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T-FUZZY SUBBIGROUPS AND NORMAL T-FUZZY SUBBIGROUPS OF BIGROUPS

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Abstract: In this article, we present the idea of fuzzy subbigroups by using t-norm T and some interesting results of them are given. By utilizing this new idea, we further introduce the notion normal fuzzy subbigroups and characterizations of them are explored. Next we investigate the intersection of them and we obtain some new results about them. Finally, we consider the image and pre image of them under group homomorphisms.

Keywords and Phrases: Groups, bigroups, fuzzy set theory, fuzzy groups, norms, homomorphisms.

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1. Introduction

Fuzzy mathematics forms a branch of mathematics related to fuzzy set theory and fuzzy logic. It started in 1965 after the publication of Zadeh's seminal work Fuzzy sets [41]. Usually, a fuzzification of mathematical concepts is based on a generalization of these concepts from characteristic functions to membership functions. Fuzzy subgroupoids and fuzzy subgroups were introduced in 1971 by Rosenfeld [39]. Hundreds of papers on related topics have been published. Recent results and references can be found in [10] and [4]. In mathematics, a t-norm (also T-norm or, unabbreviated, triangular norm) is a kind of binary operation used in the framework of probabilistic metric spaces and in multi-valued logic, specifically in fuzzy logic. A t-norm generalizes intersection in a lattice and conjunction in logic. The name triangular norm refers to the fact that in the framework of probabilistic metric spaces t-norms are used to generalize triangle inequality of ordinary metric spaces. The author by using norms, investigated some properties of fuzzy algebraic structures [15-38]. In this work, we introduce the concept of fuzzy subbigroups of a bigroup G by using t-norm T (T-fuzzy subbigroups of bigroup G). We investigate some properties of them and show the relationship between T-fuzzy subbigroups of bigroup G and subgroups of G. Next, we define the intersection of two T-fuzzy subbigroups of bigroup G and prove that intersection of any family of T-fuzzy subbigroups of bigroup G is also T-fuzzy subbigroup of bigroup G. Also we define normal of two T-fuzzy subbigroups of bigroup G and we obtain that intersection of any family of normal T-fuzzy subbigroups of bigroup G is also normal T-fuzzy subbigroup of bigroup G. Finally, we investigate T-fuzzy subbigroups of bigroup G and normal T-fuzzy subbigroups of bigroup G under homomorhisms of groups and we prove that image and pre-image of T-fuzzy subbigroups of bigroup G (normal T-fuzzy subbigroups of bigroup G) is also T-fuzzy subbigroups of bigroup G (normal T-fuzzy subbigroups of bigroup G).

2. Preliminaries

In this section we recall some of the fundamental concepts and definition, which are necessary for this paper. For details we refer readers to [1, 2, 3, 6, 7, 8, 9, 11, 13, 14, 15, 39, 40].

Proposition 2.1. *Let* $(G, \bullet)$ *be a group and* H *be a non-empty subset of* G. *Then* H *is a subgroup of* G *if and only if* $x, y \in H$ *implies* $x \bullet y^{-1} \in H$ *for all* x, y.

Definition 2.2. it Let G be a group and H be subgroup of G. We say that H is a normal subgroup of G, if we have $gH = Hg$ for all $g \in G$.

Definition 2.3. *A set* $(G, +, \circ)$ *with two binary operations* $+$ *and* $\circ$ *is called a bigroup if there exist two proper subsets* G_1 *and* G_2 *of* G *such that*
(1) $G = G_1 \cup G_2$,
(2) $(G_1, +)$ *is a group and*
(3) $(G_2, \circ)$ *is a group.*

Definition 2.4. *A non-empty subset* H *of a bigroup* $(G, +, \circ)$ *is called a subbigroup if* H *itself is a bigroup under the operations* $+$ *and* $\circ$ *defined on* G.

Definition 2.5. *A* t*-norm* T *is a function* $T : [0, 1] \times [0, 1] \to [0, 1]$ *having the following four properties:*
(T1) $T(x, 1) = x$ *(neutral element),*

(T2) $T(x,y) \leq T(x,z)$ *if* $y \leq z$ *(monotonicity),*
(T3) $T(x,y) = T(y,x)$ *(commutativity),*
(T4) $T(x,T(y,z)) = T(T(x,y),z)$ *(associativity),*
for all $x,y,z \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1,y_1) \geq T(x_2,y_2)$.

Example 2.6. (1) Standard intersection T-norm $T_m(x,y) = \min\{x,y\}$.
(2) Bounded sum T-norm $T_b(x,y) = \max\{0, x+y-1\}$.
(3) algebraic product T-norm $T_p(x,y) = xy$.
(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum T-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product T-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \frac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_{\min}(x,y)$ for all $x,y \in [0,1]$.

We say that T is idempotent if for all $x \in [0,1]$,$T(x,x) = x$.

Lemma 2.7. *Let T be a t-norm. Then*

$$T(T(x,y),T(w,z)) = T(T(x,w),T(y,z)),$$

for all $x,y,w,z \in [0,1]$.

Definition 2.8. *Let X be a non–empty set. A fuzzy subset μ of X is a function* $\mu : X \to [0,1]$.

Definition 2.9. *Let μ be a fuzzy subset of a group $(G,\bullet)$. Then μ is called a fuzzy subgroup of $(G,\bullet)$ under a t-norm T iff for all $x,y \in G$*
(1) $\mu(x \bullet y) \geq T(\mu(x),\mu(y))$
(2) $\mu(x^{-1}) \geq \mu(x)$.
Denote by T-fuzzy subgroup of $(G,\bullet)$, the set of all fuzzy subgroups of $(G,\bullet)$ under a t-norm T.

Definition 2.10. *Let μ_1, μ_2 be two T-fuzzy subgroups of $(G, \bullet)$ and $x \in G$. We define*
(1) $\mu_1 \subseteq \mu_2$ iff $\mu_1(x) \leq \mu_2(x)$,
(2) $\mu_1 = \mu_2$ iff $\mu_1(x) = \mu_2(x)$,
(3)$(\mu_1 \cap \mu_2)(x) = T(\mu_1(x), \mu_2(x))$.

