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SOME REMARKS ON GENERALIZED SUMMABILITY USING DIFFERENCE OPERATORS ON NEUTROSOPHIC NORMED SPACES

Archana Sharma and Vijay Kumar

Department of Mathematics,
Chandigarh University, Mohali, Punjab, INDIA

E-mail : dr.archanasharma1022@gmail.com, kaushikvjy@gmail.com

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Abstract: For the mth difference operator $\triangle^m$ and the admissible ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$, the purpose of this paper is to introduce generalized summability methods: $\triangle^m(\mathcal{I}_{\mathcal{N}})-$convergence and $\triangle^m(\mathcal{I}^*_{\mathcal{N}})-$convergence in neutrosophic normed spaces (briefly known as NNS). We develop some basics properties of these notions and find condition on $\mathcal{I}$ for which two methods of summability coincides. Finally, we define $\triangle^m(\mathcal{I}_{\mathcal{N}})-$Cauchy sequences in NNS and obtain the Cauchy-convergence criteria in these spaces.

Keywords and Phrases: Neutrosophic normed spaces, statistical convergence, statistical Cauchy, $\mathcal{I}-$convergence and $\mathcal{I}-$Cauchy sequences.

2020 Mathematics Subject Classification: 46S40, 11B39, 03E72, 40G15.

1. Introduction

Statistical convergence as a generalization of usual convergence was introduced by H. Fast [7] and I. J. Schoenberg [24] independently and further developed in [4], [8], [10] and [22] etc. A sequence (x_k) of numbers is said to be statistical convergent to a number L if for each $\varepsilon > 0$, $\lim_n \frac{1}{n} \ |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$. For any set $K \subseteq \mathbb{N}$, the natural density of K is denoted by $\delta(K)$ and is defined by $\lim_n \frac{1}{n} \ |\{k \leq n : k \in K\}|$. Using density, a sequence (x_k) of numbers is said to be statistical convergence to a number L if for each $\varepsilon > 0$, $\delta(K_\varepsilon) = 0$, where $K_\varepsilon = \{k \leq n : |x_k - L| \geq \varepsilon\} \subseteq \mathbb{N}$. The idea is generalized by Kostyrko et al.[15] with the help of an admissible ideal $\mathcal{I}$ called $\mathcal{I}-$convergence. Their idea attracted many mathematicians to work in this direction. For some pioneer work on $\mathcal{I}-$convergence, we refer [5, 6, 9, 14, 16, 17, 18, 19 and 20] etc. On another side, Fuzzy sets were introduced by Zadeh [27] and generalized by Atanassov [1] while observing that Zadeh's idea of fuzzy sets need more attention to handle certain problems in time domain. He called this set as intuitionistic fuzzy set. His work is followed by many authors, for instance intuitionistic fuzzy metric spaces by Park [21], intuitionistic fuzzy topological spaces by Saadati and Park [23] etc. In past decade, the ideas of statistical convergence and $\mathcal{I}-$convergence respectively have been extended in intuitionistic fuzzy normed spaces in [10] and [19]. Smarandache [26], presented a generalization of intuitionistic fuzzy sets and called it neutrosophic set. This idea is further used to define neutrosophic metric spaces and neutrosophic soft linear spaces respectively in [12] and [2]. Further, Bera and Mahapatra [3] introduced the concept of neutrosophic norm and define some sequential concepts like convergence, Cauchy and convexity in these spaces. Recently, Kirişci and Şimşek [13] extended notion of statistical convergence and study its properties in these spaces. We aim in this paper to introduce and study new kind of summability methods: $\triangle^m(\mathcal{I}_{\mathcal{N}})-$convergence and $\triangle^m(\mathcal{I}^*_{\mathcal{N}})-$convergence in NNS. We find conditions on $\mathcal{I}$ for which the two methods coincide. Later, we define some related concepts: $\triangle^m(\mathcal{I}_{\mathcal{N}})-$Cauchy, $\triangle^m(\mathcal{I}_{\mathcal{N}})-$completeness, $\triangle^m(\mathcal{I}^*_{\mathcal{N}})-$Cauchy, $\triangle^m(\mathcal{I}^*_{\mathcal{N}})-$ completeness in NNS and obtain some relationships among these notions.

2. Background and Preliminaries

We quote in this section some definition and results which form the base for present study. We begin with the following definitions of triangular norm and triangular conorm. A binary operation $\circ : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ is said to be a continuous triangular norm or $t-$norm if it satisfies the following conditions: (i) $\circ$ is associative and commutative, (ii) $\circ$ is continuous, (iii) $a \circ 1 = a$ for every $a \in [0,\ 1]$ and (iv) $a \circ b \leq c \circ d$ whenever $a \leq c$ and $b \leq d$ for each a, b, c and $d \in [0,\ 1]$. A binary operation $\bullet : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$ is said to be a continuous triangular conorm or $t-$conorm if it satisfies the following conditions:(i) $\bullet$ is associative and commutative, (ii) $\bullet$ is continuous, (iii) $a \bullet 0 = a$ for every $a \in [0,\ 1]$ and (iv) $a \bullet b \leq c \bullet d$ whenever $a \leq c$ and $b \leq d$ for each a, b, c and $d \in [0,\ 1]$. Using these definitions, Kirişci and Şimşek [13], recently defined neutrosophic normed spaces and studied statistical convergence in these spaces.

Definition 2.1. [13] *Let F be a vector space, $\mathcal{N} = \{\langle \vartheta, \mathcal{G}(\vartheta), \mathcal{B}(\vartheta), \mathcal{Y}(\vartheta)\rangle : \vartheta \in F\}$ be a normed space such that $\mathcal{N} : F \times \mathbb{R}^+ \to [0,\ 1]$ and $\circ$, $\bullet$ respectively are contin-*

uous $t-$norm and continuous $t-$conorm. Then a four touple $V = (F, \mathcal{N}, \circ, \bullet,)$ is called a neutrosophic normed spaces (NNS) if the following conditions are satisfied. For every $u, v \in F$ and $\lambda, \mu > 0$ and for every $\sigma \neq 0$ we have

(i) $0 \leq \mathcal{G}(u, \lambda) \leq 1, 0 \leq \mathcal{B}(u, \lambda) \leq 1, 0 \leq \mathcal{Y}(u, \lambda) \leq 1$ for every $\lambda \in \mathbb{R}^+$;

(ii) $\mathcal{G}(u, \lambda) + \mathcal{B}(u, \lambda) + \mathcal{Y}(u, \lambda) \leq 3$ for $\lambda \in \mathbb{R}^+$;

