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# On Bilateral Bailey Transform and its Applications

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**Abstract:** In this paper, making use of bilateral Bailey transform, certain interesting transformations of basic bilateral hypergeometric functions have been established.

**Key words and phrases:** Bailey transform, Bilateral Bailey transform, basic bilateral hypergeometric function and transformation formula.
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## 1. Introduction, Notations and Definitions

Throughout this paper we shall adopt the following notations and definitions

For any number a and q, real or complex and $|q| < 1$,

$$[\alpha;q]_n = [\alpha]_n = \begin{cases} (1-\alpha)(1-\alpha q)(1-\alpha q^2)...(1-\alpha q^{n-1}); & n > 0 \\ 1; & n = 0 \end{cases} \tag{1.1}$$

Accordingly, we have

$$[\alpha;q]_\infty = \prod_{r=0}^{\infty}(1-\alpha q^r)$$

Also

$$[a_1, a_2, ..., a_r; q]_n = [a_1;q]_n[a_2;q]_n...[a_r;q]_n.$$

and

$$[a;q]_{-n} = \frac{q^{n(n+1)/2}}{(-a)^n[q/a;q]_n} \tag{1.2}$$

Following Gasper and Rahman [2] we define a basic hypergeometric series,

$$ {}_r\Phi_s\left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix}\right]$$

$$= \sum_{n=0}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n\{(-)^n q^{n(n-1)/2}\}^{1+s-r}}{[q, b_1, b_2, ..., b_s; q]_n}, \tag{1.3}$$

where $0 < |q| < 1$ and $r \leq s+1$.

We define a basic bilateral hypergeometric function as,

$$
{}_r\Psi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right]
$$

$$
= \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, ..., a_r; q]_n z^n \{(-)^n q^{n(n-1)/2}\}^{s-r}}{[b_1, b_2, ..., b_s; q]_n}, \tag{1.4}
$$

where $|b_1.b_2....b_s/a_1.a_2....a_r| < |z| < 1$ and $r \leq s$.

Bailey in 1949 established a simple but very useful transformation, known as Bailey's Lemma, viz.,
If

$$
\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.5}
$$

and

$$
\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.6}
$$

then under suitable convergence conditions,

$$
\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.7}
$$

where $u_r, v_r, \alpha_r$ and $\delta_r$ are functions of r alone.

In order to prove certain false theta identities found in the Lost Notebook of Ramanujan, Andrews and Warnaar [1], in 2007 established two bilateral versions of Bailey's Lemma,

**(i) Symmetric bilateral Bailey's Transform**
If

$$
\beta_n = \sum_{r=-n}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.8}
$$

and

$$
\gamma_n = \sum_{r=|n|}^{\infty} \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.9}
$$

then under suitable convergence conditions,

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.10}$$

where $\alpha_r, \beta_r, \gamma_r$ and $\delta_r$ are functions of r alone.

**(ii) Asymmetric bilateral Bailey's Transform**

Let $m = max(n, -n-1)$
and if

$$\beta_n = \sum_{r=-n-1}^{n} \alpha_r u_{n-r} v_{n+r+1} \tag{1.11}$$

and

$$\gamma_n = \sum_{r=|m|}^{\infty} \delta_r u_{r-n} v_{r+n+1} \tag{1.12}$$

then under suitable convergence conditions,

$$\sum_{n=-\infty}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.13}$$

where $u_r, v_r, \alpha_r$ and $\delta_r$ are functions of r alone.

Making use of bilateral Bailey Transforms (1.8)-(1.10) and (1.11)-(1.13), Singh [4] established certain transformations for basic bilateral hypergeometric functions. The main aim of the present paper is to establish certain transformations for basic bilateral hypergeometric functions by making use of symmetric and asymmetric bilateral Bailey transforms and the following known results due to MacMahon [3],

$$\sum_{r=-n}^{n} \frac{(-)^r z^r q^{r(r-1)/2}}{[q;q]_{n-r}[q;q]_{n+r}} = \frac{[z, q/z; q]_n}{[q;q]_{2n}} \tag{1.14}$$

[MacMahon 3; p.75]

and

$$\sum_{r=-n-1}^{n} \frac{(-)^r z^{-r} q^{r^2+r}}{[q^2;q^2]_{n-r}[q^2;q^2]_{n+r+1}} = \frac{[z;q^2]_{n+1}[q^2/z;q^2]_n}{[q^2;q^2]_{2n+1}} \tag{1.15}$$