Proposition 2.11. *Let μ_1, μ_2 be two T-fuzzy subgroups of G. Then $\mu_1 \cap \mu_2$ will be T-fuzzy subgroup of G.*

Definition 2.12. *Let $f : G \to H$ be a map and $\mu : G \to [0,1]$ and $\nu : H \to [0,1]$. Following [12] $f(\mu) : H \to [0,1]$ and $f^{-1}(\nu) : G \to [0,1]$, defined by $\forall y \in H$, $f(\mu)(y) = \sup\{\mu(x) \mid x \in G, f(x) = y\}$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \emptyset$. Also $\forall x \in G$, $f^{-1}(\nu)(x) = \nu(f(x))$.*

3. Main Results

Definition 3.1. *Let $G = (G_1 \cup G_2, +, \circ)$ be a bigroup. Then a fuzzy set $\mu : G \to [0,1]$ is said to be a T-fuzzy subbigroup of bigroup G if there exist two fuzzy subsets $\mu_1 : G_1 \to [0,1]$ and $\mu_2 : G_2 \to [0,1]$ such that:*
(1) μ_1 is a T-fuzzy subgroup of $(G_1, +)$,
(2) μ_2 is a T-fuzzy subgroup of $(G_2, \circ)$ and
(3) $\mu = \mu_1 \cup \mu_2$.

Example 3.2. Consider the bigroup $G = \{\pm i, 0, \pm 1, \pm 2, \pm 3, ...\}$ under the binary operation $+$ and $\circ$ where $G_1 = \{0, \pm 1, \pm 2, \pm 3, ...\}$ and $G_2 = \{\pm 1, \pm i\}$. Define $\mu : G \to [0,1]$ by

$$\mu(x) = \begin{cases} 0.65 & \text{if } x = \pm i \\ 1 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.50 & \text{if } x \in \{\pm 1, \pm 3, ...\} \end{cases}$$

and $\mu_1 : G_1 \to [0,1]$ by

$$\mu_1(x) = \begin{cases} 1 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.50 & \text{if } x \in \{\pm 1, \pm 3, ...\}. \end{cases}$$

and

$\mu_2 : G_2 \to [0,1]$ by

$$\mu_2(x) = \begin{cases} 0.65 & \text{if } x = \pm i \\ 0.50 & \text{if } x \in \pm 1. \end{cases}$$

Let T be an algebraic product T-norm $T_p(a,b) = ab$ for all $a, b \in [0,1]$. Then μ_1 and μ_2 will be T-fuzzy subgroup of $(G_1, +)$ and $(G_2, \circ)$ respectively. Thus $\mu = \mu_1 \cup \mu_2$ will be a T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$.

Proposition 3.3. *If $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of a bigroup $G = (G_1 \cup G_2, +, \circ)$. Then*
(1) $\mu_1(-x_1) = \mu_1(x_1)$ such that $-x_1$ is an inverse element of x_1 in $(G_1, +)$.
(2) If T be idempotent t-norm, then $\mu_1(x_1) \leq \mu_1(e_{G_1})$,
(3) $\mu_2(x_2^{-1}) = \mu_2(x_2)$ and
(4) If T be idempotent t-norm, then $\mu_2(x_2) \leq \mu_2(e_{G_2})$
for all $x_1 \in G_1$ and $x_2 \in G_2$.
Proof. Let $x_1 \in G_1, x_2 \in G_2$ and μ_1 and μ_2 be two T-fuzzy subgroups of $(G_1, +)$ and $(G_2, \circ)$ respectively. Then
(1) $\mu_1(x_1) = \mu_1(-(-x_1)) \geq \mu_1(-x_1) \geq \mu_1(x_1)$ and so $\mu_1(-x_1) = \mu_1(x_1)$.
(2) Let T be idempotent t-norm. Now

$$\begin{aligned}\mu_1(e_{G_1}) &= \mu_1(x_1 - x_1)\\ &= \mu_1(x_1 + (-x_1))\\ &\geq T(\mu_1(x_1), \mu_1(-x_1))\\ &= T(\mu_1(x_1), \mu_1(x_1))\\ &= \mu_1(x_1).\end{aligned}$$

Thus $\mu_1(x_1) \leq \mu_1(e_{G_1})$.
(3) $\mu_2(x_2) = \mu_2((x_2^{-1})^{-1}) \geq \mu_2(x_2^{-1}) \geq \mu_2(x_2)$ and then $\mu_2(x_2^{-1}) = \mu_2(x_2)$.
(4) If T be idempotent t-norm, then

$$\begin{aligned}\mu_2(e_{G_2}) &= \mu_2(x_2 \circ x_2^{-1})\\ &\geq T(\mu_2(x_2), \mu_2(x_2^{-1}))\\ &= T(\mu_2(x_2), \mu_2(x_2))\\ &= \mu_2(x_2).\end{aligned}$$

Then $\mu_2(x_2) \leq \mu_2(e_{G_2})$.

Proposition 3.4. *If $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$ and T be idempotent t-norm. Then*
(1) $\mu_1(x_1 - y_1) = \mu_1(e_{G_1})$ gives us that $\mu_1(x_1) = \mu_1(y_1)$ for all $x_1, y_1 \in G_1$.
(2) $\mu_2(x_2 \circ y_2^{-1}) = \mu_2(e_{G_2})$ implies that $\mu_2(x_2) = \mu_2(y_2)$ for all $x_2, y_2 \in G_2$.
Proof. (1) Let $x_1, y_1 \in G_1$ and μ_1 be T-fuzzy subgroup of $(G_1, +)$ such that T be idempotent t-norm. Then

$$\begin{aligned}\mu_1(x_1) &= \mu_1(x_1 - y_1 + y_1)\\ &\geq T(\mu_1(x_1 - y_1), \mu_1(y_1))\\ &= T(\mu_1(e_{G_1}), \mu_1(y_1))\\ &\geq T(\mu_1(y_1), \mu_1(y_1))\end{aligned}$$

$$
\begin{aligned}
&= \mu_1(y_1) \\
&= \mu_1(y_1 - x_1 + x_1) \\
&= \mu_1(x_1 - (x_1 - y_1)) \\
&\geq T(\mu_1(x_1), \mu_1(x_1 - y_1)) \\
&= T(\mu_1(x_1), \mu_1(e_{G_1})) \\
&\geq T(\mu_1(x_1), \mu_1(x_1)) \\
&= \mu_1(x_1).
\end{aligned}
$$