(iii) $\mathcal{G}(u, \lambda) = 1$ (for $\lambda > 0$) *if and only if* $u = 0$;

(iv) $\mathcal{G}(\sigma u, \lambda) = \mathcal{G}\left(u, \frac{\lambda}{|\sigma|}\right)$;

(v) $\mathcal{G}(u, \mu) \circ \mathcal{G}(v, \lambda) \leq \mathcal{G}(u + v, \lambda + \mu)$;

(vi) $\mathcal{G}(u, .)$ *is continuous non-decreasing function*;

(vii) $\lim_{\lambda \to \infty} \mathcal{G}(u, \lambda) = 1$;

(viii) $\mathcal{B}(u, \lambda) = 0$ (for $\lambda > 0$) *if and only if* $u = 0$;

(ix) $\mathcal{B}(\sigma u, \lambda) = \mathcal{G}\left(u, \frac{\lambda}{|\sigma|}\right)$;

(x) $\mathcal{B}(u, \mu) \bullet \mathcal{B}(v, \lambda) \geq \mathcal{B}(u + v, \lambda + \mu)$;

(xi) $\mathcal{B}(u, .)$ *is continuous non-decreasing function*;

(xii) $\lim_{\lambda \to \infty} \mathcal{B}(u, \lambda) = 0$;

(xiii) $\mathcal{Y}(u, \lambda) = 0$ (for $\lambda > 0$) *if and only if* $u = 0$;

(xiv) $\mathcal{Y}(\sigma u, \lambda) = \mathcal{G}\left(u, \frac{\lambda}{|\sigma|}\right)$;

(xv) $\mathcal{Y}(u, \mu) \bullet \mathcal{B}(v, \lambda) \geq \mathcal{Y}(u + v, \lambda + \mu)$;

(xvi) $\mathcal{Y}(u, .)$ *is continuous non-decreasing function*;

(xvii) $\lim_{\lambda \to \infty} \mathcal{Y}(u, \lambda) = 0$;

If $\lambda \leq 0$, *then* $\mathcal{G}(u, \lambda) = 0$, $\mathcal{B}(u, \lambda) = 1$ and $\mathcal{Y}(u, \lambda) = 1$.

Here, $\mathcal{N}(\mathcal{G}, \mathcal{B}, \mathcal{Y})$ is called the neutrosophic norm. Some examples of neutrosophic normed spaces can be found in [13]. A sequence (a_k) in a neutrosophic normed spaces V is said to convergent if for each $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer m and $\mathcal{L} \in F$ such that $\mathcal{G}(a_k - \mathcal{L}, \lambda) > 1 - \varepsilon$, $\mathcal{B}(a_k - \mathcal{L}, \lambda) < \varepsilon$ and $\mathcal{Y}(a_k - \mathcal{L}, \lambda) < \varepsilon$ for all $k \geq m$. This is equivalent to say that $\lim_{k \to \infty} \mathcal{G}(a_k - \mathcal{L}, \lambda) = 1$, $\lim_{k \to \infty} \mathcal{B}(a_k - \mathcal{L}, \lambda) = 0$ and $\lim_{k \to \infty} \mathcal{Y}(a_k - \mathcal{L}, \lambda) = 0$ and we write in this case $\mathcal{N} - \lim_{k \to \infty} a_k = \mathcal{L}$. Moreover, the sequence (a_k) is said to be Cauchy if for each $\varepsilon > 0$ and $\lambda > 0$, there exists a positive integer p such that $\mathcal{G}(a_k - a_n, \lambda) > 1 - \varepsilon$, $\mathcal{B}(a_k - a_n, \lambda) < \varepsilon$ and $\mathcal{Y}(a_k - a_n, \lambda) < \varepsilon$ for all $k, n \geq p$.

Definition 2.2. [13] *Let V be a NNS; $0 < \varepsilon < 1$ and $\lambda > 0$. A sequence (a_k) in V is said to statistical convergent if there exists $\mathcal{L} \epsilon F$ such that* $\lim_n \frac{1}{n}|\{k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) \leq 1 - \varepsilon$ or $\mathcal{B}(a_k - \mathcal{L}, \lambda) \geq \varepsilon$ and $\mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \varepsilon\}| = 0$; *or equivalently, the natural density of the set* $A((\varepsilon, \lambda) = \{k \leq n : \mathcal{G}(a_k - \mathcal{L}, \lambda) \leq 1 - \varepsilon$ or $\mathcal{B}(a_k - \mathcal{L}, \lambda)$ and $\mathcal{Y}(a_k - \mathcal{L}, \lambda) \geq \varepsilon\}$ *is zero, i.e.,* $\delta(A(\varepsilon, \lambda)) = 0$.

Definition 2.3. [13] *Let V be a NNS; $\varepsilon > 0$ and $\lambda > 0$. A sequence (a_k) in*

V is said to statistical Cauchy if there exists $p \epsilon \mathbb{N}$ *such that* $\lim_n \frac{1}{n}|\{k \leq n : \mathcal{G}(a_k - a_p,\ \lambda) \leq 1-\varepsilon$ or $\mathcal{B}(a_k - a_p,\ \lambda) \geq \varepsilon$ and $\mathcal{Y}(a_k - a_p,\ \lambda) \geq \varepsilon\}| = 0$; *or equivalently, the natural density of the set* $A(\varepsilon,\lambda) = \{k \leq n :\ \mathcal{G}(a_k - a_p,\ \lambda) \leq 1-\varepsilon$ or $\mathcal{B}(a_k - a_p,\ \lambda) \geq \varepsilon$ and $\mathcal{Y}\ (a_k - a_p,\ \lambda) \geq \varepsilon\}$ *is zero, i.e.,* $\delta\left(A\left(\varepsilon,\ \lambda\right)\right) = 0$.

We now give a brief introduction related to $\mathcal{I}-$convergence and related concepts. For any set X, let $\mathcal{P}\left(X\right)$ denotes the power set of X. A family of sets $\mathcal{I} \subseteq \mathcal{P}\left(X\right)$ is called an ideal in X if and only if (i) $\emptyset \in \mathcal{I}$; (ii) $A,\ B \in \mathcal{I}$ implies that $A \cup B \in \mathcal{I}$ and (iii) for each $A \in \mathcal{I}$ and $B \subseteq A,$ we have $B \in \mathcal{I}$. Further, a non-empty family of sets $\mathcal{F} \subseteq \mathcal{P}\left(X\right)$ is called a filter on X if and only if (i) $\emptyset \notin \mathcal{F}$; (ii) $A,\ B \in \mathcal{F}$ implies that $A \cap B \in \mathcal{F}$ and (iii) for each $A \in \mathcal{F}$ and $B \supseteq A,$ we have $B \in \mathcal{F}$. An ideal $\mathcal{I}$ is called non-trivial if $\mathcal{I} \neq \emptyset$ and $X \notin \mathcal{I}$. A non-trivial ideal $\mathcal{I} \subseteq \mathcal{P}\left(X\right)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if it contains $\{\{x\} : x \in X\}$. If $\mathcal{I} \subseteq \mathcal{P}\left(X\right)$ be a non-trivial ideal, then the class $\mathcal{F} = \mathcal{F}\left(\mathcal{I}\right) = \left\{A^C \subseteq X : A \in \mathcal{I}\right\}$ is a filter on X and is called the filter associated with the ideal $\mathcal{I}$.

