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INTUITIONISTIC TOPOLOGICAL SPACES WITH L-GRADATIONS OF OPENNESS AND NONOPENNESS WITH RESPECT TO LT-NORM T AND LC-CONORM C ON X

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Abstract: In this paper, we assume that $L =< L, \leq, \bigwedge, \bigvee, ' >$ is a complete distributive lattice set with at least 2 elements and $(L, +)$ is also an additive group. We introduce an LT-norm T and an LC-conorm C on the lattice set L (briefly $L(T, C)$-norm). Furthermore using this norm, we define spiral LT-norm and spiral LC-conorm of any countable sequence in L. Also we introduce $IL(T, C)$-gradations of openness on X which X is an L-fuzzy subset of a nonempty set M and we prove that the set of all $IL(T, C)$-gradations of openness on X is a semicomplete lattice. We introduce intuitionistic L-fuzzy topological space with L-gradation of openness and nonopenness with respect to the $L(T, C)$-norm (briefly $ILG(T, C)$-fuzzy topological space). As an example we define an $IL(T, C)$-fuzzy subspace of $\Lambda\mathbb{R}^m$, the exterior algebra on $\mathbb{R}^m$.

Keywords and Phrases: Spiral LT-norm, intuitionistic L-fuzzy subset, intuitionistic L-fuzzy subgroup with respect to the norm $L(T, C)$-norm, intuitionistic L-gradation of closeness and noncloseness with respect to $L(T, C)$-norm.

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1. Introduction and Preliminaries

Fuzzy topology was defined by Chang [10] as a generalization of the concept of fuzzy sets introduced by Zadeh [43]. In consequence of the development of fuzzy topology, various concepts of fuzzy topology were defined, such as [11, 12, 15, 19, 22, 23, 24, 25, 38, 39, 40, 41, 42]. In 1985, Shostak [38] introduced a concept of gradation of openness of fuzzy subsets of X. Also many authers discussed graded fuzzy topological spaces. See [11, 12, 15, 25]. Many of them suggested that the properties should be considered fuzzy that is, one should be able to measure a degree to which a property holds. See [23, 24, 42, 39, 40].

In 1983 Atanassov [4], introduced intuitionistic fuzzy sets to overcome the difficulties in dealing with uncertainties. Later, with Stefka Stoeva, he [5], further generalized that concept to an intuitionistic L-fuzzy set , where L stands for some lattice coupled with a special negation. Subsequently, many mathematicians generalized this concept. For example [3, 7, 8, 13, 16, 17]. See the book [6] as a comprehensive, complete coverage of virtually all results obtained up to 2012, in the area of the theory and applications of intuitionistic fuzzy sets.

In 1960 Schweizer and Sklar [37] introduced a class of semi-groups on [0, l]. Rosenfeld [36] defined fuzzy subgroupoids and fuzzy subgroups. Anthony and Sherwood [2] redefined a fuzzy subgroup of a group using the concept of triangular norm (t-norm, for short). In mathematics, a t-norm (also T-norm or, unabbreviated, triangular norm) is a sort of binary operation used in the frame of probabilistic metric spaces and in fuzzy logic. Osman, [1], defined some products of fuzzy subgroups. Recently Rassuli [30, 31, 34, 35] defined fuzzy modules, fuzzy subrings and fuzzy subgroups, fuzzy sub-vector spaces and sub-bivector spaces under t-norms.

We investigated in [26], some properties of a novel fuzzy topological space (X, τ), where X is itself a fuzzy subset of a crisp set M. We assumed that $L =< L, \leq, \bigwedge, \bigvee, ' >$ is a complete distributive lattice set with at least 2 elements. An L-fuzzy subset D of the crisp set M, in Goguen's sense [14], is a function $D : M \to L$ and is denoted by $D \in L^M$. In order to discuss the L-fuzzification of the concepts of geometry, we introduced in [27], the concept of C^∞ L-fuzzy manifold with L-gradation of openness. Also we defined and investigated LG-paracompactness of LG-fuzzy topological metric spaces in [28] and $\mathbb{Z}_2$-graded intuitionistic L-fuzzy q-deformed quantum subspaces of A_q in [29].
The purpose of this paper is to deal with the geometric structure of intuitionistic L-fuzzy topological space with L-gradation of openness and nonopenness with respect to LT-norm T and LC-conorm C ($ILG(T, C)$-fuzzy topological space). Using t-norm defined in [9], we introduce an LT-norm T and LC-conorm C on the lattice set L. We define $IL(T, C)$-gradation of openness on the fuzzy set X and give some related properties and results. Also we establish the spiral LT-norm and spiral LC-conorm of any sequence in L and then we prove that the set of all $IL(T, C)$-gradations of openness on X is a semicomplete lattice. Our notation and

terminology for intuitionistic fuzzy sets follows that of [4, 5]. For definitions of T-norms and C-conorms we follow Rassuli [34].

2. Spiral LT-norm and Spiral LC-conorm of a Sequence in L

In this manuscript, we assume that $L =< L, \leq, \bigwedge, \bigvee, ' >$ is a complete distributive lattice set with at least 2 elements and $(L, +)$ is also an additive group.

Definition 2.1. *An LT-norm T is a function $T : L \times L \to L$ having the following four properties:*

(LT1) $T(x, 1) = x$ *(neutral element),*

(LT2) $T(x, y) \leq T(x, z)$ *if* $y \leq z$ *(monotonicity),*

(LT3) $T(x, y) = T(y, x)$ *(commutativity),*

(LT4) $T(x, T(y, z)) = T(T(x, y), z)$ *(associativity),*

for all $x, y, z \in L$.
We say that T is idempotent if for all $x \in L$, $T(x, x) = x$.

Example 2.2. (1) Standard intersection LT-norm $T_{min}(x, y) = min\{x, y\}$
(2) Bounded sum LT-norm $T_b(x, y) = max\{0, x + y - 1\}$
(3) algebraic product LT-norm $T_p(x, y) = xy$
(4) Drastic LT-norm

$$T_D(x, y) = \begin{cases} y & if \;\; x = 1 \\ x & if \;\; y = 1 \\ 0 & otherwise. \end{cases}$$

(5) Nilpotent minimum LT-norm

$$T_{nM}(x, y) = \begin{cases} min\{x, y\} & if \;\; x + y > 1 \\ 0 & otherwise. \end{cases}$$

(6) Hamacher product LT-norm

$$T_{H_0}(x, y) = \begin{cases} 0 & if \;\; x = y = 0 \\ \dfrac{xy}{x + y - xy} & otherwise. \end{cases}$$

The drastic LT-norm is the pointwise smallest LT-norm and the minimum is the pointwise largest LT-norm: $T_D(x, y) \leq T(x, y) \leq T_{min}(x, y) \; for \; all \; x, y \in L$.