[MacMahon 3; p.75]

## 2. Proof of (1.14) and (1.15)

In the first place, we shall provide the proof of the above two results of MacMahon. To prove (1.14), we consider the left side of it which can be put in the form,

$$\frac{1}{[q;q]_n^2}\ {}_1\Psi_1\left[\begin{array}{l} q^{-n};q;zq^n \\ q^{n+1} \end{array}\right] = \frac{1}{[q;q]_n^2}\frac{[q,q^{1+2n},z,q/z;q]_\infty}{[q^{1+n},q^{1+n},zq^n,q^{n+1}/z;q]_\infty} \tag{2.1}$$

by using the ${}_1\Psi_1$ summation of Ramanujan. On simplification, we get the result.

Similarly, the left side of (1.15) can be put as,

$$\frac{1}{[q^2;q^2]_n[q^2;q^2]_{n+1}}\ {}_1\Psi_1\left[\begin{array}{l} q^{-2n};q^2;q^{2n+2}/z \\ q^{2n+1} \end{array}\right]$$

from which the result follows with the help of ${}_1\Psi_1$ summation of Ramanujan.

## 3. Main Results

In this section we shall establish our main transformations. We confine to a few cases to illustrate the method.

**(a)** Taking $u_r = v_r = \dfrac{1}{[q;q]_r}$ and $\alpha_r = (-)^r z^r q^{r(r-1)/2}$ in (1.8) and making use of (1.14), we get

$$\beta_n = \frac{[z,q/z;q]_\infty}{[q;q]_{2n}}. \tag{3.1}$$

Again, setting $\delta_r = [\alpha,\beta;q]_r(q/\alpha\beta)^r$ in (1.9), we get, after some simplification

$$\gamma_n = \frac{[q/\alpha,q/\beta;q]_\infty[\alpha,\beta;q]_n(q/\alpha\beta)^n}{[q,q/\alpha\beta;q]_\infty[q/\alpha,q/\beta;q]_n}. \tag{3.2}$$

with the help of a known result (cf. Gasper-Rahman [2; App II(II.8), p. 236]).

Now, substituting the above values of $\alpha_n,\beta_n,\gamma_n$ and $\delta_n$ in (1.10), we get after some simplifications, the following interesting transformation of a bilateral series into an unilateral series

$${}_2\Psi_3\left[\begin{array}{l} \alpha,\beta;q;zq/\alpha\beta \\ q/\alpha,q/\beta,0 \end{array}\right] = \frac{[q,q/\alpha\beta;q]_\infty}{[q/\alpha,q/\beta;q]_\infty}\ {}_4\Phi_3\left[\begin{array}{l} \alpha,\beta,z,q/z;q;q/\alpha\beta \\ -q,\sqrt{q},-\sqrt{q} \end{array}\right], \tag{3.3.}$$

where $|zq/\alpha\beta| < 1$.

**(b)** Next, taking $u_r = v_r = \dfrac{1}{[q^2;q^2]_r}$ and $\alpha_r = (-)^r z^{-r} q^{r^2+r}$ in (1.11), we get with the help of (1.15),

$$\beta_n = \frac{[z,q^2]_{n+1}[q^2/z;q^2]_n}{[q^2;q^2]_{2n+1}}. \tag{3.4}$$

Again, taking m=n and $\delta_r = [\alpha, \beta; q^2]_r (q^4/\alpha\beta)^r$ in (1.12), we get after some simplifications

$$\gamma_n = \frac{[q^4/\alpha, q^4/\beta; q^2]_\infty [\alpha, \beta; q^2]_n (q^4/\alpha\beta)^n}{[q^2, q^4/\alpha\beta; q^2]_\infty [q^4/\alpha, q^4/\beta; q^2]_n} \tag{3.5}$$

The substitution of the above values of $\alpha_n, \beta_n, \gamma_n$ and $\delta_n$ in (1.13), leads to the yet another interesting transformation of a bilateral series into an unilateral series,

$$ {}_2\Psi_3 \left[ \begin{array}{l} \alpha, \beta; q^2; q^6/\alpha\beta z \\ q^4/\alpha, q^4/\beta; 0 \end{array} \right]$$

$$= \frac{[q^2, q^4/\alpha\beta; q^2]_\infty (1-z)}{[q^4/\alpha, q^4/\beta; q^2]_\infty (1-q^2)} \, {}_4\Phi_3 \left[ \begin{array}{l} zq^2, q^2/z, \alpha, \beta; q^2; q^4/\alpha\beta \\ -q^2, q^{3/2}, -q^{3/2} \end{array} \right] \tag{3.6}$$

## 4. Special Cases

In this section we shall deduce few special cases of the results established in the previous section.