Let w denotes the set of all sequences in the neutrosophic normed space $V = (F, \mathcal{N},\ \circ, \bullet,\ )$. Define $\triangle^m : w \to w$ by $\triangle^0 a_k = a_k; \triangle^1 a_k = a_k - a_{k+1}\ ; \cdots \triangle^m a_k = \triangle^{m-1}\ (\ \triangle a_k) = \triangle^{m-1}\ (a_k - a_{k+1}),\ m \geq 2$ and for all $k \in \mathbb{N}$. We now turn our attention to the main results.

3. $\Delta^m(\mathcal{I}_\mathcal{N})$-convergence

Definition 3.1. *Let* $\mathcal{I} \subseteq \mathcal{P}\left(\mathbb{N}\right)$ *is an admissible ideal, V be a NNS;* $0 < \varepsilon < 1$ *and* $\lambda > 0$. *A sequence* (a_k) *in V is said to* $\triangle^m(\mathcal{I}_\mathcal{N})-$*convergent to a if there exists* $\mathcal{L} \in F$ *such that* $A(\varepsilon) = \{k \in \mathbb{N} : \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) \leq 1-\varepsilon$ or $\mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \lambda) \geq \varepsilon\} \in \mathcal{I}$. *In this case we write* $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = a$.

With particular choice of $m = 0$ and the ideal $\mathcal{I} = \{K \subseteq \mathbb{N} : K$ is an finite set$\}$, $\triangle^m(\mathcal{I}_\mathcal{N})-$convergent coincides with the statistical convergence of [13] in NNS.

By a lacunary sequence we mean an increasing integer sequence $\theta = (p_r)$ with $p_0 = 0$ and $h_r = p_r - p_{r-1} \to \infty$ as $r \to \infty$. If we denote $I_r = (p_{r-1},\ p_r]$ and $q_r = \frac{p_r}{p_{r-1}}$, then for any set $K \subseteq \mathbb{N}$ the lacunary density of the set K is denoted by $\delta_\theta\left(K\right)$ and is defined by $\delta_\theta\left(K\right) = \frac{1}{h_r}\left|p \in I_r : p \in K\right|$ provided the limit exists. For the choice $m = 0$ and the ideal $\mathcal{I} = \{K \subseteq \mathbb{N} : \delta_\theta\left(K\right) = 0\}$, then $\triangle^m(\mathcal{I}_\mathcal{N})-$convergence in this case coincides with lacunary statistical convergence of [11] in NNS.

For $K \subseteq \mathbb{N}$, K is said to be uniformly dense if $u(K) = \lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{\infty} \chi_K(j + p) = a$ uniformly p or equivalently $\lim_{n\to\infty} \frac{1}{n}\left|K \cap \{p+1,\ p+2,\ \cdots p+n\}\right| = a$ uniformly in p where $p = 0,\ 1,\ 2,\ 3\cdots$ and χ_K is the characteristic function. If we take ideal $\mathcal{I} = \{K \subseteq \mathbb{N} : u\left(K\right) = 0\}$ and $m = 0$, then $\mathcal{I}$ is an admissible ideal and the corresponding $\triangle^m(\mathcal{I}_\mathcal{N})-$convergence coincides with uniform statistical convergence in NNS.

The following lemma is a direct implication of the Definition 3.1 together with properties of ideal and filter.

Lemma 3.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal and V be a NNS, then for every $\varepsilon > 0$ and $\lambda > 0$, the following conditions are equivalent.*

(i) $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = a$.

(ii) $\{k \in \mathbb{N} : \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) \leq 1-\varepsilon\} \in \mathcal{I}$; $\{k \in \mathbb{N} : \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) \geq \varepsilon\} \in \mathcal{I}$ *and* $\{k \in \mathbb{N} : \mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \lambda) \geq \varepsilon\} \in \mathcal{I}$

(iii) $\{k \in \mathbb{N} : \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) > 1-\varepsilon \text{ and } \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) < \varepsilon,\ \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$.

(iv) $\{k \in \mathbb{N} : \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) > 1-\varepsilon\} \in \mathcal{F}(\mathcal{I})$, $\{k \in \mathbb{N} : \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$ *and* $\{k \in \mathbb{N} : \mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \lambda) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$.

(v) $\mathcal{I} - \lim_k \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) = 1$ *and* $\mathcal{I} - \lim_k \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) = 0$, $\mathcal{I} - \lim_k \mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \lambda) = 0$.

We next formulate the following theorem of uniqueness.

Theorem 3.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ be an admissible ideal and V be a NNS. If (a_k) is a sequence in V such that $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = \mathcal{L}_1$ and $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = \mathcal{L}_2$, then $\mathcal{L}_1 = \mathcal{L}_2$.*