Then $\mu_1(x_1) = \mu_1(y_1)$.
(2) Let $x_2, y_2 \in G_2$ and μ_2 be T-fuzzy subgroup of $(G_2, \circ)$ and T be idempotent t-norm. Now

$$
\begin{aligned}
\mu_2(x_2) &= \mu_2(x_2 \circ y_2^{-1} \circ y_2) \\
&\geq T(\mu_2(x_2 \circ y_2^{-1}), \mu_2(y_2)) \\
&= T(\mu_2(e_{G_2}), \mu_2(y_2)) \\
&\geq T(\mu_2(y_2), \mu_2(y_2)) \\
&= \mu_2(y_2) \\
&= \mu_2(y_2 \circ x_2^{-1} \circ x_2) \\
&= \mu_2((x_2 \circ y_2^{-1})^{-1} \circ x_2) \\
&\geq T(\mu_2((x_2 \circ y_2^{-1})^{-1}), \mu_2(x_2)) \\
&= T(\mu_2(x_2 \circ y_2^{-1}), \mu_2(x_2)) \\
&= T(\mu_2(e_{G_2}), \mu_2(x_2)) \\
&\geq T(\mu_2(x_2), \mu_2(x_2)) \\
&= \mu_2(x_2).
\end{aligned}
$$

Therefore $\mu_2(x_2) = \mu_2(y_2)$.

Proposition 3.5. *Let $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of a bigroup $G = (G_1 \cup G_2, +, \circ)$.*
(1) If $\mu_1(x_1 - y_1) = 1$, then $\mu_1(x_1) = \mu_1(y_1)$ for all $x_1, y_1 \in G_1$.
(2) If $\mu_2(x_2 \circ y_2^{-1}) = 1$, then $\mu_2(x_2) = \mu_2(y_2)$ for all $x_2, y_2 \in G_2$.
Proof. (1) Let $x_1, y_1 \in G_1$. Then

$$
\begin{aligned}
\mu_1(x_1) &= \mu_1(x_1 - y_1 + y_1) \\
&\geq T(\mu_1(x_1 - y_1), \mu_1(y_1)) \\
&= T(1, \mu_1(y_1)) \\
&= \mu_1(y_1)
\end{aligned}
$$

$$\begin{aligned}&= \mu_1(-y_1)\\&= \mu_1(-x_1 + x_1 - y_1)\\&\geq T(\mu_1(-x_1), \mu_1(x_1 - y_1))\\&= T(\mu_1(-x_1), 1)\\&= \mu_1(-x_1)\\&= \mu_1(x_1).\end{aligned}$$

Thus $\mu_1(x_1) = \mu_1(y_1)$.
(2) If $x_2, y_2 \in G_2$, then

$$\begin{aligned}\mu_2(x_2) &= \mu_2(x_2 \circ y_2^{-1} \circ y_2)\\&\geq T(\mu_2(x_2 \circ y_2^{-1}), \mu_2(y_2))\\&= T(1, \mu_2(y_2))\\&= \mu_2(y_2)\\&= \mu_2(y_2^{-1})\\&= \mu_2(x_2^{-1} \circ x_2 \circ y_2^{-1})\\&\geq T(\mu_2(x_2^{-1}), \mu_2(x_2 \circ y_2^{-1}))\\&= T(\mu_2(x_2^{-1}), 1)\\&= \mu_2(x_2^{-1})\\&= \mu_2(x_2).\end{aligned}$$

Thus $\mu_2(x_2) = \mu_2(y_2)$.

Proposition 3.6. *If $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$. Then*
(1) $H_1 = \{x_1 \in G_1 \mid \mu_1(x_1) = 1\}$ is either empty or a subgroup of $(G_1, +)$.
(2) $H_2 = \{x_2 \in G_2 \mid \mu_2(x_2) = 1\}$ is either empty or a subgroup of $(G_2, \circ)$.
(3) $H = H_1 \cup H_2$ is either empty or a subbigroup of G.
Proof. If H_1 and H_2 be empty, then $H = H_1 \cup H_2$ will be empty.
(1) Let $x_1, y_1 \in H_1$ then $\mu_1(x_1) = \mu_1(y_1) = 1$. As μ_1 is a T-fuzzy subgroup of $(G_1, +)$, so $\mu_1(x_1 - y_1) \geq T(\mu_1(x_1), \mu_1(-y_1)) = T(\mu_1(x_1), \mu_1(y_1)) = T(1, 1) = 1$. Thus $\mu_1(x_1 - y_1) = 1$ and then $x_1 - y_1 \in H_1$ and then H_1 will be subgroup of $(G_1.+)$.
(2) If $x_2, y_2 \in H_2$, then $\mu_2(x_2) = \mu_2(y_2) = 1$. Since μ_2 is a T-fuzzy subgroup of $(G_2, \circ)$, so $\mu_2(x_2 \circ y_2^{-1}) \geq T(\mu_2(x_2), \mu_2(y_1^{-1})) = T(\mu_2(x_2), \mu_2(y_2)) = T(1, 1) = 1$. This implies that $\mu_2(x_2 \circ y_2^{-1}) = 1$ and so $x_2 \circ y_2^{-1} \in H_2$ and H_2 will be subgroup of G_2.

(3) From (1) and (2) we have that H_1 and H_2 are subgroup of $(G_1, +)$ and $(G_2, \circ)$ respectively. Then $H = H_1 \cup H_2$ will be a subbigroup of $G = (G_1 \cup G_2, +, \circ)$.