Definition 2.3. *An LC-conorm C is a function $C : L \times L \to L$ having the following four properties:*

(LC1) $C(x, 0) = x$ *(neutral element),*

(LC2) $C(x, y) \leq C(x, z)$ *if* $y \leq z$ *(monotonicity),*

(LC3) $C(x, y) = C(y, x)$ *(commutativity),*

(LC4) $C(x, C(y, z)) = C(C(x, y), z)$ *(associativity),*

for all $x, y, z \in L$.
We say that the LC*-conorm* C *is idempotent if for all* $x \in L$, $C(x, x) = x$.

Example 2.4. (1) Standard union LC-conorm $C_{max}(x, y) = max\{x, y\}$
(2) Bounded sum LC-conorm $C_b(x, y) = max\{1, x + y\}$
(3) Algebraic product LC-conorm $C_p(x, y) = x + y - xy$
(4) Drastic LC-conorm

$$C_D(x, y) = \begin{cases} y & if \;\; x = 1 \\ x & if \;\; y = 1 \\ 1 & otherwise. \end{cases}$$

(5) Nilpotent maximum LC-conorm

$$C_{nM}(x, y) = \begin{cases} max\{x, y\} & if \;\; x + y < 1 \\ 1 & otherwise. \end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity)

$$C_{H_2}(x, y) = \frac{x + y}{1 + xy}.$$

Note that for all LC-conorm C, we have $C_{max}(x, y) \leq C(x, y) \leq C_D(x, y)$ for all $x, y \in L$.

Lemma 2.5. *Consider an* $L(T, C)$*-norm. Then for all* $x, y, z, w \in L$ *we have*

$$T(x, y) \leq x \wedge y, \tag{2.1}$$

$$C(x, y) \geq x \vee y, \tag{2.2}$$

$$T\big(T(x, y), T(z, w)\big) = T\big(T(x, z), T(y, w)\big), \tag{2.3}$$

$$C\big(C(x, y), C(z, w)\big) = C\big(C(x, z), C(y, w)\big), \tag{2.4}$$

Proof. Using $(LT1)$ and $(LT2)$ we have $T(x,y) \leq T(x,1) = x$. Also using $(LT3)$ we have $T(x,y) = T(y,x) \leq T(y,1) = y$. Thus $T(x,y) \leq x \wedge y$.
Similarly using $(LC1)$ and $(LC2)$ we have $C(x,y) \geq C(x,0) = x$ and using $(LC3)$ we have $C(x,y) = C(y,x) \geq C(y,0) = y$. Therefore $C(x,y) \geq x \vee y$.

To prove (2.3) with frequent use of $(LT4)$ and using $(LT3)$, we see

$$\begin{aligned} T\big(T(x,y),T(z,w)\big) &= T\big(T(T(x,y),z),w)\big) \\ &= T\big(T(x,T(y,z)),w)\big) \\ &= T\big(T(x,T(z,y)),w)\big) \\ &= T\big(T(T(x,z),y),w)\big) \\ &= T\big(T(x,z),T(y,w)\big). \end{aligned}$$

The proof of (2.4) is similar.

Definition 2.6. *Let T be an LT-norm and $\{x_i | i \in \mathbb{N}\}$ be a countable subset of L. Define*

$$T^1_{\circledS}(\{x_i\}) = x_1, \quad T^2_{\circledS}(\{x_i\}) = T(x_1,x_2)$$
$$T^3_{\circledS}(\{x_i\}) = T(T(x_1,x_2),x_3), \quad T^4_{\circledS}(\{x_i\}) = T\big(T(T(x_1,x_2),x_3),x_4\big)$$
$$T^k_{\circledS}(\{x_i\}) = T\big(\dots T(T(x_1,x_2),x_3),\dots,x_k\big)$$

Then we define

$$T^\infty_{\circledS}(\{x_i\}) = \lim_{k\to\infty} T^k_{\circledS}(\{x_i\}) \tag{2.5}$$

called spiral LT-norm of $\{x_i\}$.

Lemma 2.7. *Let T be an LT-norm. Then the definition of spiral LT-norm of a countable subset $\{x_i | i \in \mathbb{N}\}$ of L, is well defined. Also we have*

$$T^\infty_{\circledS}(\{x_i\}) \leq x_i, \ \forall i \in \mathbb{N}. \tag{2.6}$$

Proof. Using $(LT1)$ and $(LT2)$, we have

$$T^2_{\circledS}(\{x_i\}) \leq T(x_1,1) = x_1 = T^1_{\circledS}(\{x_i\})$$
$$T^3_{\circledS}(\{x_i\}) = T(T(x_1,x_2),x_3) \leq T(T(x_1,x_2),1) = T(x_1,x_2) = T^2_{\circledS}(\{x_i\})$$

By contradiction on k, we can prove that $\{T^k_{\circledS}(\{x_i\})\}$ is a decreasing sequence in L. Since we assumed that the lattis L is complete so $\lim_{k\to\infty} T^k_{\circledS}(\{x_i\})$ exists. Because of $(LT4)$, the associativity of T, this definition is independent of the ordering of the elements of this subset. Hence definition of spiral LT-norm of $\{x_i\}$ is well defined.

Definition 2.8. *Let C be an LT-conorm and $\{x_i | i \in \mathbb{N}\}$ be a countable subset of L. Define*

$$C^1_{Ⓢ}(\{x_i\}) = x_1, \quad C^2_{Ⓢ}(\{x_i\}) = C(x_1, x_2)$$
$$C^3_{Ⓢ}(\{x_i\}) = C(T(x_1, x_2), x_3), \quad T^4_{Ⓢ}(\{x_i\}) = T\big(T(T(x_1, x_2), x_3), x_4\big)$$
$$C^k_{Ⓢ}(\{x_i\}) = C\big(\ldots C(C(x_1, x_2), x_3), \ldots, x_k\big)$$

Then we define

$$C^\infty_{Ⓢ}(\{x_i\}) = \lim_{k\to\infty} C^k_{Ⓢ}(\{x_i\}) \tag{2.7}$$

called spiral LC-conorm of $\{x_i\}$.

Lemma 2.9. *Let C be an LC-conorm. Then the definition of spiral LC-conorm of any countable subset of L, is well defined. Also we have*

$$C^\infty_{Ⓢ}(\{x_i\}) \le x_i, \ \forall i \in \mathbb{N}. \tag{2.8}$$

Proof. We can prove this lemma similarly to the proof of Lemma 2.7.

3. Intuitionistic L-fuzzy Subgroups with Respect to the $L(T, C)$-norm

Definition 3.1. *Let M be a nonempty set. An intuitionistic L-fuzzy subset B of M is defined as an object having the form $B = \{\langle x, \mu_B(x), \nu_B(x)\rangle \mid x \in M\}$ or $B = (\mu_B, \nu_B)$, where the functions $\mu_B : M \to L$ and $\nu_B : M \to L$ denote the degree of membership and the degree of non-membership of each element $x \in M$ to the set B, respectively s.t. $0 \le \mu_B(x) + \nu_B(x) \le 1$ for each $x \in M$. We denote all intuitionistic L-fuzzy subsets of M by IL^M.*

We assume that $k = \mathbb{R}$, $\mathbb{C}$ or any field with characteristic≥ 2.