Proof. Suppose that $\mathcal{L}_1 \neq \mathcal{L}_2$ and let $\varepsilon > 0$. Choose $\mu > 0$ such that $(1-\varepsilon) \circ (1-\varepsilon) > 1-\mu$ and $\varepsilon \bullet \varepsilon < \mu$. For $\lambda > 0$, we define the following sets. $K_{\mathcal{G}_1}(\varepsilon,\ \lambda) = \{k \in \mathbb{N}{:}\mathcal{G}(\triangle^m a_k - \mathcal{L}_1,\ \frac{\lambda}{2}) \leq 1-\varepsilon\}$, $K_{\mathcal{G}_2}(\varepsilon,\ \lambda) = \{k \in \mathbb{N}{:}\mathcal{G}(\triangle^m a_k - \mathcal{L}_2,\ \frac{\lambda}{2}) \leq 1-\varepsilon\}$, $K_{\mathcal{B}_1}(\varepsilon,\ \lambda) = \{k \in \mathbb{N}{:}\mathcal{B}(\triangle^m a_k - \mathcal{L}_1,\ \frac{\lambda}{2}) \geq \varepsilon\}$, $K_{\mathcal{B}_2}(\varepsilon,\ \lambda) = \{k \in \mathbb{N}{:}\mathcal{B}(\triangle^m a_k - \mathcal{L}_2,\ \frac{\lambda}{2}) \geq \varepsilon\}$, $K_{\mathcal{Y}_1}(\varepsilon,\ \lambda) = \{k \in \mathbb{N}{:}\mathcal{Y}(\triangle^m a_k - \mathcal{L}_1,\ \frac{\lambda}{2}) \geq \varepsilon\}$ and $K_{\mathcal{Y}_2}(\varepsilon,\ \lambda) = \{k \in \mathbb{N}{:}\mathcal{Y}(\triangle^m a_k - \mathcal{L}_2,\ \frac{\lambda}{2}) \geq \varepsilon\}$ Since $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = \mathcal{L}_1$ and $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = \mathcal{L}_2$ so by Lemma 3.1, sets $K_{\mathcal{G}_1}(\varepsilon,\ \lambda);\ K_{\mathcal{B}_1}(\varepsilon,\ \lambda);\ K_{\mathcal{Y}_1}(\varepsilon,\ \lambda)$ and $K_{\mathcal{G}_2}(\varepsilon,\ \lambda);\ K_{\mathcal{B}_2}(\varepsilon,\ \lambda)$ and $K_{\mathcal{Y}_2}(\varepsilon,\ \lambda)$ belongs to $\mathcal{I}$. Define a set $K_\mathcal{N}(\mu,\ \lambda)$ by $K_\mathcal{N}(\mu,\ \lambda) = \{\{\{K_{\mathcal{G}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{G}_2}(\mu,\ \lambda)\}\} \cap \{\{K_{\mathcal{B}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{B}_2}(\mu,\ \lambda)\}\} \cap \{\{K_{\mathcal{Y}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{Y}_2}(\mu,\ \lambda)\}\}\}$; then $K_\mathcal{N}(\mu,\ \lambda) \in \mathcal{I}$ which immediately gives $\{\mathbb{N} - K_\mathcal{N}(\mu,\ \lambda)\} \in \mathcal{F}(\mathcal{I})$. Then $\{\mathbb{N} - K_\mathcal{N}(\mu,\ \lambda)\}$ is a nonempty set as otherwise $\{\mathbb{N} - K_\mathcal{N}(\mu,\ \lambda)\} \in \mathcal{I}$. Let $k \in \{\mathbb{N} - K_\mathcal{N}(\mu,\ \lambda)\}$, then we have the following possibilities: (i) $k \in \mathbb{N} - \{\{K_{\mathcal{G}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{G}_2}(\mu,\ \lambda)\}\}$. (ii) $k \in \mathbb{N} - \{\{K_{\mathcal{B}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{B}_2}(\mu,\ \lambda)\}\}$ and (iii) $k \in \mathbb{N} - \{\{K_{\mathcal{Y}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{Y}_2}(\mu,\ \lambda)\}\}$.

Assume (i) holds, then $k \notin \{K_{\mathcal{G}_1}(\mu,\ \lambda)\} \cup \{K_{\mathcal{G}_2}(\mu,\ \lambda)\}$ which gives $k \notin K_{\mathcal{G}_1}(\mu,\ \lambda)$ and $k \notin K_{\mathcal{G}_2}(\mu,\ \lambda)$. This implies that

$$\mathcal{G}(\triangle^m a_k - \mathcal{L}_1,\ \tfrac{\lambda}{2}) > 1-\varepsilon \quad \text{and} \quad \mathcal{G}(\triangle^m a_k - \mathcal{L}_2,\ \tfrac{\lambda}{2}) > 1-\varepsilon$$

Now,

$$\mathcal{G}(\mathcal{L}_1-\mathcal{L}_2,\lambda)\geq \mathcal{G}(\triangle^m a_k-\mathcal{L}_1,\frac{\lambda}{2})\circ\mathcal{G}(\triangle^m a_k-\mathcal{L}_2,\frac{\lambda}{2})>(1-\varepsilon)\circ(1-\varepsilon)>1-\mu. \quad (3.1)$$

Since μ is arbitrary and (3.1) holds for every $\lambda>0$, it follows that $\mathcal{G}\left(\mathcal{L}_1-\mathcal{L}_2,\ \lambda\right)=1$ and therefore $\mathcal{L}_1=\mathcal{L}_2$.

We now assume (ii) holds, then $k\notin K_{\mathcal{B}_1}\left(\varepsilon,\ \lambda\right)$ and $k\notin K_{\mathcal{B}_2}\left(\mu,\ \lambda\right)$ and therefore we have

$\mathcal{B}(\triangle^m a_k-\mathcal{L}_1,\ \frac{\lambda}{2})<\varepsilon$ and $\mathcal{B}(\triangle^m a_k-\mathcal{L}_2,\ \frac{\lambda}{2})<\varepsilon$.

Now,

$$\mathcal{B}(\mathcal{L}_1-\mathcal{L}_2,\ \lambda)\leq\ \mathcal{B}(\triangle^m a_k-\mathcal{L}_1,\ \frac{\lambda}{2})\bullet\mathcal{B}(\triangle^m a_k-\mathcal{L}_2,\ \frac{\lambda}{2})<\varepsilon\bullet\varepsilon<\mu. \quad (3.2)$$

As μ is arbitrary and (3.2) holds for every $\lambda>0$, we must have $\mathcal{B}\left(\mathcal{L}_1-\mathcal{L}_2,\ \lambda\right)=0$, which gives immediately $\mathcal{L}_1=\mathcal{L}_2$.

Finally, if we assume (iii) holds then as in case (ii) one have $\mathcal{L}_1=\mathcal{L}_2$. This completes the proof of the Theorem.

Theorem 3.2. *Let $\mathcal{I}\subseteq\mathcal{P}\left(\mathbb{N}\right)$ is an admissible ideal, V be a NNS and (a_k), be any sequences in V such that $\mathcal{N}-\lim_k\ (\triangle^m a_k)=\mathcal{L}$, then $\triangle^m(\mathcal{I}_{\mathcal{N}})-\lim_k a_k=\mathcal{L}$.*

Proof. Assume $\mathcal{N}-\lim_k\ (\triangle^m a_k)=\mathcal{L}$. Then for each $\varepsilon>0$ and $\lambda>0$ there exists a positive integer m_0 such that $\mathcal{G}(\triangle^m a_k-\mathcal{L},\ \lambda)>1-\varepsilon$, $\mathcal{B}(\triangle^m a_k-\mathcal{L},\ \lambda)<\varepsilon$ and $\mathcal{Y}(\triangle^m a_k-\mathcal{L},\ \lambda)<\varepsilon$ for all $k\geq\ m_0$. It follows that the set $\{k\in\mathbb{N}:\mathcal{G}(\triangle^m a_k-\mathcal{L},\ \lambda)\leq 1-\varepsilon$ or $\mathcal{B}(\triangle^m a_k-\mathcal{L},\ \lambda)\geq\varepsilon$, $\mathcal{Y}(\triangle^m a_k-\mathcal{L},\ \lambda)\geq\varepsilon\}$; is a finite set and therefore belongs to $\mathcal{I}$. Hence $\triangle^m(\mathcal{I}_{\mathcal{N}})-\lim_k a_k=\mathcal{L}$.