Proposition 3.7. *If $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of a bigroup $G = (G_1 \cup G_2, +, \circ)$ and T be idempotent t-norm. Then*
(1) $H_1 = \{x_1 \in G_1 \mid \mu_1(x_1) = \mu_1(e_{G_1})\}$ is a subgroup of $(G_1, +)$.
(2) $H_2 = \{x_2 \in G_2 \mid \mu_2(x_2) = \mu_2(e_{G_2})\}$ is a subgroup of $(G_2, \circ)$.
(3) $H = H_1 \cup H_2$ is a subbigroup of G.
Proof. (1) Since $e_{G_1} \in H_1$ so H_1 is not empty. Let $x_1, y_1 \in H_1$ then $\mu_1(x_1) = \mu_1(y_1) = \mu_1(e_{G_1})$. From part(2) Proposition 3.3 we get that $\mu_1(x_1 - y_1) \leq \mu_1(e_{G_1})$. Now as μ_1 is a T-fuzzy subgroup of $(G_1, +)$, so $\mu_1(x_1 - y_1) \geq T(\mu_1(x_1), \mu_1(-y_1)) = T(\mu_1(x_1), \mu_1(y_1)) = T(\mu_1(e_{G_1}), \mu_1(e_{G_1})) = \mu_1(e_{G_1})$ and then $\mu_1(x_1 - y_1) \geq \mu_1(e_{G_1})$. Therefore $\mu_1(x_1 - y_1) = \mu_1(e_{G_1})$ so that $x_1 - y_1 \in H_1$ and then H_1 will be subgroup of $(G_1, +)$.
(2) We know that $e_{G_2} \in H_2$ then H_2 is not empty. If $x_2, y_2 \in H_2$, then $\mu_2(x_2) = \mu_2(y_2) = \mu_2(e_{G_2})$. Part(4) Proposition 3.3 give us that $\mu_2(x_2 \circ y_2^{-1}) \leq \mu_1(e_{G_1})$. Since μ_2 is a T-fuzzy subgroup of $(G_2, \circ)$, so $\mu_2(x_2 \circ y_2^{-1}) \geq T(\mu_2(x_2), \mu_2(y_2^{-1})) = T(\mu_2(x_2), \mu_2(y_2)) = T(\mu_2(e_{G_2}), \mu_2(e_{G_2})) = \mu_2(e_{G_2})$ and then $\mu_2(x_2 \circ y_2^{-1}) \geq \mu_2(e_{G_2})$. Therefore $\mu_2(x_2 \circ y_2^{-1}) = \mu_2(e_{G_2})$ so that $x_2 \circ y_2^{-1} \in H_2$ and then H_2 will be subgroup of $(G_2, \circ)$.
(3) By (1) and (2) we obtained that H_1 and H_2 are subgroup of $(G_1, +)$ and $(G_2, \circ)$ respectively. Then $H = H_1 \cup H_2$ will be a subbigroup of $G = (G_1 \cup G_2, +, \circ)$.

Proposition 3.8. *If $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of a bigroup $G = (G_1 \cup G_2, +, \circ)$ and T be idempotent t-norm. Then*
(1) $\mu_1(x_1 + y_1) = T(\mu_1(x_1), \mu_1(y_1))$ for all $x_1, y_1 \in G_1$ such that $\mu_1(x_1) \neq \mu_1(y_1)$.
(2) $\mu_2(x_2 \circ y_2) = T(\mu_2(x_2), \mu_2(y_2))$ for all $x_2, y_2 \in G_2$ such that $\mu_2(x_2) \neq \mu_2(y_2)$.
Proof. (1) Let $x_1 \in G_1$ such that for all $y_1 \in G_1$ we have that $\mu_1(y_1) < \mu_1(x_1) \leq 1$. Then

$$\mu_1(y_1) = T(\mu_1(y_1), \mu_1(y_1)) \leq T(\mu_1(x_1), \mu_1(y_1)) \leq T(\mu_1(y_1), 1) = \mu_1(y_1)$$

and so $\mu_1(y_1) = T(\mu_1(x_1), \mu_1(y_1))$. Now

$$\begin{aligned}
\mu_1(y_1) &= \mu_1(-x_1 + x_1 + y_1) \\
&\geq T(\mu_1(-x_1), \mu_1(x_1 + y_1)) \\
&= T(\mu_1(x_1), \mu_1(x_1 + y_1)) \\
&\geq T(\mu_1(x_1 + y_1), \mu_1(x_1 + y_1)) \\
&= \mu_1(x_1 + y_1) \\
&\geq T(\mu_1(x_1), \mu_1(y_1))
\end{aligned}$$

$$= \mu_1(y_1).$$

Therefore $\mu_1(x_1 + y_1) = \mu_1(y_1) = T(\mu_1(x_1), \mu_1(y_1))$.
(2) Let $x_2 \in G_2$ such that for all $y_2 \in G_2$ we have that $\mu_2(y_2) < \mu_2(x_2) \leq 1$. Then $\mu_2(y_2) = T(\mu_2(y_2), \mu_2(y_2)) \leq T(\mu_2(x_2), \mu_2(y_2)) \leq T(\mu_2(y_2), 1) = \mu_2(y_2)$ and so $\mu_2(y_2) = T(\mu_2(x_2), \mu_2(y_2))$. Then

$$\begin{aligned}
\mu_2(y_2) &= \mu_2(x_2^{-1} \circ x_2 \circ y_2) \\
&\geq T(\mu_2(x_2^{-1}), \mu_2(x_2 \circ y_2)) \\
&= T(\mu_2(x_2), \mu_2(x_2 \circ y_2)) \\
&\geq T(\mu_2(x_2 \circ y_2), \mu_2(x_2 \circ y_2)) \\
&= \mu_2(x_2 \circ y_2) \\
&\geq T(\mu_2(x_2), \mu_2(y_2)) \\
&= \mu_2(y_2).
\end{aligned}$$

Thus $\mu_2(x_2 \circ y_2) = \mu_2(y_2) = T(\mu_2(x_2), \mu_2(y_2))$.

