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THE EULER METHOD IN THE BLOW-UP NUMERICAL SOLUTIONS FOR A REACTION-DIFFUSION PROBLEMS WITH BOUNDARY CONDITIONS

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Abstract: This paper concerns the study of the numerical approximation for the following initial-boundary value problem

$$\begin{cases} u_t(x,t) - u_{xx}(x,t) = \gamma e^{u(a,t)}, \quad x \in (0,1), \quad t \in (0,T), \\ u(0,t) = 0, \quad u_x(1,t) = 0, \quad t \in (0,T), \\ u(x,0) = u_0(x) \geq 0, \quad x \in [0,1], \end{cases}$$

where $u_0 \in C^1([0,1])$, $u_0(0) = 0$, $u_0'(1) = 0$. $a \in (0,1)$, γ is a positif parameter. We find some conditions under which the solution of a semidiscrete form of the above problem blows up in a finite time and estimate its semidiscrete blow-up time. We study the asymptotic behavior of a semi-discrete numerical approximation. We also prove the convergence of the semidiscrete blow-up time to the theoretical one. A similar study has been also undertaken for a discrete form of the above problem. Finally, we give some numerical results to illustrate our analysis.Also obtaining results on the convergence of the numerical blow-up times to the theoretical limit

when the mesh parameter is small enough.

Keywords and Phrases: Semidiscretization in space, Quasilinear reaction diffusion equation, blow-up, numerical blow-up time, Euler method, Asymptotic behaviour.

2020 Mathematics Subject Classification: 35B40, 35B50, 35K60, 65M06.

1. Introduction and Definitions

Consider the following initial-boundary value problem

$$u_t(x,t) - u_{xx}(x,t) = \gamma e^{u(a,t)}, x \in (0,1), \qquad t \in (0,T), \tag{1.1}$$

$$u(0,t) = 0, \quad u_x(1,t) = 0, t \in (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \geq 0, x \in [0,1], \tag{1.3}$$

which models the temperature distribution of a large number of physical phenomena from physics, chemistry and biology. The diffusion and the boundary condition have the tendency to decrease the solution. The initial data $u_0(x)$ is a continuous and non decreasing function in [0,1], $u_0(0) = 0$, $u_0'(1) = 0$. Here $(0,T)$ is the maximal time interval of existence of the solution u. The time T may be finite or infinite. When T is infinite, we say that the solution u exists globally. When T is finite, the solution u develops a singularity in a finite time, namely

$$\lim_{t \to T} \|u(x,t)\|_\infty = +\infty,$$

where $\|u(x,t)\|_\infty = \max_{0 \leq x \leq 1} |u(x,t)|$.
In this case, we say that the solution u blows up in a finite time and the time T is called the blow-up time of the solution u.
Blow-up phenomena for reaction diffusion problems in bounded domains have been studied for the first time in a seminal paper by Kaplan [12].
Solutions of nonlinear reaction diffusion equations which blow up in a finite time have been the subject of investigation of many authors (see [1], [2], [15-16], [19-21] and the references cited therein). In particular, the above problem has been studied and existence and uniqueness of a classical solution has been proved. Under some assumptions, it is also shown that the classical solution blows up in a finite time and its blow-up time has been estimated.
This problem arises in combustion theory where blow-up phenomena play an important role. In this paper we are interesting in the numerical study of the above

problem. The rest of the paper is organized as follows. In the next section, we give some result which will be used later. In the third section, under some assumptions, we show that the solution of the semidiscrete problem blows up in a finite time and estimate its semidiscrete blow-up time. In the fourth section, we show that, under some additional hypotheses, the semidiscrete blow-up time goes to the real one when the mesh size goes to zero. In the fifth section we obtain similar results as in section 3 and 4 using a discrete scheme. Finally, in the last section, we give some numerical results to illustrate our analysis.