In next Theorem, we give linear property of $\mathcal{I}-$convergence in a NNS.

Theorem 3.3. *Let $\mathcal{I}\subseteq\mathcal{P}\left(\mathbb{N}\right)$ is an admissible ideal, V be a NNS and (a_k), (b_k) be two sequences in V such that $\triangle^m(\mathcal{I}_{\mathcal{N}})-\lim_k a_k=\mathcal{L}_1$ and $\triangle^m(\mathcal{I}_{\mathcal{N}})-\lim_k b_k=\mathcal{L}_2$, then*

(i) $\triangle^m(\mathcal{I}_{\mathcal{N}})-\lim_k(a_k+b_k)=\mathcal{L}_1\ +\mathcal{L}_2$

(ii) $\triangle^m(\mathcal{I}_{\mathcal{N}})-\lim_k(\beta\ a_k)=\beta\ \mathcal{L}_1$ *for* $\beta\neq 0$

Proof. The proof of the Theorem follows parallel lines of Theorem 3.1 so we skipped here.

4. $\mathcal{I}^*-$ convergence

Kirişci and Şimşek [13] proved that *"Let V be a NNS. $S_{\mathcal{N}}-\lim_k a_k=\mathcal{L}$ if and only if there exists a increasing index sequence $J=\{j_1,j_2,j_3\cdots\}\subseteq\mathbb{N}$, while $\delta\left(J\right)=1$, $\mathcal{N}-\lim_k a_{j_k}=\mathcal{L}$."* We use this result together via applying the difference operator $\triangle^m$ to define a new type of convergence called $\triangle^m(I^*)-$

convergence in NNS.

Definition 4.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal and V be a NNS. A sequence (a_k) in V is said to $\triangle^m(\mathcal{I}_{\mathcal{N}}^*)-$convergent to $\mathcal{L}$ if and only if there exist a set $J = \{j_1, j_2, j_3 \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $\mathcal{N} - \lim_n \triangle^m a_{j_n} = \mathcal{L}$. In this case we write $\triangle^m(\mathcal{I}_{\mathcal{N}}^*) - \lim_k a_k = \mathcal{L}$.*

Next Theorem gives relationship between $\triangle^m(\mathcal{I})-$convergence and $\triangle^m(\mathcal{I}^*)-$ convergence in NNS.

Theorem 4.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal, V be a NNS and (a_k) be any sequences in V such that $\triangle^m(\mathcal{I}_{\mathcal{N}}^*) - \lim_k a_k = \mathcal{L}$, then $\triangle^m(\mathcal{I}_{\mathcal{N}}) - \lim_k a_k = \mathcal{L}$.*
Proof. Since $\triangle^m(\mathcal{I}_{\mathcal{N}}^*) - \lim_k a_k = \mathcal{L}$ so there exist a set $J = \{j_1, \ j_2, \ j_3 \cdots\} \in \mathcal{F}(\mathcal{I})$ such that $\mathcal{N} - \lim_n \triangle^m a_{j_n} = \mathcal{L}$. For each $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer p_0, such that $\mathcal{G}(\triangle^m a_{j_n} - \mathcal{L}, \ \lambda) > 1 - \varepsilon$, $\mathcal{B}(\triangle^m a_{J_n} - \mathcal{L}, \ \lambda) > \varepsilon$ and $\mathcal{Y}(\triangle^m a_{J_n} - \mathcal{L}, \ \lambda) > \varepsilon$ for every $n \geq p_0$. If we take a set $P = \mathbb{N} - J$, then $P \in \mathcal{I}$ and therefore we have the containment $A(\varepsilon, \ \lambda) = \{k \in \mathbb{N} : \ \mathcal{G}(\triangle^m a_k - \mathcal{L}, \ \lambda) \leq 1 - \varepsilon$ or $\mathcal{B}(\triangle^m a_k - \mathcal{L}, \ \lambda) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - \mathcal{L}, \ \lambda) \geq \varepsilon\} \subseteq P \cup \{j_1, \ \ j_2, \ j_3 \cdots j_{p_0}\}$. Since $P \in \mathcal{I}$ and $\{j_1, \ \ j_2, \ j_3 \cdots j_{p_0}\}$ is a finite set so their union must be $\mathcal{I}$ which immediately gives $A(\varepsilon, \ \lambda) \in \mathcal{I}$. Hence, $\triangle^m(\mathcal{I}_{\mathcal{N}}) - \lim_k a_k = \mathcal{L}$.

We now describe the Lemma 4 of [20].

Lemma 4.1. *Let $\{K_i\}_{i=1}^{\infty}$be a countable collection of subsets of $\mathbb{N}$ such that $K_i \in \mathcal{F}(\mathcal{I})$ for each i where $\mathcal{F}(\mathcal{I})$ is a filter associate with an admissible ideal $\mathcal{I}$ satisfying property (AP). Then there exists a set $K \subseteq \mathbb{N}$ such that $K \in \mathcal{F}(\mathcal{I})$ and the set $K - K_i$is finite for each i.*

