq r_1(x)q(Tx, Ty) + r_2(x)[q(TQy, Tx) + q(TPx, Ty)]$
$+r_3(x)[q(TPx, Tx) + q(TQy, Ty)]$

forall $x, y \in X$ Then the mapping P and Q have a unique common fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{P^{2k}x\}$and $\{Q^{2k+1}x\}$ converges to the common fixed point. If $u = Pu = Qu$. Then $q(Tu, Tu) = \theta$.

Proof. Choose $x_0 \in X$.We define the iterative sequences $\{x_{2k}\}$ and $\{x_{2n+1}\}$ by

$$x_{2k+1} = Px_{2K} = P^{2K}x_0, \qquad x_{2K+2} = Qx_{2K+1} = Q^{2K+1}x_0. \tag{10}$$

Then we have

$$\begin{aligned}
&q(Tx_{2k}, Tx_{2k+1})) = q(TPx_{2k-1}, TQx_{2k})) \\
&\leq r_1(x_{2k-1})q(Tx_{2k-1}, Tx_{2k}) + r_2(x_{2k-1})[q(TQx_{2k}, Tx_{2k-1}) + q(TPx_{2k-1}, Tx_{2k})] \\
&+ r_3(x_{2k-1})[q(TPx_{2k-1}, Tx_{2k-1}) + q(TQx_{2k}, Tx_{2k})] \\
&= r_1(Px_{2k-1})q(Tx_{2k-1}, Tx_{2k}) + r_2(Px_{2k-1})[q(Tx_{2k+1}, Tx_{2k-1}) + q(Tx_{2k}, Tx_{2k})] \\
&+ r_3(Px_{2k-1})[q(Tx_{2k}, Tx_{2k-1}) + q(Tx_{2k+1}, Tx_{2k})] \\
&= r_1(x_{2k-2})q(Tx_{2k-1}, Tx_{2k}) + r_2(x_{2k-2})[q(Tx_{2k+1}, Tx_{2k-1}) + q(Tx_{2k}, Tx_{2k+1})] \\
&+ r_3(Px_{2k-1})[q(Tx_{2k}, Tx_{2k-1}) + q(Tx_{2k+1}, Tx_{2k})] \\
&\leq r_1(x_{2k-2})q(Tx_{2k-1}, Tx_{2k}) + r_2(x_{2k-2})[q(Tx_{2k-1}, Tx_{2k}) + q(Tx_{2k}, Tx_{2k+1})] \\
&+ r_3(x_{2k-2})[q(Tx_{2k-1}, Tx_{2k}) + q(Tx_{2k}, Tx_{2k+1})]
\end{aligned}$$

Continuing in this manner, we can get

$$\begin{aligned}
q(Tx_{2k}, Tx_{2k+1})) \leq\ & [r_1(x_0) + r_2(x_0) + r_3(x_0)]q(Tx_{2k-1}, Tx_{2k}) \\
& + [r_2(x_0) + r_2(x_0)]q(Tx_{2k}, Tx_{2k+1})
\end{aligned}$$

and hence

$$\begin{aligned}
q(Tx_{2k}, Tx_{2k+1}) &\leq \frac{[r_1(x_0) + r_2(x_0) + r_3(x_0)]}{1 - r_2(x_0) - r_3(x_0)} q(Tx_{2k-1}, Tx_{2k}) \\
&= \xi q(Tx_{2k-1}, Tx_{2k}) \\
&\leq \xi^2 q(Tx_{2k-2}, Tx_{2k-1}) \\
&\leq \xi^n q(Tx_0, Tx_1)
\end{aligned}$$

where $\xi = \frac{[r_1(x_0)+r_2(x_0)+r_3(x_0)]}{1-r_2(x_0)-r_3(x_0)} < 1$. Note that

$$\begin{aligned}
q(TPx_{2k-1}, TQx_{2k}) &= q(Tx_{2k}, Tx_{2k-1}) \\
&= \xi q(Tx_{2k-1}, Tx_{2k})
\end{aligned} \tag{11}$$

Let $k > n \geq 1$. Then it follows that,

$$\begin{aligned}
q(Tx_{2k}, Tx_{2n})) &\leq q(d(x_{2k}, x_{2k+1}) + q(Tx_{2k+1}, Tx_{2n+2} + \cdots + q(Tx_{2k-1}, x_{2k}] \\
&\leq \xi^{2k} + \xi^{2k+1} + \xi^{2k+2} + \cdots + \xi^{2n})(Tx_0, Tx_1) \\
&\leq \frac{\xi^{2k}}{1-\xi} q(Tx_0, Tx_1) \to \infty, as\ n \to \infty. \qquad (12)
\end{aligned}$$