2. Semi-discrete Problem and its Properties

In this section, we give some lemmas about the discrete maximum principle for localized parabolic problems and reveal certain properties concerning the semidiscrete solution. After discretizing the spatial coordinate x. Let I be a positive integer,$a = 1$ and let $h = 1/I$. Define the grid $x_i = ih$, $0 \leq i \leq I$ and approximate the solution u of the problem (1.1)–(1.3) by the solution $U_h(t) = (U_0(t), U_1(t), \ldots, U_I(t))^T$ of the following semidiscrete equations

$$\frac{dU_i(t)}{dt} = \delta^2 U_i(t) + \gamma e^{U_I(t)}, 1 \leq i \leq I, \qquad t \in (0, T_b^h), \tag{1.4}$$

$$U_0(t) = 0, t \in (0, T_b^h), \tag{1.5}$$

$$U_i(0) = \varphi_i \geq 0, 0 \leq i \leq I, \tag{1.6}$$

where

$$\varphi_{i+1} \geq \varphi_i, \quad 0 \leq i \leq I-1,$$

$$\delta^2 U_i(t) = \frac{U_{i-1}(t) - 2U_i(t) + U_{i+1}(t)}{h^2}, \quad 1 \leq i \leq I-1,$$

$$\delta^2 U_I(t) = \frac{2U_{I-1}(t) - 2U_I(t)}{h^2}.$$

Here $(0, T_b^h)$ is the maximal time interval on which $\|U_h(t)\|_\infty$ is finite where $\|U_h(t)\|_\infty = \max_{0 \leq i \leq I} |U_i(t)|$. When T_b^h is finite, we say that the solution u of (2.1)-(2.3) blows up in a finite time.

In [5], the authors have considered the problem (1.1)-(1.3) in the case where the boundary conditions are replaced by the Dirichlet boundary conditions and the initial data is symmetric. They have considered a scheme as the one given in (2.1)-(2.3). They have shown that the semidiscrete solution blows up in a finite time and its blow-up time goes to the real one when the mesh size tends to zero. Let us notice that when u vanishes, the source $e^u = 1 \neq 0$. In this case, it is not easy to have the convergence of the semidiscrete blow-up time. In fact, if e^u is replaced by u^p with p¿1, one establishes that there exists a constant $A > 0$ such that

$$\frac{dU_i(t)}{dt} \geq AU_i^p(t), 0 \leq i \leq I, \tag{1.7}$$

and this estimate is crucial to obtain the convergence of the semidiscrete blow-up time. It is not possible to establish the estimate in (2.4) when the term of the source is γe^u. In this case, one introduces an auxiliary function and obtain an estimate of the following form

$$\frac{dU_i(t)}{dt} \geq c_i(t)e^{U_i(t)}, \quad 0 \leq i \leq I.$$

(see [5] for the details). In this paper, firstly, we show that under some assumptions, the solution of the semidiscrete problem defined in (2.1)-(2.3) blows up in a finite time and estimate its semidiscrete blow-up time. We also show that the semidiscrete blow-up time converges to the real one when the mesh size goes to zero. A similar study has been also undertaken for a full discrete form of (1.1)-(1.3). Let us notice that in [5], only the semidiscrete scheme has been analyzed. At the end of the paper, we have shown how one may treat the case Dirichlet boundary condition. One may find in [8] similar studies concerning other parabolic problems. Let us notice that many authors have used numerical methods to study the phenomenon of blow-up but there are only a few studies on the convergence of the numerical blow-up time for solutions which blow up in L^∞ norm. Four instance in [10], the authors have proved the convergence of numerical blow-up time for solutions which blow up in L^p norm with $1 < p < \infty$. The following lemma is a discrete form of the maximum principle for localized reaction diffusion problems.

Lemma 2.1. *Let $U_h \in \mathbb{R}^{I+1}$ such that $U_h \geq 0$. Then we have*

$$\delta^2 e^{U_I} \geq e^{U_I}\delta^2 U_I(t).$$

Proof. Apply Taylor's expansion to obtain

$$\delta^2 e^{U_I} = e^{U_I}\delta^2 U_I + \frac{(U_{I-1} - U_I)^2}{h^2}e^{\eta_I}.$$

Use the fact that $U_h \geq 0$ to complete the rest of the proof.
To end this section, let us give another property of the operator δ^2.