Theorem 4.2. *If $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal with property (AP) then the concepts of $\triangle^m(\mathcal{I}_{\mathcal{N}})-$*convergence *and $\triangle^m(\mathcal{I}_{\mathcal{N}}^*) -$* convergence *in neutrosophic normed spaces coincide.*
Proof. To prove the result it is sufficient to show that if $\mathcal{I}$ is an admissible ideal with property *(AP)* then $\triangle^m(\mathcal{I}_{\mathcal{N}})-$convergence implies $\triangle^m(\mathcal{I}_{\mathcal{N}}^*)-$convergence in neutrosophic normed spaces. Let (a_k) be any sequences in Vsuch that $\triangle^m(\mathcal{I}_{\mathcal{N}}) - \lim_k a_k = \mathcal{L}$. By definition, for every $\varepsilon > 0$ and $\lambda > 0$ we have $A(\varepsilon, \ \lambda) = \{k \in \mathbb{N} : \ \mathcal{G}(\triangle^m a_k - \mathcal{L}, \ \lambda) \leq 1 - \varepsilon$ or $\mathcal{B}(\triangle^m a_k - \mathcal{L}, \lambda) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - \mathcal{L}, \ \lambda) \geq \varepsilon\} \in \mathcal{I}$ For $\mu \in \mathbb{N}$, we define sets $K(\mu, \ \lambda)$ and $P(\mu, \ \lambda)$ by $K(\mu, \ \lambda) = \{k \in \mathbb{N} : \mathcal{G}(\triangle^m a_k - \mathcal{L}, \ \lambda) > 1 - \frac{1}{\mu}$ and $\mathcal{B}(\triangle^m a_k - \mathcal{L}, \ \lambda) < \frac{1}{\mu}$ *or* $\mathcal{Y}(\triangle^m a_k - \mathcal{L}, \ \lambda) < \frac{1}{\mu}\}$ and $P(\mu, \ \lambda) = \{k \in \mathbb{N} : \ \mathcal{G}(\triangle^m a_k - \mathcal{L}, \ \lambda) \leq 1 - \frac{1}{\mu}$ *or* $\mathcal{B}(\triangle^m a_k - \mathcal{L}, \ \lambda) > \frac{1}{\mu}$, $\mathcal{Y}(\triangle^m a_k - \mathcal{L}, \ \lambda) > \frac{1}{\mu}\}$. Since $\triangle^m(\mathcal{I}_{\mathcal{N}}) - \lim_k a_k = \mathcal{L}$, so for $\lambda > 0$ and $\mu \in \mathbb{N}$, *i.e* $\mu = 1, \ 2, \ 3, \ \cdots$, $P(\mu, \ \lambda) \in \mathcal{I}$ which immediately gives $K(\mu, \ \lambda) \in \mathcal{F}(\mathcal{I})$. Thus $K(1, \ \lambda)$, $K(2, \ \lambda)$, $K(3, \ \lambda)$, $\cdots$ is a sequence of sets in $\mathcal{F}(\mathcal{I})$. As the ideal

satisfy the property *(AP)* so by Lemma 4.1, there exists a set $K \subseteq \mathbb{N}$ such that $K = \{k_1,\ k_2,\ k_3 \cdots\} \in \mathcal{F}(\mathcal{I})$ and the set $\{K - K(\mu,\ \lambda)\}$ is finite for $\mu = 1,\ 2,\ 3,\ \cdots$. Now to prove the result it is sufficient to show that $\mathcal{N} - \lim_j \triangle^m a_{k_j} = \mathcal{L}$. Suppose that $\mathcal{N} - \lim_j \triangle^m a_{k_j} \neq \mathcal{L}$. Then there is some $\varepsilon_1 > 0$ and a positive integer p such that for all $j \geq p_0$ $\mathcal{G}(\triangle^m a_{k_j} - \mathcal{L},\ \lambda) \leq 1 - \varepsilon_1$ or $\mathcal{B}(\triangle^m a_{k_j} - \mathcal{L},\ \lambda) \geq \varepsilon_1$ and $\mathcal{Y}(\triangle^m a_{k_j} - \mathcal{L},\ \lambda) \geq \varepsilon_1$; which immediately implies that the set $\{k_j \in \mathbb{N} : \mathcal{G}(\triangle^m a_{k_j} - \mathcal{L},\ \lambda) > 1 - \varepsilon_1\ ,\ \mathcal{B}(\triangle^m a_{k_j} - \mathcal{L},\ \lambda) < \varepsilon_1$ or $\mathcal{Y}(\triangle^m a_{k_j} - \mathcal{L},\ \lambda) < \varepsilon_1\}$ is a finite set and must be in $\mathcal{I}$ and therefore we obtain a contradiction as it belongs to $\mathcal{F}(\mathcal{I})$. Hence, $\mathcal{N} - \lim_j\ \triangle^m a_{k_j} = \mathcal{L}$ and this completes the proof of the Theorem.

5.$\triangle^m(\mathcal{I}_\mathcal{N})-$ Completeness in *NNS*

In this section, we introduce the concepts of $\Delta^m(\mathcal{I}_\mathcal{N})-$Cauchy and $\Delta^m(\mathcal{I}^*_\mathcal{N})-$Cauchy sequences in NNS and define corresponding completeness.

Definition 5.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal and V be a NNS. A sequence (a_k) in V is said to $\triangle^m(\mathcal{I}_\mathcal{N}) -$ Cauchy if and only if for every $\varepsilon > 0$ and $\lambda > 0$ there exists a positive integer p such that the set $\{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - a_p,\ \lambda) \leq 1 - \varepsilon$ or $\mathcal{B}(\triangle^m a_k - a_p,\ \lambda) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - a_p,\ \lambda) \geq \varepsilon\}$ is in $\mathcal{I}$.*

We next give the generalized Cauchy convergence criteria in neutrosophic normed spaces.

Theorem 5.1. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal, V be a NNS and (a_k) be any sequences in V such that $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = \mathcal{L}$, then (a_k) is $\triangle^m(\mathcal{I}_\mathcal{N}) -$Cauchy.*