Proposition 3.9. *If $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroup of a bigroup $G = (G_1 \cup G_2, +, \circ)$ and T be idempotent t-norm.*
(1) Let $x_1 \in G_1$ then $\mu_1(x_1 + y_1) = \mu_1(y_1)$ if and only if $\mu_1(x_1) = \mu_1(e_{G_1})$ for all $y_1 \in G_1$.
(2) Let $x_2 \in G_2$ then $\mu_2(x_2 \circ y_2) = \mu_2(y_2)$ if and only if $\mu_2(x_2) = \mu_2(e_{G_2})$ for all $y_2 \in G_2$.
Proof. (1) Necessity: let $x_1 \in G_1$ and $\mu_1(x_1 + y_1) = \mu_1(y_1)$ for all $y_1 \in G_1$. Now set $y_1 = e_{G_1}$ then $\mu_1(x_1 + e_{G_1}) = \mu_1(e_{G_1})$ and so $\mu_1(x_1) = \mu_1(e_{G_1})$.
Sufficiency: assume that $\mu_1(x_1) = \mu_1(e_{G_1})$ for all $y_1 \in G_1$ then by Proposition 3.3 (part 2) we get that $\mu_1(x_1) = \mu_1(e_{G_1}) \geq \mu_1(y_1), \mu_1(x_1 + y_1)$. Now

$$\begin{aligned}
\mu_1(x_1 + y_1) &\geq T(\mu_1(x_1), \mu_1(y_1)) \\
&\geq T(\mu_1(y_1), \mu_1(y_1)) \\
&= \mu_1(y_1) \\
&= \mu_1(-x_1 + x_1 + y_1) \\
&\geq T(\mu_1(-x_1), \mu_1(x_1 + y_1)) \\
&= T(\mu_1(x_1), \mu_1(x_1 + y_1)) \\
&\geq T(\mu_1(x_1 + y_1), \mu_1(x_1 + y_1)) \\
&= \mu_1(x_1 + y_1).
\end{aligned}$$

Thus $\mu_1(x_1 + y_1) = \mu_1(y_1)$.
(2) Necessity: assume $x_2 \in G_2$ and $\mu_2(x_2 \circ y_2) = \mu_2(y_2)$ for all $y_2 \in G_2$. Now if we

let $y_2 = e_{G_2}$, then $\mu_2(x_2 \circ e_{G_2}) = \mu_2(e_{G_2})$ and therefore $\mu_2(x_2) = \mu_2(e_{G_2})$.
Sufficiency: as $\mu_2(x_2) = \mu_2(e_{G_2})$ for all $y_2 \in G_2$ so by Proposition 3.3 (part 4) we obtain that $\mu_2(x_2) = \mu_2(e_{G_2}) \geq \mu_2(y_2), \mu_2(x_2 \circ y_2)$. Thus

$$\begin{aligned}\mu_2(x_2 \circ y_2) &\geq T(\mu_2(x_2), \mu_2(y_2))\\ &\geq T(\mu_2(y_2), \mu_2(y_2))\\ &= \mu_2(y_2)\\ &= \mu_2(x_2^{-1} \circ x_2 \circ y_2)\\ &\geq T(\mu_2(x_2^{-1}), \mu_2(x_2 \circ y_2))\\ &= T(\mu_2(x_2), \mu_2(x_2 \circ y_2))\\ &\geq T(\mu_2(x_2 \circ y_2), \mu_2(x_2 \circ y_2))\\ &= \mu_2(x_2 \circ y_2).\end{aligned}$$

Thus $\mu_2(x_2 \circ y_2) = \mu_2(y_2)$.

Definition 3.10. *Let $\mu = \mu_1 \cup \mu_2$ and $\nu = \nu_1 \cup \nu_2$ be two T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$. Define the intersection μ and ν by $\beta = \mu \cap \nu = (\mu_1 \cup \mu_2) \cap (\nu_1 \cup \nu_2) = (\mu_1 \cap \nu_1) \cup (\mu_2 \cap \nu_2) = \beta_1 \cup \beta_2$ such that $\beta_1 = \mu_1 \cap \nu_1 : G_1 \to [0, 1]$ and $\beta_2 = \mu_2 \cap \nu_2 : G_2 \to [0, 1]$.*

Now we prove that the intersection of two T-fuzzy subbigroups is also T-fuzzy subbigroup.

Proposition 3.11. *Let $\mu = \mu_1 \cup \mu_2$ and $\nu = \nu_1 \cup \nu_2$ be two T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$. Then $\beta = \mu \cap \nu = \beta_1 \cup \beta_2$ will be T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$ such that $\beta_1 = \mu_1 \cap \nu_1 : G_1 \to [0, 1]$ and $\beta_2 = \mu_2 \cap \nu_2 : G_2 \to [0, 1]$.*

Proof. (1) We prove that $\beta_1 = \mu_1 \cap \nu_1 : G_1 \to [0, 1]$ is T-fuzzy subgroup of $(G_1, +)$. Now for all $x_1, y_1 \in G_1$ we have that
(a)

$$\begin{aligned}\beta_1(x_1 + y_1) &= (\mu_1 \cap \nu_1)(x_1 + y_1)\\ &= T(\mu_1(x_1 + y_1), \nu_1(x_1 + y_1))\\ &\geq T(T(\mu_1(x_1), \mu_1(y_1)), T(\nu_1(x_1), \nu_1(y_1)))\\ &= T(T(\mu_1(x_1), \nu_1(x_1)), T(\mu_1(y_1), \nu_1(y_1)))\\ &= T((\mu_1 \cap \nu_1)(x_1), (\mu_1 \cap \nu_1)(y_1))\\ &= T(\beta_1(x_1), \beta_1(y_1)).\end{aligned}$$

Then $\beta_1(x_1 + y_1) \geq T(\beta_1(x_1), \beta_1(y_1))$.

(b)

$$\begin{aligned}\beta_1(-x_1) &= (\mu_1 \cap \nu_1)(-x_1)\\ &= T(\mu_1(-x_1), \nu_1(-x_1))\\ &\geq T(\mu_1(x_1), \nu_1(x_1))\\ &= (\mu_1 \cap \nu_1)(x_1)\\ &= \beta_1(x_1).\end{aligned}$$

So $\beta_1(-x_1) \geq \beta_1(x_1)$.