Definition 3.2. *Let V be a k-vector space. An intuitionistic L-fuzzy subset $B = (\mu_B, \nu_B)$ of V is called an intuitionistic L-fuzzy subspace with respect to the $L(T, C)$-norm if*

$$\mu_B(\gamma x + \lambda y) \ge T\big(\mu_B(x), \mu_B(y)\big), \quad \nu_B(\gamma x + \lambda y) \le C\big(\nu_B(x), \nu_B(y)\big)$$

for any $x, y \in V$ and $\gamma, \lambda \in k$. Then we can write briefly $B = (\mu_B, \nu_B)$ is an $IL(T, C)$-fuzzy subspace of V or $B = (\mu_B, \nu_B) \in ILFTC(V)$.

Definition 3.3. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic L-fuzzy subsets of a nonempty set M. We define intuitionistic L-fuzzy subsets $A \cap B$, $A \cup B$ by*

$$\mu_{A\cap B}(x) = T\big(\mu_A(x), \mu_B(x)\big), \quad \nu_{A\cap B}(x) = C\big(\nu_A(x), \nu_B(x)\big)$$

$$\mu_{A\cup B}(x) = C\big(\mu_A(x), \mu_B(x)\big), \quad \nu_{A\cup B}(x) = T\big(\nu_A(x), \nu_B(x)\big).$$

The intuitionistic L-fuzzy subsets $A+B$ and $\gamma.A$ of V for each $\gamma \in k$, $x \in X$, are defined by

$$\mu_{A+B}(x) = \begin{cases} \sup_{x=a+b} \{T\big(\mu_A(a), \mu_B(b)\big)\} & if\ x = a+b \\ 0 & elsewhere \end{cases}$$

$$\nu_{A+B}(x) = \begin{cases} \inf_{x=a+b} \{C\big(\nu_A(a), \nu_B(b)\big)\} & if\ x = a+b, \\ 0 & elsewhere \end{cases}$$

and

$$\mu_{\gamma.A}(x) = \begin{cases} \mu_A(\frac{1}{\gamma}x) & if\ \gamma \neq 0 \\ 1 & if\ \gamma = 0,\ x = 0 \\ 0 & if\ \gamma = 0,\ x \neq 0 \end{cases}$$

$$\nu_{\gamma.A}(x) = \begin{cases} \nu_A(\frac{1}{\gamma}x) & if\ \gamma \neq 0 \\ 0 & if\ \gamma = 0,\ x = 0 \\ 1 & if\ \gamma = 0,\ x \neq 0. \end{cases}$$

Further if $A \cap B = \tilde{0}$, then $A+B$ is said to be the direct sum and denoted by $A \oplus B$.

Lemma 3.4. *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IL(T,C)$-fuzzy subspaces of V. Then $A + B = (\mu_{A+B}, \nu_{A+B})$, $A \cap B = (\mu_{A\cap B}, \nu_{A\cap B})$ and $\gamma.A = (\mu_{\gamma.A}, \nu_{\gamma.A})$ for each $\gamma \in k$, are also $IL(T,C)$-fuzzy subspaces of V.*

Definition 3.5. *Let $B = (\mu_B, \nu_B)$ be an $IL(T,C)$-fuzzy subspace of a group G. Then $B = (\mu_B, \nu_B)$ is called an intuitionistic L-fuzzy subgroup of G, with respect to the $L(T,C)$-norm, (briefly $IL(T,C)$-fuzzy subgroup of G) if it satisfies two following conditions:*

i) $\mu_B(xy) \geq T\big(\mu_A(x), \mu_B(x)\big), \quad \nu_B(xy) \leq C\big(\nu_A(x), \nu_B(x)\big)$

ii) $\mu_B(x^{-1}) \geq \mu_B(x), \quad \nu_B(x^{-1}) \leq \nu_B(x).$

for any $x, y \in G$.

Example 3.6. The set of natural numbers, $\mathbb{N}$, partially ordered by divisibility, is a distributive lattice set, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. Let $L = \mathbb{N} \cup \{\infty\}$. Then L is a complete lattice.

Let $G = \{1, -1, i, -i\}$ be a group with respect to multiplication. Define L-fuzzy

subsets
$\mu_B, \nu_B : G \to L$ as

$$\mu_B(x) = \begin{cases} 5 \;\; if \;\; x = 1 \\ 4 \;\; if \;\; x = -1 \\ 3 \;\; if \;\; x = \pm i \end{cases} \qquad \nu_B(x) = \begin{cases} 7 \;\; if \;\; x = 1 \\ 8 \;\; if \;\; x = -1 \\ 9 \;\; if \;\; x = \pm i \end{cases}$$

If $T(x, y) = T_{min}(x, y)$ and $C(x, y) = C_{max}(x, y)$ for all $x, y \in G$, then $B = (\mu_B, \nu_B)$ is an $IL(T, C)$-fuzzy subgroup of G.

Definition 3.7. *Let f be a mapping from a nonempty set M to a nonempty set M'. Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be intuitionistic L-fuzzy subsets of M and M' respectively. Then the inverse image of $B = (\mu_B, \nu_B)$ under f, is an intuitionistic L-fuzzy subset $f^{-1}[B] = (\mu_{f^{-1}[B]}, \nu_{f^{-1}[B]})$ defined by*

$$\mu_{f^{-1}[B]}(x) = \mu_B(f(x)), \quad \nu_{f^{-1}[B]}(x) = \nu_B(f(x))$$

for all $x \in V$ and the image of $A = (\mu_A, \nu_A)$ under f is an intuitionistic L-fuzzy subset $f[A] = (\mu_{f[A]}, \nu_{f[A]})$ defined by

$$\mu_{f[A]}(y) = \begin{cases} \sup \; \{\mu_A(x) | \; x \in f^{-1}(y)\} & if \;\; y \in f(V) \\ 0 & if \;\; y \notin f(V) \end{cases}$$

$$\nu_{f[A]}(y) = \begin{cases} \inf \; \{\nu_A(x) | \; x \in f^{-1}(y)\} & if \;\; y \in f(V) \\ 0 & if \;\; y \notin f(V). \end{cases}$$

for all $y \in V'$.