Proof. Assume that $\triangle^m(\mathcal{I}_\mathcal{N}) - \lim_k a_k = \mathcal{L}$. For every $\varepsilon > 0$ and $\lambda > 0$, choose $\mu > 0$ such that $(1-\varepsilon) \circ (1-\varepsilon) > 1 - \mu$ and $\varepsilon \bullet \varepsilon < \mu$. Then, $A(\varepsilon,\ \lambda) = \{k \in \mathbb{N} : \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) \leq 1 - \varepsilon$ or $\mathcal{B}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) \geq \varepsilon\} \in \mathcal{I}$ and $A^C(\varepsilon,\ \lambda) = \{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) > 1 - \varepsilon\ ,\ \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon$ or $\mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$ and therefore, is a non-empty set. Let $p \in A^C(\varepsilon,\ \lambda)$, then we have $\mathcal{G}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) > 1 - \varepsilon\ ,\ \mathcal{B}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon$ or $\mathcal{Y}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon$ Let, $B(\mu,\ \lambda) = \{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - a_p,\ \lambda) \leq 1 - \mu$ or $\mathcal{B}(\triangle^m a_k - a_p,\ \lambda) \geq \mu$ and $\mathcal{Y}(\triangle^m a_k - a_p,\ \lambda) \geq \mu\}$ is in $\mathcal{I}$. We shall show that $B(\mu,\ \lambda) \subseteq A(\mu,\ \lambda)$. For this, let $k_0 \in B(\mu,\ \lambda) - A(\mu,\ \lambda)$, then we have $\mathcal{G}(\triangle^m a_{k_0} - a_p),\ \lambda) \leq 1 - \mu$ and $\mathcal{G}(\triangle^m a_{k_0} - \mathcal{L}, \frac{\lambda}{2}) > 1 - \mu$. In particular $\mathcal{G}(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}) > 1 - \varepsilon$. Now, $1 - \mu\ \geq \mathcal{G}(\triangle^m a_{k_0} - a_p),\ \lambda) \geq \mathcal{G}(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}) \circ \mathcal{G}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) > (1-\varepsilon) \circ (1-\varepsilon) > 1 - \mu$ which is not possible. If $\mathcal{B}\left(\triangle^m(a_{k_0} - a_{p)},\ \lambda\right) \geq \mu$ and $\mathcal{B}\left(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}\right) < \mu$. In particular $\mathcal{B}\left(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}\right) < \varepsilon$. Now, $\mu \leq \mathcal{B}(\triangle^m(a_{k_0} - a_p),\ \lambda) \leq \mathcal{B}(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}) \bullet \mathcal{B}(\triangle^m a_m - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon \bullet \varepsilon < \mu$ which is not possible. Finally, if $\mathcal{Y}(\triangle^m(a_{k_0} - a_p),\ \lambda) \geq \mu$ and $\mathcal{Y}(a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}) < \mu$. In particular $\mathcal{Y}\left(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}\right) < \varepsilon$. Now, $\mu \leq \mathcal{Y}(\triangle^m(a_{k_0} - a_p),\ \lambda) \leq \mathcal{Y}(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}) \bullet \mathcal{Y}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon \bullet \varepsilon < \mu$

which is not possible. Thus, in every case, $B(\mu,\ \lambda) \subseteq A(\mu,\ \lambda)$. Since $A(\mu,\ \lambda) \in \mathcal{I}$ therefore $B(\mu,\ \lambda) \in \mathcal{I}$ as $B(\mu,\ \lambda) \subseteq A(\mu,\ \lambda)$. Hence, (a_k) is $\triangle^m(\mathcal{I}_\mathcal{N})-$Cauchy sequence.

Definition 5.2. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal and V be a NNS. V is said to be ideal complete or $\triangle^m(\mathcal{I}_\mathcal{N})-$complete if every $\triangle^m(\mathcal{I}_\mathcal{N})-$Cauchy sequence in V is $\triangle^m(\mathcal{I}_\mathcal{N})-$convergent.*

Theorem 5.2. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal, the NNS V is $\triangle^m(\mathcal{I}_\mathcal{N})-$complete.*

Proof. Let (a_k) be any $\triangle^m(\mathcal{I}_\mathcal{N})-$Cauchy sequence in V. To prove the result, we have to prove that (a_k) is $\triangle^m(\mathcal{I}_\mathcal{N})-$convergent in V. Suppose that (a_k) is not $\triangle^m(\mathcal{I}_\mathcal{N})-$convergent in V. Let $\varepsilon > 0$ and $\lambda > 0$. Choose $\mu > 0$ such that $(1-\varepsilon)\circ(1-\varepsilon) > 1-\mu$ and $\varepsilon \bullet \varepsilon < \mu$. Since, (a_k) is $\triangle^m(\mathcal{I}_\mathcal{N})-$Cauchy so there exists a positive integer p such that $A(\varepsilon,\ \lambda) = \{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - a_p),\ \lambda) \leq 1-\varepsilon$ or $\mathcal{B}(\triangle^m a_k - a_p),\ \lambda) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - a_p),\ \lambda) \geq \varepsilon\} \in \mathcal{I}$, and $\emptyset \neq A^C(\varepsilon,\ \lambda) = \{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - a_p),\ \lambda) > 1-\varepsilon\ ,\ \mathcal{B}(\triangle^m a_k - a_p),\lambda) < \varepsilon$ or $\mathcal{Y}(\triangle^m a_k - a_p),\ \lambda) < \varepsilon\} \in \mathcal{F}(\mathcal{I})$. Let, $B(\varepsilon,\ \lambda)$ be define by $B(\varepsilon,\ \lambda) = \{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) \leq 1-\varepsilon$ or $\mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) \geq \varepsilon$ and $\mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \lambda) \geq \varepsilon\}$. Since, (a_k) is not $\triangle^m(\mathcal{I}_\mathcal{N})-$convergent in V so $B(\varepsilon,\ \lambda) \notin \mathcal{I}$ and therefore in $\mathcal{F}(\mathcal{I})$, which immediately implies that $B^C(\varepsilon,\ \lambda) = \{k \in \mathbb{N} :\ \mathcal{G}(\triangle^m a_k - \mathcal{L},\ \lambda) > 1-\varepsilon,\quad \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \lambda) < \varepsilon$ or $\mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \lambda) < \varepsilon\} \in \mathcal{I}$. Now we shall show that $A^C(\varepsilon,\ \lambda) \subseteq B^C(\varepsilon,\ \lambda)$. Let $k_0 \in A^C(\varepsilon,\ \lambda)$, then $\mathcal{G}(\triangle^m(a_{k_0} - a_p),\ \lambda) > 1-\varepsilon\ ,\quad \mathcal{B}(\triangle^m a_{k_0} - a_p,\ \lambda) < \varepsilon$ or $\mathcal{Y}(\triangle^m a_{k_0} - a_p,\ \lambda) < \varepsilon$. Now, as in Theorem 5.1, we have $\mathcal{G}(\triangle^m a_{k_0} - a_p),\ \lambda) \geq \mathcal{G}(\triangle^m a_{k_0} - \mathcal{L},\ \frac{\lambda}{2}) \circ \mathcal{G}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) > (1-\varepsilon)\circ(1-\varepsilon) > 1-\mu$; $\mathcal{B}(\triangle^m(a_k - a_p),\ \lambda) \leq \mathcal{B}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) \bullet \mathcal{B}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon \bullet \varepsilon < \mu$ and $\mathcal{Y}(\triangle^m(a_k - a_p),\ \lambda) \leq \mathcal{Y}(\triangle^m a_k - \mathcal{L},\ \frac{\lambda}{2}) \bullet \mathcal{Y}(\triangle^m a_p - \mathcal{L},\ \frac{\lambda}{2}) < \varepsilon \bullet \varepsilon < \mu$. This shows that $k_0 \in B^C(\mu\ ,\ \lambda)$ and therefore we have $A^C(\varepsilon,\ \lambda) \subseteq B^C(\varepsilon,\ \lambda)$. Since $B^C(\varepsilon,\ \lambda) \in \mathcal{I}$ so $A^C(\varepsilon,\ \lambda) \in \mathcal{I}$, which is a contradiction as $A^C(\varepsilon,\ \lambda) \in \mathcal{F}(\mathcal{I})$. Hence, (a_k) is $\triangle^m(\mathcal{I}_\mathcal{N})-$convergent in V and therefore V is $\triangle^m(\mathcal{I}_\mathcal{N})-$complete.