Therefore (a) and (b) give us that β_1 will be T-fuzzy subgroup of $(G_1, +)$.
(2) We show that $\beta_2 = \mu_2 \cap \nu_2 : G_2 \to [0, 1]$ can be T-fuzzy subgroup of $(G_2, \circ)$. If $x_2, y_2 \in G_2$, then
(a)

$$\begin{aligned}\beta_2(x_2 \circ y_2)) &= (\mu_2 \cap \nu_2)(x_2 \circ y_2)\\ &= T(\mu_2(x_2 \circ y_2), \nu_2(x_2 \circ y_2))\\ &\geq T(T(\mu_2(x_2), \mu_2(y_2)), T(\nu_2(x_2), \nu_2(y_2)))\\ &= T(T(\mu_2(x_2), \nu_2(x_2)), T(\mu_2(y_2), \nu_2(y_2)))\\ &= T((\mu_2 \cap \nu_2)(x_2), (\mu_2 \cap \nu_2)(y_2))\\ &= T(\beta_2(x_2), \beta_2(y_2)).\end{aligned}$$

Then $\beta_2(x_2 \circ y_2)) \geq T(\beta_2(x_2), \beta_2(y_2))$.
(b)

$$\begin{aligned}\beta_2(x_2^{-1}) &= (\mu_2 \cap \nu_2)(x_2^{-1})\\ &= T(\mu_2(x_2^{-1}), \nu_2(x_2^{-1}))\\ &\geq T(\mu_2(x_2), \nu_2(x_2))\\ &= (\mu_2 \cap \nu_2)(x_2)\\ &= \beta_2(x_2)\end{aligned}$$

then $\beta_2(x_2^{-1}) \geq \beta_2(x_2)$. Thus from (a) and (b) we obtain that β_2 will be T-fuzzy subgroup of $(G_2, \circ)$.

Corollary 3.12. *The intersection of family of T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$ is a T-fuzzy subbigroup of $G = (G_1 \cup G_2, +, \circ)$.*

Definition 3.13. *Let $\mu = \mu_1 \cup \mu_2$ be a T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$. We say that $\mu = \mu_1 \cup \mu_2$ is normal if for all $x_1, x_2 \in G_1$ and*

$x_2, x_2 \in G_2$ we have that $\mu_1(x_1 + y_1 - x_1) = \mu_1(y_1)$ and $\mu_2(x_2 \circ y_2 \circ x_2^{-1}) = \mu_2(y_2)$.

Example 3.14. Let $G_1 = (\mathbb{Z}, +)$ and $G_2 = (\mathbb{R} - 0, \circ)$ be two groups. Then $G = (G_1 \cup G_2, +, \circ)$ will be subbigroup. Define $\mu_1 : G_1 \to [0, 1]$ by

$$\mu_1(x) = \begin{cases} 0.60 & \text{if } x \in \{0, \pm 2, \pm 4, ...\} \\ 0.50 & \text{if } x \in \{\pm 1, \pm 3, ...\} \end{cases}$$

and

$\mu_2 : G_2 \to [0, 1]$ by

$$\mu_2(x) = \begin{cases} 0.65 & \text{if } x \in \{\pm 2, \pm 4, ...\} \\ 0.50 & \text{if } x \in \{\pm 1, \pm 3, ...\}. \end{cases}$$

Let T be a Bounded sum T-norm $T_b(a, b) = \max\{0, a + b - 1\}$ for all $a, b \in [0, 1]$. Then $\mu = \mu_1 \cup \mu_2$ will be a T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$. Also since for all $x_1, x_2 \in G_1$ and $x_2, x_2 \in G_2$ we have that $\mu_1(x_1 + y_1 - x_1) = \mu_1(y_1)$ and $\mu_2(x_2 \circ y_2 \circ x_2^{-1}) = \mu_2(y_2)$ so $\mu = \mu_1 \cup \mu_2$ will be a normal T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$.

We claim that the intersection of two normal T-fuzzy subbigroups is also normal T-fuzzy subbigroup.

Proposition 3.15. *Let $\mu = \mu_1 \cup \mu_2$ and $\nu = \nu_1 \cup \nu_2$ be two normal T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$. Then $\beta = \mu \cap \nu = \beta_1 \cup \beta_2$ will be normal T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$ such that $\beta_1 = \mu_1 \cap \nu_1 : G_1 \to [0, 1]$ and $\beta_2 = \mu_2 \cap \nu_2 : G_2 \to [0, 1]$.*

Proof. As Proposition 3.11 $\beta = \mu \cap \nu = \beta_1 \cup \beta_2$ is T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$. Let $x_1, y_1 \in G_1$. Then

$$\begin{aligned} \beta_1(x_1 + y_1 - x_1) &= (\mu_1 \cap \nu_1)(x_1 + y_1 - x_1) \\ &= T(\mu_1(x_1 + y_1 - x_1), \nu_1(x_1 + y_1 - x_1)) \\ &= T(\mu_1(y_1), \nu_1(y_1)) \\ &= (\mu_1 \cap \nu_1)(y_1) \\ &= \beta_1(y_1). \end{aligned}$$

Also if $x_1, y_1 \in G_2$, then

$$\begin{aligned} \beta_2(x_1 \circ y_1 \circ x_1^{-1}) &= (\mu_2 \cap \nu_2)(x_1 \circ y_1 \circ x_1^{-1}) \\ &= T(\mu_2(x_1 \circ y_1 \circ x_1^{-1}), \nu_2(x_1 \circ y_1 \circ x_1^{-1})) \\ &= T(\mu_2(y_1), \nu_1(y_1)) \\ &= (\mu_2 \cap \nu_2)(y_1) \\ &= \beta_2(y_1). \end{aligned}$$

Corollary 3.16. *The intersection of family of normal T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$ is a T-fuzzy subbigroup of $G = (G_1 \cup G_2, +, \circ)$.*

4. Homomorphisms and T-fuzzy Subbigroups of Bigroups

In this section we investigate T-fuzzy subbigroups of bigroups under homomorphisms.