Proposition 3.8. *Let f be a linear mapping from the k-vector space V to the k-vector space V'. If $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are $IL(T, C)$-fuzzy subspaces of V and V' respectively. Then $f^{-1}[B] = (\mu_{f^{-1}[B]}, \; \nu_{f^{-1}[B]})$ and $f[A] = (\mu_{f[A]}, \; \nu_{f[A]})$ are $IL(T, C)$-fuzzy of V and V' respectively.*

Proof. For each $x, z \in V$ and $\gamma, \delta \in k$, we have

$$\begin{aligned} T\big(\mu_{f^{-1}[B]}(x), \mu_{f^{-1}[B]}(z)\big) &= T\big(\mu_B(f(x)), \mu_B(f(z))\big) \\ &\leq \mu_B\big(\gamma f(x) + \delta f(z)\big), \\ &= \mu_B\big(f(\gamma x + \delta z)\big), \\ &= \mu_{f^{-1}[B]}(\gamma x + \delta z) \end{aligned}$$

Similarly we can prove

$$C\big(\nu_{f^{-1}[B]}(x), \nu_{f^{-1}[B]}(z)\big) \geq \nu_{f^{-1}[B]}(\gamma x + \delta z).$$

Hence $f^{-1}[B] = (\mu_{f^{-1}[B]},\ \nu_{f^{-1}[B]})$ is an $IL(T, C)$-fuzzy subspaces of V.
To prove that $f[A] = (\mu_{f[A]},\ \nu_{f[A]})$ is an $IL(T, C)$-fuzzy subspaces of V', we show that for each $y, w \in V'$ and $\gamma, \delta \in k$, we have

$$T\big(\mu_{f[A]}(y), \mu_{f[A]}(w)\big) \leq \mu_{f[A]}(\gamma y + \delta w).$$

1) If $y, w \in f(V)$, then we have

$$\begin{aligned} T\big(\mu_{f[A]}(y), \mu_{f[A]}(w)\big) &= T\big(\sup\ \{\mu_A(x)|\ x \in f^{-1}(y)\},\ \sup\ \{\mu_A(z)|\ z \in f^{-1}(w)\}\big) \\ &= \sup\ \{T\big(\mu_A(x), \mu_A(z)\big)|\ x \in f^{-1}(y), z \in f^{-1}(w)\} \\ &\leq \sup\ \{\mu_A\big(\gamma x + \delta z\big)|\ x \in f^{-1}(y), z \in f^{-1}(w)\} \\ &\leq \sup\ \{\mu_A(t)|\ t \in f^{-1}(\gamma y + \delta w)\} \\ &= \mu_{f[A]}(\gamma y + \delta w). \end{aligned}$$

2) If $y \in f(V)$ and $w \notin f(V)$, we have

$$T\big(\mu_{f[A]}(y), \mu_{f[A]}(w)\big) = T\big(\sup\ \{\mu_A(x)|\ x \in f^{-1}(y)\},\ 0\big) = 0 \leq \mu_{f[A]}(\gamma y + \delta w).$$

3) If $y, w \notin f(V)$, we have

$$T\big(\mu_{f[A]}(y), \mu_{f[A]}(w)\big) = T\big(0,\ 0\big) = 0 \leq \mu_{f[A]}(\gamma y + \delta w).$$

Similarly we can show that

$$C\big(\mu_{f[A]}(y), \mu_{f[A]}(w)\big) \geq \nu_{f[A]}(\gamma y + \delta w).$$

Proposition 3.9. *Let $f : V \to V'$ be a linear mapping from the k-vector spaces. Then for any $IL(T, C)$-fuzzy subspaces $A = (\mu_A, \nu_A)$ and $D = (\mu_D, \nu_D)$ of V and all $\lambda \in k$, we have*

1) $f[A + D] = f[A] + f[D]$,

2) $f[\lambda A] = \lambda f[A]$.

Proof. 1) Let $w \in V'$. We want to show that $a = b$ where $a = \mu_{f[A+D]}(w)$ and $b = \mu_{f[A]+f[D]}(w)$. Suppose first that $w \notin Imf$. Then $a = 0$. Also if $x, y \in V'$ with $x+y = w$, then at least one of the x, y is not in Imf and thus $\mu_{f[A]}(x) \wedge \mu_{f[D]}(y) = 0$. So using (2.1) we have
$T(\mu_{f[A]}(x), \mu_{f[D]}(y)) = 0$. Hence $b = 0 = a$.
Assume next that $w \in Imf$. Given $\varepsilon > 0$, there exists $z \in V$ with $f(z) = w$ such that $\mu_{A+D}(z) > a - \varepsilon$. Then there exist $x, y \in V$ with $x + y = z$, such that $T\big(\mu_A(x), \mu_D(y)\big) > a - \varepsilon$. Since $f(x) + f(y) = w$, we have

$$b = \sup_{w=u+v} \{T\big(\mu_{f[A]}(u), \mu_{f[D]}(v)\big)\}$$

$$\geq T\big(\mu_{f[A]}(f(x)), \mu_{f[D]}(f(y))\big)$$

$$\geq T\big(\mu_A(x), \mu_D(y)\big)$$

$$> a - \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we get $b \geq a$. On the other hand given $\varepsilon > 0$, there exists u_1, u_2 with $u_1 + u_2 = w$ such that

$$b - \varepsilon < T\big(\mu_{f[A]}(u_1), \mu_{f[D]}(u_2)\big)$$

Taking $\varepsilon < b$ (if $b = 0$ then $a = 0$ and we have nothing to prove), we have that $u_1, u_2 \in Imf$. Therefore there exist x_1, x_2 in V with $u_1 = f(x_1)$, $u_2 = f(x_2)$ such that

$$b - \varepsilon < T\big(\mu_A(x_1), \mu_D(x_2)\big).$$

Since $f(x_1 + x_2) = w$, we get $a > b - \varepsilon$ and hence $a \geq b$, because $\varepsilon > 0$ was arbitrary. So $a = b$. Similarly we can prove that $\nu_{f[A+D]} = \nu_{f[A]+f[D]}$.

2) Let $w \in V'$, $c = \mu_{f[\lambda A]}(w)$ and $d = \mu_{\lambda f[A]}(w)$. If $w \notin Imf$. Then $c = d = 0$. Assume that $w \in Imf$. If $\lambda \neq 0$,

$$c = \sup \{\mu_{\lambda A}(x) |\ f(x) = w\}$$

$$= \sup \{\mu_A(\tfrac{1}{\lambda}x) |\ f(x) = w\}$$

$$= \sup \{\mu_A(y) |\ f(\lambda y) = w\}$$

$$= \sup \{\mu_A(y) |\ \lambda f(y) = w\}$$

$$= \mu_{\lambda f[A]}(w) = d.$$

Next suppose that $\lambda = 0$. If $w \neq 0$, then $c = 0$ and $d = \mu_{0f[A]}(w) = 0$.
If $w = 0$, we have

$$c = \sup\ \{\mu_{0A}(x)|\ f(x) = 0\}$$

$$= \sup\ \{1|\ f(x) = 0\}$$

$$= \sup\ \{\mu_A(y)|\ y \in V\}$$

$$= \mu_{0f[A]}(0) = d.$$

In a similar manner, we can show that $\nu_{f[\lambda A]} = \nu_{\lambda f[A]}$ and this completes the proof.

4. Intuitionistic L-fuzzy Topological Space with L-gradation of Openness and Nonopenness with respect to the $L(T, C)$-norm

Definition 4.1. *Let X be an intuitionistic L-fuzzy subset of M. We denote the set of all intuitionistic L-fuzzy subsets of M which are less or equal to X (called IL-fuzzy subsets of X) by IL_X^M. If τ as a collection of intuitionistic L-fuzzy subsets of X, satisfies the following conditions:*

1) $X,\ \phi \in \tau$,

2) $\{A_i\}_{i\in I} \subseteq \tau \ \Rightarrow\ \bigcup_{i\in I} A_i \in \tau$,

3) $A, B \in \tau \ \Rightarrow\ A \cap B \in \tau$,

then (X, τ) is called an intuitionistic L-fuzzy topological space (ILfts).