From, the above discussion we have the following Theorem.

Theorem 5.3. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal, and V be a NNS. For any sequence (a_k) in V, the following conditions are equivalents.*

(i) (a_k) *is* $\triangle^m(\mathcal{I}_\mathcal{N})-$*convergent in* V;

(ii) (a_k) $\triangle^m(\mathcal{I}_\mathcal{N})-$*Cauchy in* V *and*

(iii) V *is* $\triangle^m(\mathcal{I}_\mathcal{N})-$*complete.*

Definition 5.3. *Let $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ is an admissible ideal and V be a NNS. A sequence (a_k) in V is said to $\triangle^m(\mathcal{I}^*_\mathcal{N})-$Cauchy if and only if there exist a set*

$J = \{j_1, j_2, j_3 \cdots\} \in \mathcal{F}(\mathcal{I})$ *such that the subsequence* $(\triangle^m a_{j_n})$ *is* $\mathcal{N}-$*Cauchy.*

Definition 5.4. *Let* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is an admissible ideal and* V *be a* NNS. V *is said to be* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*complete if every* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*Cauchy sequence in* V *is* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*convergent.*

Theorem 5.4. *Let* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is an admissible ideal,* V *be a* NNS *and* (a_k) *be any sequences in* V *such that* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*Cauchy, then it is* $\triangle^m(\mathcal{I}_\mathcal{N})-$*Cauchy.*

Theorem 5.5. *If* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is an admissible ideal with property (AP) then the concepts of* $\triangle^m(\mathcal{I}_\mathcal{N})-$*Cauchy and* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*Cauchy in neutrosophic normed spaces coincide.*

Theorem 5.6. *Let* $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$ *is an admissible ideal and* NNS. *For any sequence* (a_k) *in* V*, the following conditions are equivalents.*

(i) (a_k) *is* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*convergent in* V;

(ii) (a_k) $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*Cauchy in* V *and*

(iii) V *is* $\triangle^m(\mathcal{I}^*_\mathcal{N})-$*complete.*

References

[1] Atanassov K. K., Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1986), 87-96.

[2] Bera T. and Mahapatra N. K., Neutrosophic soft linear spaces, Fuzzy Information and Engineering, 9(2017), 299-324.

[3] Bera T. and Mahapatra N. K., Neutrosophic soft normed linear spaces, Neutrosophic Sets and Systems, 23(2018), 52-71.

[4] Connor J. S., The statistical and strong $p-$Cesaro convergence of sequences, Analysis, 8(1988), 47-63.

[5] Demirci K., $\mathcal{I}-$limit superior and inferior, Math. Commun., 6(2001), 165-172.

[6] Dems K., On $\mathcal{I}-$Cauchy sequences, Real Analysis Exchange, 30(2004), 123-128.

[7] Fast H., Surla convergence statistique, Coll. Math., 2(1951), 241-24.

[8] Fridy J. A., On statistical convergence, Analysis, 5(1985), 301-313.

[9] Hazarika B., Kumar V. and Lafuerza-Guillén B., Generalized ideal convergence in Intuitionistic fuzzy normed linear spaces, Filomat, 27, 811-820.

[10] Karakus S., Demirci K. and Duman O., Statistical convergence on Intuitionistic fuzzy normed spaces, Chaos Solitons Fractals, 35(2008), 763-769.

[11] Khan V. A., Khan M. D. and Ahmad M., Some New Type of Lacunary Statistically Convergent Sequences in Neutrosophic Normed Space, Neutrosophic Sets and Systems, Vol. 42, 2021.

[12] Kirisci M., and Simsek N., Neutrosophic metric spaces, arXiv:1907.00798.

[13] Kirisci M., and Simsek N., Neutrosophic normed spaces and statistical convergence, The Journal of Analysis, https://doi.org/10.1007/s41478-020-00234-0

[14] Komisarski A., Point wise $\mathcal{I}$-convergence and $\mathcal{I}^*$-convergence in measure of sequences of functions, J. Math. Anal. Appl., 340(2008), 770-779.

[15] Kostyrko P., Salat T., Wilczynski W., $\mathcal{I}-$convergence, Real Anal. Exchange, 26(2000/2001), 669-686.

[16] Kostyrko P., Macaj M., Salat T. and, Sleziak M., $\mathcal{I}$-convergence and extremal $\mathcal{I}$-limit points, Math. Slovaca, 4(2005), 443-464.

[17] Kumar V., On $\mathcal{I}$ and $\mathcal{I}^*$-convergence of double sequences, Math. Commun, 12(2007), 171-181.

[18] Kumar V. and Lafuerza-Guillén B., On ideal convergence of double sequences in probabilistic normed spaces Acta Mathematica Sinica, 29(2012), 1689-1700.

[19] Mursaleen M., Mohiuddine S. A. and Edely O. H. H., On ideal convergence of double sequences in Intuitionistic fuzzy normed spaces, Comput. Math. Appl., 59(2010), 603-611.

[20] Nabiev A., Pehlivan S. and Gurdal M., On $\mathcal{I}-$Cauchy sequences, Taiwanese Journal of Mathematics, 2(2007), 569-576.

[21] Park J. H., Intuitionistic fuzzy metric spaces, Chaos, Solitons & Fractals, 22(2004), 1039-46.

[22] Salat T., On statistically convergent sequences of real numbers, Math. Slovaca, 30(1980), 139-150.

[23] Saadati R., Park J. H., On the Intuitionistic fuzzy topological spaces, Chaos, Solitons & Fractals, 27(2006), 331-44.

[24] Schoenberg I. J., The inerrability of certain functions and related summability methods, Amer. Math. Monthly, 66(1959), 361-375.

[25] Schweizer B., Sklar A., Statistical metric spaces, Pacific J Math, 10(1960), 314-44.

[26] Smarandache F., Neutrosophic set, a generalisation of the Intuitionistic fuzzy sets, International Journal of Pure and Applied Mathematics, 24(2005), 287-297.

[27] Zadeh L. A., Fuzzy sets, Inform Control, 8(1965), 338-53.