Definition 4.1. *Let $\mu = \mu_1 \cup \mu_2$ and $\nu = \nu_1 \cup \nu_2$ be two T-fuzzy subbigroups of bigroup $G = (G_1 \cup G_2, +, \circ)$ and $H = (H_1 \cup H_2, +, \circ)$ respectively and $f : G = (G_1 \cup G_2, +, \circ) \to H = (H_1 \cup H_2, +, \circ)$ be a mapping. Define*

$$f(\mu) = f(\mu_1 \cup \mu_2) = f(\mu_1) \cup f(\mu_2) : (H_1 \cup H_2, +, \circ) \to [0, 1]$$

by

$$\begin{aligned} f(\mu)(y_1, y_2) &= f(\mu_1 \cup \mu_2)(y_1, y_2) \\ &= (f(\mu_1) \cup f(\mu_2))(y_1, y_2) \\ &= \sup\{\mu(x_1, x_2) \mid x_1 \in G_1, x_2 \in G_2, f(x_1) = y_1, f(x_2) = y_2\} \\ &= \sup\{(\mu_1 \cup \mu_2)(x_1, x_2) \mid x_1 \in G_1, x_2 \in G_2, f(x_1) = y_1, f(x_2) = y_2\} \\ &= \sup\{\mu_1(x_1) \cup \mu_2(x_2) \mid x_1 \in G_1, x_2 \in G_2, f(x_1) = y_1, f(x_2) = y_2\} \end{aligned}$$

for all $y_1 \in H_1$ and $y_2 \in H_2$ with $f^{-1}(y_1), f^{-1}(y_2) \neq \emptyset$.
Also for all $x_1 \in G_1$ and $x_2 \in G_2$ define

$$\begin{aligned} f^{-1}(\nu)(x_1, x_2) &= f^{-1}(\nu_1 \cup \nu_2)(x_1, x_2) \\ &= f^{-1}(\nu_1)(x_1) \cup f^{-1}(\nu_2)(x_2) \\ &= \nu_1(f(x_1)) \cup \nu_2(f(x_2)). \end{aligned}$$

Proposition 4.2. *Let $\mu = \mu_1 \cup \mu_2$ be T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$ and $H = (H_1 \cup H_2, +, \circ)$ be a bigroup. If $f : G \to H$ be a group epimomorphism(surjective homomorphism), then $f(\mu) = f(\mu_1) \cup f(\mu_2)$ will be T-fuzzy subbigroup of bigroup $H = (H_1 \cup H_2, +, \circ)$.*

Proof. Let $y_1, y_2 \in H_1$ and $x_1, x_2 \in G_1$ with $f^{-1}(y_1), f^{-1}(y_2) \neq \emptyset$ and $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

(1) We must prove that $f(\mu_1) : (H_1, +) \to [0, 1]$ is a T-fuzzy subgroup of $(H_1, +)$. As μ_1 is T-fuzzy subgroup of $(G_1, +)$ so

$$\begin{aligned} f(\mu_1)(y_1 + y_2) &= \sup\{\mu_1(x_1 + x_2) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &\geq \sup\{T(\mu_1(x_1), \mu_1(x_2)) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &= T(\sup\{\mu_1(x_1) \mid y_1 = f(x_1)\}, \sup\{\mu_1(x_2) \mid y_2 = f(x_2)\}) \\ &= T(f(\mu_1)(y_1), f(\mu_1)(y_2)) \end{aligned}$$

and

$$\begin{aligned} f(\mu_1)(-y_1) &= \sup\{\mu_1(-x_1) \mid -y_1 = f(-x_1)\} \\ &= \sup\{\mu_1(x_1) \mid -y_1 = -f(x_1)\} \\ &= \sup\{\mu_1(x_1) \mid y_1 = f(x_1)\} \\ &= f(\mu_1)(y_1). \end{aligned}$$

Thus $f(\mu_1)$ will be a T-fuzzy subgroup of $(H_1, +)$.
(2) Let $y_1, y_2 \in H_2$ and $x_1, x_2 \in G_2$ with $f^{-1}(y_1), f^{-1}(y_2) \neq \emptyset$ and $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Now we prove that $f(\mu_2) : (H_2, \circ) \to [0, 1]$ is a T-fuzzy subgroup of $(H_2, \circ)$. Since μ_2 is T-fuzzy subgroup of $(G_2, \circ)$ so we can obtain that

$$\begin{aligned} f(\mu_2)(y_1 \circ y_2) &= \sup\{\mu_2(x_1 \circ x_2) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &\geq \sup\{T(\mu_2(x_1), \mu_2(x_2)) \mid y_1 = f(x_1), y_2 = f(x_2)\} \\ &= T(\sup\{\mu_2(x_1) \mid y_1 = f(x_1)\}, \sup\{\mu_2(x_2) \mid y_2 = f(x_2)\}) \\ &= T(f(\mu_2)(y_1), f(\mu_2)(y_2)) \end{aligned}$$

and

$$\begin{aligned} f(\mu_2)(y_1^{-1}) &= \sup\{\mu_2(x_1^{-1}) \mid y_1^{-1} = f(x_1^{-1})\} \\ &= \sup\{\mu_2(x_1) \mid y_1^{-1} = f(x_1)^{-1}\} \\ &= \sup\{\mu_2(x_1) \mid y_1 = f(x_1)\} \\ &= f(\mu_2)(y_1). \end{aligned}$$

Then $f(\mu_2)$ will be a T-fuzzy subgroup of $(H_2, \circ)$.
Therefore (1) and (2) will give that $f(\mu) = f(\mu_1) \cup f(\mu_2)$ is T-fuzzy subbigroups of bigroup $H = (H_1 \cup H_2, +, \circ)$.

Proposition 4.3. *Let $\nu = \nu_1 \cup \nu_2$ be T-fuzzy subbigroup of bigroup $H = (H_1 \cup H_2, +, \circ)$ and $G = (G_1 \cup G_2, +, \circ)$ be a bigroup. If $f : G \to H$ be a group homomorphism, then $f^{-1}(\nu) = \nu_1(f) \cup \nu_2(f)$ will be T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$.*
Proof. Let $x_1, x_2 \in G_1$.
(1) We prove that $f^{-1}(\nu_1) = \nu_1(f) : G_1 \to [0, 1]$ is a T-fuzzy subgroup of group $(G_1, +)$. Since ν_1 is a T-fuzzy subgroup of group $H = (H_1, +)$ so

$$\begin{aligned} f^{-1}(\nu_1)(x_1 + x_2) &= \nu_1(f(x_1 + x_2)) \\ &= \nu_1(f(x_1) + f(x_2)) \\ &\geq T(\nu_1(f(x_1)), \nu_1(f(x_2))) \\ &= T(f^{-1}(\nu_1)(x_1), f^{-1}(\nu_1)(x_2)). \end{aligned}$$

Also

$$f^{-1}(\nu_1)(-x_1) = \nu_1(f(-x_1)) = \nu_1(-f(x_1)) = \nu_1(f(x_1)) = f^{-1}(\nu_1)(x_1).$$