Example 4.2. Let $M = \mathbb{R}^n$ and $X = \tilde{1}$ be a constant intuitionistic L-fuzzy subset of M. Let $B(a, r, b, c) = (\mu_{B(a,r,b,c)}, \nu_{B(a,r,b,c)})$ be an intuitionistic L-fuzzy subset of M that $\mu_{B(a,r,b,c)}$ and $\nu_{B(a,r,b)}$ are equal to 0 and 1 outside or on the sphere $B(a, r)$ and equal to the functions b and c on M with values in L, respectively, where $0 \leq b + c \leq 1$. We call the intuitionistic L-fuzzy topology induced by

$$\beta_{ILn} = \{B(a, r, b, c), a \in \mathbb{R}^n, r \in \mathbb{R}^+, b, c : B(a, r) \to L, are\ functions\ s.t.\ 0 \leq b + c \leq 1\}$$

the intuitionistic L-fuzzy Euclidean topology of dimension n and denote it by τ_{ILn}. Therefore we have the IL-fuzzy Euclidean topological space $(\tilde{1}_{\mathbb{R}^n},\ \tau_{ILn})$.

Definition 4.3. *Let $\mathfrak{T},\ \mathfrak{T}^* : IL_X^M \to L$, be two mappings satisfying:*

(i $\forall A = (\mu_A, \nu_A) \in IL_X^M\ ,\ \ 0 \leq \mathfrak{T}(A) + \mathfrak{T}^*(A) \leq 1$

(ii $,\mathfrak{T}(X) = \mathfrak{T}(\tilde{0}) = 1,\ \ \mathfrak{T}^*(X) = \mathfrak{T}^*(\tilde{0}) = 0$

(iii $\forall A=(\mu_{A},\nu_{A}),\ B=(\mu_{A},\nu_{A})\in IL_X^M$

$\mathfrak{T}(A\cap B)\geq T\big(\mathfrak{T}(A),\mathfrak{T}(B)\big),\quad \mathfrak{T}^*(A\cap B)\leq C\big(\mathfrak{T}^*(A)\vee\mathfrak{T}^*(B)\big),$

(iv $,\forall\{A_j=(\mu_{A_j},\nu_{A_j}),\ j\in J\}\subseteq IL_X^M$

$\mathfrak{T}(\bigcup_{j\in J}A_j)\geq\bigwedge_{i,j\in J}T\big(\mathfrak{T}(A_i),\mathfrak{T}(A_j)\big),\quad \mathfrak{T}^*(\bigcup_{j\in J}A_j)\leq\bigvee_{i,j\in J}C\big(\mathfrak{T}^*(A_i)\vee \mathfrak{T}^*(A_j)\big)$

Then $(\mathfrak{T},\ \mathfrak{T}^*)$ *is called an intuitionistic* L*-gradation of openness and nonopenness with respect to the* $L(T,C)$*-norm, (briefly* $IL(T,C)$*-gradation of openness) and* $(X,\ \mathfrak{T},\ \mathfrak{T}^*)$ *is called an* $ILG(T,C)$*-fuzzy topological space (*$ILG(T,C)$*-fts).*

Example 4.4. Let $M=\mathbb{R}^n$ and $X=\tilde{1}$ be a constant intuitionistic L-fuzzy subset of M. As two useful examples, we define $\mathfrak{T}_{ILn},\mathfrak{T}^*_{ILn}\ :\ IL_X^M\to L$, by

$$\mathfrak{T}_{ILn}(B)=\begin{cases}1 & B\in\tau_{_{ILn}},\\ 0 & elsewhere.\end{cases}\qquad \mathfrak{T}^*_{ILn}(B)=\begin{cases}0 & B\in\tau_{_{ILn}},\\ 1 & elsewhere.\end{cases}\tag{4.1}$$

and $\mathfrak{T}_{Linf},\mathfrak{T}^*_{Linf}\ :\ L_X^M\to L$, by

$$\mathfrak{T}_{Linf}(B)=\begin{cases}1 & B=\tilde{0},\\ inf\{B(x):x\in M\} & \tilde{0}\neq B\in\tau_{_{ILn}},\\ 0 & elsewhere,\end{cases}\tag{4.2}$$

$$\mathfrak{T}^*_{Linf}(B)=\begin{cases}0 & B=\tilde{0},\\ sup\{B(x):x\in M\} & \tilde{0}\neq B\in\tau_{_{ILn}},\\ 1 & elsewhere,\end{cases}\tag{4.3.}$$

Obviously both are intuitionistic L-gradation of openness and nonopenness w.r.t. the $L(T_{min},C_{max})$-norm. In general if $(\mathfrak{T},\ \mathfrak{T}^*)$ is any $IL(T,C)$-gradation of openness on $1_{\mathbb{R}^n}$, such that $supp\mathfrak{T}=\tau_{_{ILn}}$, then we call $(1_{\mathbb{R}^n},\ \mathfrak{T}_{ILn})$ the $ILG(T,C)$-fuzzy Euclidean topological space.

Proposition 4.5. *Let* $(X,\mathfrak{T},\ \mathfrak{T}^*)$ *be an* $ILG(T,C)$*-fuzzy topological space. For any* $r,s\in[0,1]$*, such that* $0\leq r+s\leq 1$ *we define* $\mathfrak{T}_{r,s}=\{A\in L_X^M:\mathfrak{T}(A)\geq r,\ \mathfrak{T}^*(A)\leq s\}$*. Then* $(X,\mathfrak{T}_{r,s})$ *is a fuzzy topological space.*

Proof. Since Dom$\mathfrak{T}=L_X^M$, then for all $A\in supp\mathfrak{T}$, we have A is less than or equal to X. Hence $suppA\subseteq suppX$. Also we have

i) $\mathfrak{T}(\tilde{0})=\mathfrak{T}(X)=1\geq r,\ \mathfrak{T}^*(\tilde{0})=\mathfrak{T}(X)=0\leq s\ \Rightarrow\ \phi,\ X\in\mathfrak{T}_{r,s}.$

ii) For any $A, B \in \mathfrak{T}_{r,s}$, using the condition (iii) of Definition 4.3 and $(LT2)$ we have

$$\mathfrak{T}(A \cap B) \geq T\big(\mathfrak{T}(A), \mathfrak{T}(B)\big) \geq T(r,r) = r,$$

$$\mathfrak{T}^*(A \cap B) \leq C\big(\mathfrak{T}^*(A) \vee \mathfrak{T}^*(B)\big) \leq C(s,s) = s.$$

Thus $A \cap B \in \mathfrak{T}_{r,s}$.

iii) For all family $\{A_j = (\mu_{A_j}, \nu_{A_j}),\ j \in J\} \subseteq IL_X^M$, we have

$$\mathfrak{T}(\bigcup_{j\in J} A_j) \geq \bigwedge_{i,j\in J} T\big(\mathfrak{T}(A_i), \mathfrak{T}(A_j)\big) \geq \bigwedge_{i,j\in J} T(r,r)) = r$$

$$\mathfrak{T}^*(\bigcup_{j\in J} A_j) \leq \bigvee_{i,j\in J} C\big(\mathfrak{T}^*(A_i) \vee \mathfrak{T}^*(A_j)\big) \leq \bigvee_{i,j\in J} C(s,s) = s.$$

Hence $\bigcup_{j\in J} A_j \in \mathfrak{T}_{r,s}$.