**(i)** If we set z=q in (3.3), we get

$$ {}_2\Psi_3 \left[ \begin{array}{l} \alpha, \beta; q; q^2\alpha\beta \\ q/\alpha, q/\beta, 0 \end{array} \right] = \frac{[q, q/\alpha\beta; q]_\infty}{[q/\alpha, q/\beta; q]_\infty} \tag{4.1}$$

**(ii)** If we let $\alpha, \beta \to \infty$ in (4.1), we get the following well known Euler's identity

$$\sum_{r=-\infty}^{\infty} (-)^r q^{r(3r+1)/2} = [q; q]_\infty \tag{4.2}$$

**(iii)** Agian, with $\alpha = -1$ and $\beta = -q$, (4.1) leads to

$$\sum_{r=-\infty}^{\infty} (-)^r q^{r(r+1)/2} = 0 \tag{4.3}$$

**(iv)** Next, with $\alpha = \beta = -1$, (4.1) again leads to,

$$\sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(r+3)/2}}{(1+q^r)^2} = \frac{[q; q]_\infty^2}{4[-q; q]_\infty^2} \tag{4.4}$$

**(v)** Further, if we let $\beta \to \infty$ in (4.1), we get

$$\sum_{r=-\infty}^{\infty} \frac{q^{r(r+1)} [\alpha; q]_r}{\alpha^r [q/\alpha; q]_r} = \frac{[q; q]_\infty}{[q/\alpha; q]_\infty} \tag{4.5}$$

**(vi)** If we set $\alpha = -1$ in the above, we get

$$\sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(r+1)}}{1+q^r} = \frac{[q;q]_\infty}{2[-q;q]_\infty} \tag{4.6}$$

Now, compairing (4.4) and (4.6), we get the following intersting identity,

$$\sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(r+3)/2}}{(1+q^r)^2} = \left\{ \sum_{r=-\infty}^{\infty} \frac{(-)^r q^{r(r+1)}}{1+q^r} \right\}^2 \tag{4.7}$$

**(vii)** Again, with $\alpha = \beta = -q$ in (4.1), we get

$$\sum_{r=-\infty}^{\infty} (-)^r (1+q^r)^2 q^{r(r-1)/2} = 0 \tag{4.8}$$

**(viii)** Next, if we take z=1 in (3.6), we get

$${}_2\Psi_3 \left[ \begin{matrix} \alpha, \beta; q^2; q^6/\alpha\beta \\ q^4/\alpha, q^4/\beta, 0 \end{matrix} \right] = 0 \tag{4.9}$$

and with $z = q^2$ (3.6) yields

$${}_2\Psi_3 \left[ \begin{matrix} \alpha, \beta; q^2; q^6/\alpha\beta \\ q^4/\alpha, q^4/\beta, 0 \end{matrix} \right] = \frac{[q^2, q^4/\alpha\beta; q^2]_\infty}{[q^4/\alpha, q^4/\beta; q^2]_\infty} \tag{4.10}$$

If we let $\alpha, \beta \to \infty$ in the above relation, we again get Euler's identity (4.2) with q replaced by $q^2$.

**(ix)** Further, if we take z=-1 in (3.6) and then let $\alpha, \beta \to \infty$, we get on simplification

$$\sum_{r=0}^{\infty} \frac{[-q^2;q^2]_r q^{2r(r+1)}}{[q^2;q^2]_r [q^3;q^4]_r} = \frac{[q^{12};q^{12}]_\infty [-q^6;q^6]_\infty}{[q^4;q^4]_\infty} \tag{4.11}$$

**(x)** Lastly, if we take $z = -1$ in (3.6) and then $\alpha = q^{3/2}$ and $\beta = -q^{3/2}$ in it, we get the following interesting q series identity,

$$\sum_{r=-\infty}^{\infty} \frac{(-)^r [q^3;q^4]_r q^{r(r+2)}}{[q^{10};q^4]_r} = \frac{2[q^4, -q; q^2]_\infty}{[q^{10};q^4]_r} \sum_{r=0}^{\infty} \frac{[-q^2;q^2]_r (-q)^r}{[q^2;q^2]_r} \tag{4.12}$$

It is evident that a number of other special cases of our main results can also be deduced similarly.

## References

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