Thus, Lemma 2.9(iii) shows that $\{Tx_{2k}\}$ is Cauchy sequence in X. Since X is complete, there exists $v \in X$ such that $x_{2k} \to v$ as $n \to \infty$. Since T is subsequentially convergent, $\{x_{2k}\}$ has convergent sequence. So, there are $x^* \in X$ and $\{x_{2ki}\}$ such that $x_{2ki} \to x^*$ as $i \to \infty$. Since T is continuous. So, we obtain $\lim\limits_{n\to\infty} Tx_{2ki} \to Tx^*$. The uniqueness of the limit implies that $Tx^* = v$. Then by (q_3), we have

$$q(Tx_{2k}, Tx^*) \leq \frac{\lambda^k}{1-\lambda} q(Tx_0, Tx_1) \qquad (13)$$

Now using (6),we have

$$\begin{aligned}
q(Tx_k, TPx^*) &= q(TPx_{2k-1}, TPx^*) \\
&\leq hq(Tx_{2k-1}, Tx^*) \\
&\leq \xi \frac{\lambda^{2k-1}}{1-\lambda} q(Tx_0, Tx_1) \\
&\leq \frac{\xi^{2k}}{1-\xi} q(Tx_0, Tx_1). \qquad (14)
\end{aligned}$$

By Lemma 2.9(i), (8) and (9), we have $Tx^* = TPx^*$. Since T is one to one. So, then $x^* = Px^*$. Thusx^* is fixed point of P. Similarly, we can prove that x^* is fixed point of Q. Therefore, x^* is common fixed point of P and Q.
Moreover, suppose that $u = Pu = Qu$, then we have

$$\begin{aligned}
&q(Tu, Tu) \leq q(TPu, TQu) \\
&\leq r_1(u)q(Tu, Tu) + r_2(u)[q(TQu, Tu) + q(TPu, Tu)] + r_3(u)[q(TPu, Tu) + q(TQu, Tu)] \\
&= r_1(u)q(Tu, Tu) + r_2(u)[q(Tu, Tu) + q(Tu, Tu)] + r_3(u)[q(Tu, Tu) + q(Tu, Tu)] \\
&= [(r_1 + 2r_2 + 2r_3)](x_0)q(Tu, Tu).
\end{aligned}$$

Since $[(r_1 + 2r_2 + 2r_3)](x_0) < 1$, Lemma 2.6(1) shows that $q(Tu, Tu) = \theta$. Next, we prove that the uniqueness of the common fixed point. Suppose that, there is another common fixed point of y^* of P and Q. Then we have

$$\begin{aligned} q(Tx^*, Ty^*) &\leq q(TPx^*, TQy^*) \\ &\leq r_1(x^*)q(Tx^*, Ty^*) + r_2(x^*)[q(TQy^*, Tx^*) + q(TPx^*, Ty^*)] \\ &+ r_3(x^*)[q(TPx^*, Tx^*) + q(TQy^*, Ty^*)] \\ &= r_1(x^*)q(Tx^*, Ty^*) + r_2(x^*)[q(Ty^*, Tx^*) + q(Tx^*, Ty^*)] \\ &+ r_3(x^*)[q(Tx^*, Tx^*) + q(Ty^*, Ty^*)] \\ &= (r_1 + 2r_2)(x^*)q(Tx^*, Ty^*) \\ &\leq (r_1 + 2r_2 + 2r_3)(x^*)q(Tx^*, Ty^*) \end{aligned}$$

Since $(r_1 + 2r_2 + 2r_3)(x^*) < 1$, then by Lemma 2.6(1), we have $q(Tx^*, Ty^*) = \theta$. Also, we have $q(Tx^*, Tx^*) = \theta$. Thus 2.9(i), $Tx^* = Ty^*$.Since T is one to one. So, then $x^* = Px^* = Qx^*$. Therefore, the common fixed point is unique.

Theorem 3.4. *Let (X, d) be a complete cone metric space, P be solid cone and q be c-distance on X. and $T : X \to X$ be an one to one, continuous function and subsequentilly convergent and $P, Q : X \to X$ be a pair of mappings. In addition, suppose that there exists mapping $r_1, r_2, r_3, r_4 : X \to [0, 1)$ such that the following conditions hold:*

(a) $r_1(Px) \leq r_1(x), r_2(Px) \leq r_2(x)$, *for all* $x \in X$;

(b) $(r_1 + r_2 + r_3 + 2r_4)(x) < 1$ *for all* $x \in X$;

(c) $q(TPx, TQy) \leq r_1(x)q(Tx, Ty) + r_2(x)q(TPx, Tx) + r_3(x)q(TQy, Ty)]$
$+r_4(x)[q(TPx, Ty) + q(TQy, Tx)]$

for all $x, y \in X$ Then the mapping P and Q have a unique common fixed point $x^* \in X$ and for any $x \in X$, iterative sequence $\{P^{2k}x\}$ and $\{Q^{2k+1}x\}$ converges to the common fixed point. If $u = Pu = Qu$. Then $q(Tu, Tu) = \theta$.

Proof. The proof of this theorem is same as previous theorem.

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