Therefore $(X, \mathfrak{T}_{r,s})$ is a fuzzy topological space.

Definition 4.6. *Let L be a lattice set. If any countable subset $\{x_i \mid i \in J \subseteq \mathbb{N}\}$ of L, has an infimum in L, then L is called a semicomplete lattice.*

Proposition 4.7. *We assume that X is an intuitionistic L-fuzzy subset of M and T, C are respectively LT-norm and LC-conorm on L . Let $\mathfrak{M}_{\mathfrak{T},\mathfrak{T}^*}(X)$ be the set of all $IL(T,C)$-gradations of openness on X. We write $(\mathfrak{T}_1, \mathfrak{T}_1^*) \leq (\mathfrak{T}_2, \mathfrak{T}_2^*)$ if we have $\mathfrak{T}_1(A) \leq \mathfrak{T}_2(A)$, $\mathfrak{T}_1^*(A) \geq \mathfrak{T}_2^*(A)$, for all $A \in L_X^M$. Then $(\mathfrak{M}_{\mathfrak{T},\mathfrak{T}^*}(X), \leq)$ is a semicomplete lattice.*

Proof. It is clear that $\leq$ between functions from IL_X^M to L, is an equivalence relation. Hence $(\mathfrak{M}_{\mathfrak{T}}(X), \mathfrak{T}^* \ \leq)$ is a partially ordered set. We define

$$\mathfrak{T}_0(\tilde{0}) = \mathfrak{T}_0(X) = 1, \quad \mathfrak{T}_0^*(\tilde{0}) = \mathfrak{T}_0^*(X) = 0,$$

$$\mathfrak{T}_0(A) = 0, \ \ , \mathfrak{T}_0^*(A) = 1, \quad \forall A \in L_X^M - \{\tilde{0}, X\},$$

$$\mathfrak{T}_1(A) = 1, \quad \mathfrak{T}_1^*(A) = 0, \quad \forall A \in L_X^M.$$

Then $(\mathfrak{T}_0, \mathfrak{T}_0^*)$ and $(\mathfrak{T}_1, \mathfrak{T}_1^*)$ are two $IL(T,C)$-gradation of openness on X. Since we have

$$\mathfrak{T}_0(A) \leq \mathfrak{T}(A) \leq \mathfrak{T}_1(A), \ \ \mathfrak{T}_0^*(A) \geq \mathfrak{T}^*(A) \geq \mathfrak{T}_1^*(A), \quad \forall A \in L_X^M,$$

then $(\mathfrak{T}_0, \mathfrak{T}_0^*)$, $(\mathfrak{T}_1, \mathfrak{T}_1^*)$ are respectively, $0, 1$ in the lattice set $\mathfrak{M}_{\mathfrak{T},\mathfrak{T}^*}(X)$.

We show that every countable subset $\{(\mathfrak{T}_j, \mathfrak{T}_j^*),\ j \in \mathbb{N}\}$ of $\mathfrak{M}_{\mathfrak{T},\mathfrak{T}^*}(X)$ has an infimum in it.

Define $(\mathfrak{T}, \mathfrak{T}^*)$ by $\mathfrak{T}(A) = T^{\infty}_{ⓢ}(\{\mathfrak{T}_i(A)\})$ and $\mathfrak{T}^*(A) = C^{\infty}_{ⓢ}(\{\mathfrak{T}_i^*(A)\})$. Then we have

$$0 \leq \mathfrak{T}(A) + \mathfrak{T}^*(A) = T^{\infty}_{ⓢ}(\{\mathfrak{T}_i(A)\}) + C^{\infty}_{ⓢ}(\{\mathfrak{T}_i^*(A)\}) \leq \mathfrak{T}_1(A) + \mathfrak{T}_1^*(A) \leq 1$$

for all $A \in IL_X^M$. Since for each $i \in \mathbb{N}$ we have

$$\mathfrak{T}_i(X) = \mathfrak{T}_i(\tilde{0}) = 1, \quad \mathfrak{T}_i^*(X) = \mathfrak{T}_i^*(\tilde{0}) = 0,$$

hence

$$T\big(\mathfrak{T}_1(X), \mathfrak{T}_2(X)\big) = T(1,1) = 1 \implies T\big(T(\mathfrak{T}_1(X), \mathfrak{T}_2(X)), \mathfrak{T}_3(X)\big) = T(1,1) = 1,$$

$$C\big(\mathfrak{T}_1^*(X), \mathfrak{T}_2^*(X)\big) = C(0,0) = 0 \implies C\big(C(\mathfrak{T}_1^*(X), \mathfrak{T}_2^*(X)), \mathfrak{T}_3^*(X)\big) = C(0,0) = 0,$$

By contradiction on k, we can show $T^{k}_{ⓢ}(\{\mathfrak{T}_i(X)\}) = 1$ and $C^{k}_{ⓢ}(\{\mathfrak{T}_i(X)\}) = 0$ for each $k \in \mathbb{N}$.
Therefore $\mathfrak{T}(X) = 1$ and $\mathfrak{T}^*(X) = 0$. Similarly we can show $\mathfrak{T}(\tilde{0}) = 1$ and $\mathfrak{T}^*(\tilde{0}) = 0$.
Also for each $A, B \in IL_X^M$ we have

$$\begin{aligned}
T^{3}_{ⓢ}(\{\mathfrak{T}_i(A \cap B)\}) &= T\Big(T\big(\mathfrak{T}_1(A \cap B)), \mathfrak{T}_2(A \cap B))\big), \mathfrak{T}_3(A \cap B)\Big)\\
&\geq T\Big(T\Big(T(\mathfrak{T}_1(A), \mathfrak{T}_1(B)), T\big(\mathfrak{T}_2(A), \mathfrak{T}_2(B)\big)\Big), T\big(\mathfrak{T}_3(A), \mathfrak{T}_3(B)\big)\Big)\\
&= T\Big(T\Big(T\big(\mathfrak{T}_1(A), \mathfrak{T}_2(A)\big), T\big(\mathfrak{T}_1(B), \mathfrak{T}_2(B)\big)\Big), T\big(\mathfrak{T}_3(A), \mathfrak{T}_3(B)\big)\Big) \quad \textit{by } (3.1)\\
&= T\Big(T\Big(T\big(\mathfrak{T}_1(A), \mathfrak{T}_2(A)\big), \mathfrak{T}_3(A)\Big),\ T\Big(T\big(\mathfrak{T}_1(B), \mathfrak{T}_2(B)\big), \mathfrak{T}_3(B)\Big)\Big) \quad \textit{by } (3.1)\\
&= T\Big(T^{3}_{ⓢ}(\{\mathfrak{T}_i(A)\}), T^{3}_{ⓢ}(\{\mathfrak{T}_i(B)\}),\Big)
\end{aligned}$$