Thus $f^{-1}(\nu_1) = \nu_1(f)$ is a T-fuzzy subgroup of group $(G_1, +)$.
(2) Now prove that $f^{-1}(\nu_2) = \nu_2(f) : G_2 \to [0, 1]$ is a T-fuzzy subgroup of group $(G_2, \circ)$. Since ν_2 is a T-fuzzy subgroup of group $H = (H_2, \circ)$ then

$$\begin{aligned} f^{-1}(\nu_2)(x_1 \circ x_2) &= \nu_2(f(x_1 \circ x_2)) \\ &= \nu_2(f(x_1) \circ f(x_2)) \\ &\geq T(\nu_2(f(x_1)), \nu_2(f(x_2))) \\ &= T(f^{-1}(\nu_2)(x_1), f^{-1}(\nu_2)(x_2)). \end{aligned}$$

Also

$$f^{-1}(\nu_2)(x_1^{-1}) = \nu_2(f(x_1^{-1})) = \nu_2(f(x_1)^{-1}) = \nu_2(f(x_1)) = f^{-1}(\nu_2)(x_1).$$

Thus $f^{-1}(\nu_2) = \nu_2(f)$ is a T-fuzzy subgroup of group $(G_2, \circ)$.
Now (1) and (2) show that $f^{-1}(\nu) = \nu_1(f) \cup \nu_2(f)$ will be T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$.

Proposition 4.4. *Let $\mu = \mu_1 \cup \mu_2$ be normal T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$ and $H = (H_1 \cup H_2, +, \circ)$ be a bigroup. If $f : G \to H$ be a group epimomorphism(surjective homomorphism), then $f(\mu) = f(\mu_1) \cup f(\mu_2)$ will be normal T-fuzzy subbigroup of bigroup $H = (H_1 \cup H_2, +, \circ)$.*
Proof. By Proposition 4.2 we have that $f(\mu) = f(\mu_1) \cup f(\mu_2)$ is T-fuzzy subbigroup of bigroup $H = (H_1 \cup H_2, +, \circ)$. Let $y_1, y_2 \in H_1$ and $x_1, x_2 \in G_1$ with $f^{-1}(y_1), f^{-1}(y_2) \neq \emptyset$ and $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then $f(\mu_1)(y_1 + y_2 - y_1) = \sup\{\mu_1(g_1) \mid g_1 \in G_1, f(g_1) = y_1 + y_2 - y_1\} = \sup\{\mu_1(g_1) \mid g_1 \in G_1, f(g_1) = f(x_1) + f(x_2) - f(x_1)\} = \sup\{\mu_1(g_1) \mid g_1 \in G_1, f(g_1) = f(x_1 + x_2 - x_1)\} = \sup\{\mu_1(x_1 + x_2 - x_1) \mid g_1 \in G_1, f(g_1) = f(x_2) = y_2\} \sup\{\mu_1(x_2) \mid g_1 \in G_1, f(g_1) = f(x_2 = y_2)\} = f(\mu_1)(y_2)$. Let $y_1, y_2 \in H_2$ and $x_1, x_2 \in G_2$ with $f^{-1}(y_1), f^{-1}(y_2) \neq \emptyset$ and $y_1 = f(x_1)$ and $y_2 = f(x_2)$. Then

$$\begin{aligned} f(\mu_2)(y_1 \circ y_2 \circ y_1^{-1}) &= \sup\{\mu_2(g_2) \mid g_2 \in G_2, f(g_1) = y_1 + y_2 - y_1\} \\ &= \sup\{\mu_2(g_2) \mid g_2 \in G_2, f(g_2) = f(x_1) \circ f(x_2) \circ f(x_1)^{-1}\} \\ &= \sup\{\mu_2(g_2) \mid g_2 \in G_2, f(g_2) = f(x_1 \circ x_2 \circ x_1^{-1})\} \\ &= \sup\{\mu_2(x_1 \circ x_2 \circ x_1^{-1}) \mid g_2 \in G_2, f(g_2) = f(x_2) = y_2\} \\ &= \sup\{\mu_2(x_2) \mid g_2 \in G_2, f(g_2) = f(x_2) = y_2\} \\ &= f(\mu_2)(y_2). \end{aligned}$$

Thus $f(\mu) = f(\mu_1) \cup f(\mu_2)$ will be normal T-fuzzy subbigroup of bigroup $H = (H_1 \cup H_2, +, \circ)$.

Proposition 4.5. *Let $\nu = \nu_1 \cup \nu_2$ be normal T-fuzzy subbigroup of bigroup $H = (H_1 \cup H_2, +, \circ)$ and $G = (G_1 \cup G_2, +, \circ)$ be a bigroup. If $f : G \to H$ be a group homomorphism, then $f^{-1}(\nu) = f^{-1}(\nu_1) \cup f^{-1}(\nu_2)$ will be normal T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$.*

Proof. From Proposition 4.3 we get that $f^{-1}(\nu) = f^{-1}(\nu_1) \cup f^{-1}(\nu_2)$ is T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$. Let $x_1, x_2 \in G_1$. Then

$$\begin{aligned} f^{-1}(\nu_1)(x_1 + x_2 - x_1) &= \nu_1(f(x_1 + x_2 - x_1)) \\ &= \nu_1(f(x_1) + f(x_2) - f(x_1)) \\ &= \nu_1(f(x_2)) \\ &= f^{-1}(\nu_1)(x_2). \end{aligned}$$

Now let $x_1, x_2 \in G_2$ then

$$\begin{aligned} f^{-1}(\nu_2)(x_1 \circ x_2 \circ x_1^{-1}) &= \nu_2(f(x_1 \circ x_2 \circ x_1^{-1})) \\ &= \nu_2(f(x_1) \circ f(x_2) \circ f(x_1)^{-1}) \\ &= \nu_2(f(x_2)) \\ &= f^{-1}(\nu_2)(x_2). \end{aligned}$$

Therefore $f^{-1}(\nu) = f^{-1}(\nu_1) \cup f^{-1}(\nu_2)$ will be normal T-fuzzy subbigroup of bigroup $G = (G_1 \cup G_2, +, \circ)$.

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