By contradiction on k, we can show for each $k \in \mathbb{N}$ we have

$$T^{k}_{ⓢ}(\{\mathfrak{T}_i(A \cap B)\}) \geq T\Big(T^{k}_{ⓢ}(\{\mathfrak{T}_i(A)\}), T^{k}_{ⓢ}(\{\mathfrak{T}_i(B)\}),\Big)$$

Therefore

$$\begin{aligned}\mathfrak{T}(A\cap B) &= T^{\infty}_{\circledS}(\{\mathfrak{T}_i(A\cap B)\})\\ &= \lim_{k\to\infty} T^{k}_{\circledS}(\{\mathfrak{T}_i(A\cap B)\})\\ &\geq \lim_{k\to\infty} T\Big(T^{k}_{\circledS}(\{\mathfrak{T}_i(A)\}), T^{k}_{\circledS}(\{\mathfrak{T}_i(B)\}),\Big) \quad \textit{by } (3.1)\\ &\geq T\Big(T^{\infty}_{\circledS}(\{\mathfrak{T}_i(A)\}), T^{\infty}_{\circledS}(\{\mathfrak{T}_i(B)\})\Big)\\ &= T\big(\mathfrak{T}(A), \mathfrak{T}(B)\big),\end{aligned}$$

Similarly we can prove that $\mathfrak{T}^*(A\cap B) \leq C\big(\mathfrak{T}^*(A) \vee \mathfrak{T}^*(B)\big)$.
For any arbitrary family $\{A_k,\ k\in K\}\subseteq IL^M_X$, we have

$$\mathfrak{T}_j(\bigcup_{k\in K} A_k) \geq \bigwedge_{k,l\in K} T\big(\mathfrak{T}_j(A_k), \mathfrak{T}_j(A_l)\big),$$

for each $j\in\mathbb{N}$. Hence

$$\begin{aligned} &T^{3}_{\circledS}(\{\mathfrak{T}_i(\textstyle\bigcup_{k\in K} A_k)\}) = T\Big(T\big(\mathfrak{T}_1(\bigcup_{k\in K} A_k)), \mathfrak{T}_2(\bigcup_{k\in K} A_k))\big), \mathfrak{T}_3(\bigcup_{k\in K} A_k)\Big)\\ &\geq T\Big(T\Big(\textstyle\bigwedge_{k,l\in K} T\big(\mathfrak{T}_1(A_k), \mathfrak{T}_1(A_l)\big),\ \bigwedge_{k,l\in K} T\big(\mathfrak{T}_2(A_k), \mathfrak{T}_2(A_l)\big)\Big), \bigwedge_{k,l\in K} T\big(\mathfrak{T}_3(A_k), \mathfrak{T}_3(A_l)\big)\Big)\\ &\geq \textstyle\bigwedge_{k,l\in K} T\Big(T\Big(T\big(\mathfrak{T}_1(A_k), \mathfrak{T}_1(A_l)\big),\ T\big(\mathfrak{T}_2(A_k), \mathfrak{T}_2(A_l)\big)\Big), T\big(\mathfrak{T}_3(A_k), \mathfrak{T}_3(A_l)\big)\Big)\\ &= \textstyle\bigwedge_{k,l\in K} T\Big(T\Big(T\big(\mathfrak{T}_1(A_k), \mathfrak{T}_2(A_k)\big),\ T\big(\mathfrak{T}_2(A_l), \mathfrak{T}_2(A_l)\big)\Big), T\big(\mathfrak{T}_3(A_k), \mathfrak{T}_3(A_l)\big)\Big)\\ &= \textstyle\bigwedge_{k,l\in K} T\Big(T\Big(T\big(\mathfrak{T}_1(A_k), \mathfrak{T}_2(A_k)\big), \mathfrak{T}_3(A_k)\Big),\ T\Big(T\big(\mathfrak{T}_1(A_l), \mathfrak{T}_2(A_l)\big), \mathfrak{T}_3(A_l)\Big)\Big)\\ &= \textstyle\bigwedge_{k,l\in K} T\Big(T^{3}_{\circledS}(\{\mathfrak{T}_i(A_k)\}), T^{3}_{\circledS}(\{\mathfrak{T}_i(A_l)\})\Big)\end{aligned}$$

By contradiction on k, we can show for each $k\in\mathbb{N}$ we have

$$T^{k}_{\circledS}(\{\mathfrak{T}_i(\bigcup_{j} A_j)\}) \geq \bigwedge_{k,l\in K} T\Big(T^{k}_{\circledS}(\{\mathfrak{T}_i(A_k)\}), T^{k}_{\circledS}(\{\mathfrak{T}_i(A_l)\})\Big)$$

Therefore

$$\begin{aligned}\mathfrak{T}(\bigcup_j A_j) &= T^{\infty}_{\circledS}(\{\mathfrak{T}_i(\bigcup_j A_j)\}) \\ &= \lim_{k\to\infty} T^{k}_{\circledS}(\{\mathfrak{T}_i(\bigcup_j A_j)\}) \\ &\geq \lim_{k\to\infty} \bigwedge_{k,l\in K} T\Big(T^{k}_{\circledS}(\{\mathfrak{T}_i(A_k)\}), T^{k}_{\circledS}(\{\mathfrak{T}_i(A_l)\})\Big) \\ &= \bigwedge_{k,l\in K} \lim_{k\to\infty} T\Big(T^{k}_{\circledS}(\{\mathfrak{T}_i(A_k)\}), T^{k}_{\circledS}(\{\mathfrak{T}_i(A_l)\})\Big) \\ &= \bigwedge_{k,l\in K} T\Big(T^{\infty}_{\circledS}(\{\mathfrak{T}_i(A_k)\}), T^{\infty}_{\circledS}(\{\mathfrak{T}_i(A_l)\})\Big) \\ &= \bigwedge_{k,l\in K} T\big(\mathfrak{T}(A_k), \mathfrak{T}(A_l)\big).\end{aligned}$$

Similarly we can prove that

$$\mathfrak{T}^*(\bigcup_{j\in J} A_j) \leq \bigvee_{k,l\in K} C\big(\mathfrak{T}^*(A_k), \mathfrak{T}^*(A_l)\big).$$

Hence $(\mathfrak{T}, \mathfrak{T}^*) \in \mathfrak{M}_{\mathfrak{T},\mathfrak{T}^*}(X)$. Therefore this lattis set is semicomplete.

Definition 4.8. *Let* $\mathfrak{C}, \mathfrak{C}^* : L_X^M \to L$ *satisfy following conditions:*

i) $0 \leq \mathfrak{C}(A) + \mathfrak{C}^*(A) \leq 1$, *for all* $A \in L_X^M$.

ii) $\mathfrak{C}(X) = \mathfrak{C}(\tilde{0}) = 1$, $\mathfrak{C}^*(X) = \mathfrak{C}^*(\tilde{0}) = 0$

iii—) For all $A, B \in L_X^M$ *we have*

$$\mathfrak{C}(A \cup B) \geq \mathfrak{C}(A) \wedge \mathfrak{C}(B) \mathfrak{C}^*(A \cup B) \leq \mathfrak{C}^*(A) \vee \mathfrak{C}^*(B)$$

iv) For all family $\{A_j,\ j \in J\} \subseteq L_X^M$
$\mathfrak{C}(\bigcap_{j\in J} A_j) \geq \bigwedge_{j\in J} \mathfrak{C}(A_j)$ $\quad \mathfrak{C}^*(\bigcap_{j\in J} A_j) \leq \bigvee_{j\in J} \mathfrak{C}^*(A_j)$.

then the pair $(\mathfrak{C},\ \mathfrak{C}^*)$ *is called an intuitionistic* L*-gradation of closeness and non-closeness with respect to the* $L(T, C)$*-norm on* X *(briefly* $IL(T, C)$*-gradation of closeness).*

Proposition 4.9. *Let* $(\mathfrak{T}, \mathfrak{T}^*)$ *and* $(\mathfrak{C}, \mathfrak{C}^*)$ *be* $IL(T, C)$*-gradation of openness and closeness on* X *respectively, then*

i) The pair $(\mathfrak{T}_{\mathfrak{C}}, \mathfrak{T}^*_{\mathfrak{C}^*})$ *defined by* $\mathfrak{T}_{\mathfrak{C}}(A) = \mathfrak{C}(X - A)$ *and* $\mathfrak{T}^*_{\mathfrak{C}^*}(A) = \mathfrak{C}^*(X - A)$ *is an*
IL-fuzzy of X defined by

$$\mu_{X-A}(p) = \mu_X(p) - \mu_A(p), \qquad \nu_{X-A}(x) = 1 - \mu_X(p) + \mu_A(p).$$

ii) The pair $(\mathfrak{C}_{\mathfrak{T}}, \mathfrak{C}^*_{\mathfrak{T}^*})$ *defined by* $\mathfrak{C}_{\mathfrak{T}}(A) = \mathfrak{T}(X - A)$ *and* $\mathfrak{C}^*_{\mathfrak{T}^*}(A) = \mathfrak{T}^*(X - A)$, *is an $IL(T, C)$-gradation of closeness on X.*

iii) We have $(\mathfrak{C}_{\mathfrak{T}_{\mathfrak{C}}}, \mathfrak{C}^*_{\mathfrak{T}^*_{\mathfrak{C}^*}}) = (\mathfrak{C}, \mathfrak{C}^*)$ *and* $(\mathfrak{T}_{\mathfrak{C}_{\mathfrak{T}}}, \mathfrak{T}^*_{\mathfrak{C}^*_{\mathfrak{T}^*}}) = (\mathfrak{T}, \mathfrak{T}^*)$.

Proof. The proof is straightforward.

Example 4.10. Let $E = \Lambda\mathbb{R}^m$ be an exterior algebra on $\mathbb{R}^m$ with anticommutative generators $\{\xi_1, \ldots, \xi_m\}$. Hence $\xi_i^2 = 0$, and $\xi_j \wedge \xi_i = -\xi_i \wedge \xi_j$. Then each $\xi \in E$ has the form

$$\xi = \sum_{1 \le i_1 < \ldots < i_k \le m} \alpha_{i_1 \ldots i_k}\ \xi_{i_1} \wedge \ldots \wedge \xi_{i_k}, \qquad \alpha_{i_1 \ldots i_k} \in \mathbb{R}$$

Let $B(r_1, \ldots, r_m, t_1, \ldots, t_m) = (\mu_B, \nu_B)$ be an intuitionistic L-fuzzy subset of E, defined by

$$\mu_B(\xi_i) = r_i, \qquad \nu_B(\xi_i) = t_i, \quad r_i, t_i \in L, \quad s.t. \quad 0 \le r_i + t_j \le 1,$$

for all $1 \le i, j \le m$. Hence we have

$$\mu_B(\xi) = \sup_{1 \le i_1 < \ldots < i_k \le m} \left\{ T\Big(\ldots T(T(r_{i_1}, r_{i_2}), r_{i_3}), \ldots, r_{i_k} \Big) \right\} \tag{4.4}$$

$$\nu_B(\xi) = \inf_{1 \le i_1 < \ldots < i_k \le m} \left\{ C\Big(\ldots C(C(t_{i_1}, t_{i_2}), t_{i_3}), \ldots, t_{i_k} \Big) \right\}. \tag{4.5}$$

Then $B = (\mu_B, \nu_B)$ is an $IL(T, C)$-fuzzy subspace of E.
Proof. Let $r = \max\{r_1, r_2, \ldots, r_m\}$ and $s = \min\{s_1, s_2, \ldots, s_m\}$. Then $\forall \xi \in E$,

$$0 \le \mu_B(\xi) + \nu_B(\xi) \le r + s \le 1.$$

Also for each $\xi,\ \eta \in E$ and $\gamma, \alpha \in k$, we have

$$\eta = \sum_{1 \le j_1 < \ldots < j_l \le m} \beta_{j_1 \ldots j_l}\ \xi_{j_1} \wedge \ldots \wedge \xi_{j_l}, \qquad \beta_{j_1 \ldots j_l} \in \mathbb{R}$$

$$T\big(\mu_B(\xi), \mu_B(\eta)\big) = T\Bigg(\sup_{1\le i_1<\ldots<i_k\le m} \big\{T\big(\ldots T(T(r_{i_1}, r_{i_2}), r_{i_3}), \ldots, r_{i_k}\big)\big\},$$

$$\sup_{1\le j_1<\ldots<j_l\le m} \big\{T\big(\ldots T(T(r_{j_1}, r_{j_2}), r_{j_3}), \ldots, r_{j_k}\big)\big\}\Bigg)$$

$$\le \sup_{1\le i_1<\ldots<i_k\le m} \big\{T\big(\ldots T(T(r_{i_1}, r_{i_2}), r_{i_3}), \ldots, r_{i_k}\big)\big\},$$

$$\bigvee \sup_{1\le j_1<\ldots<j_l\le m} \big\{T\big(\ldots T(T(r_{j_1}, r_{j_2}), r_{j_3}), \ldots, r_{j_k}\big)\big\} \quad by\ (3.1)$$

$$= \mu_B(\gamma\ \xi + \alpha\ \eta),$$

Similarly we can prove

$$C\big(\nu_B(\xi), \nu_B(\eta)\big) \ge \nu_B(\gamma\ \xi + \alpha\ \eta).$$

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