Lemma 2.2. *Let U_h and $U_h \in C^1([0,T],\ R^{I+1})$ if $\delta^+(U_i)\ \delta^+(V_i) \geq 0$ and $\delta^-(U_i)$ $\delta^-(V_i) \geq 0$*

$$\delta^2(U_i V i) \geq U_i \delta^2(V_i) + V_i \delta^2(U_i),$$

where $\delta^+(U_i) = \frac{U_{i+1}-U_i}{h}$ and $\delta^-(U_i) = \frac{U_{i-1}-U_i}{h}$.
Proof. A straightforward computation yields

$$\begin{aligned} h^2\delta^2(U_iV_i) &= U_{i+1}V_{i+1} - 2U_iV_i + U_{i-1}V_{i-1} = (U_{i+1}-U_i)(V_{i+1}-V_i) \\ &\quad + V_i(U_{i+1}-U_i) + U_i(V_{i+1}-V_i) + U_iV_i - 2U_iV_i \\ &\quad + (U_{i-1}-U_i)(V_{i-1}-V_i) + (U_{i-1}-U_i)V_i + U_i(V_{i-1}-V_i) + U_iV_i. \end{aligned}$$

Which implies that

$$\delta^2(U_iV_i) = \delta^+(U_i)\delta^+(V_i) + \delta^-(U_i)\delta^-(V_i) + V_i\delta^2(U_i).$$

Using the assumption of the lemma. We obtain the desired result.

3. Full Discretizations of the Problem

In this section, we study the phenomenon of blow-up, using a full discrete explicit scheme of (1.1)-(1.3). Approximate the solution $u(x,t)$ of the continuous problem by the solution $U_h^n = (U_0^{(n)}, U_1^{(n)}, \ldots, U_I^{(n)})^T$ of the following explicit scheme

$$\delta_t U_i^{(n)} = \delta^2 U_i^{(n)} + \gamma e^{U_i^{(n)}}, 0 \leq i \leq I, \tag{3.1}$$

$$U_0^{(n)} = 0, \tag{3.2}$$

$$U_i^{(0)} = \varphi_i \geq 0, 0 \leq i \leq I, \tag{3.3}$$

where $n \geq 0$ $\varphi_{i+1} > \varphi_i$ $0 \leq i \leq I-1$,

$$\delta_t U_i^{(n)} = \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n},$$

In order to permit the discrete solution to reproduce the properties of the continuous one when the time t approaches the blow-up time T, we need to adapt the size of the time step so that we take

$$\Delta t_n = \min\{\frac{h^2}{3}, \tau e^{-\|U_h^{(n)}\|_\infty}\}, \quad 0 < \tau < 1.$$

Let us notice that the restriction on the time step ensures the positivity of the discrete solution.

Definition 3.1. *We say that the solution $U_h^{(n)}$ of the explicit scheme blows up in a finite time if* $\lim_{n\to\infty} \|U_h^{(n)}\|_\infty = \infty$, *and the series* $\sum_{n=0}^{\infty} \Delta t_n$ *converges. The quantity* $\sum_{n=0}^{\infty} \Delta t_n$ *is called the numerical blow-up time of the discrete solution.*

Lemma 3.1. *Let $U_h^{(n)}$ be the solution of (3.1)-(3.3). Then we have* $U_{i+1}^{(n)} > U_i^{(n)}$, $0 \le i \le I-1$.

Proof. Let $Z_i^{(n)} = U_{i+1}^{(n)} - U_i^{(n)}$, $0 \le i \le I-1$. Obviously, $Z_0^{(n)} > 0$. A routine computation reveals that

$$\frac{Z_i^{(n+1)} - Z_i^{(n)}}{\Delta t_n} = \frac{Z_{i+1}^{(n)} - 2Z_i^{(n)} + Z_{i-1}^{(n)}}{h^2} + e^{U_{i+1}^{(n)}} - e^{U_i^{(n)}}, \quad 0 \le i \le I-2,$$

$$\frac{Z_{I-1}^{(n+1)} - Z_{I-1}^{(n)}}{\Delta t_n} = \frac{-3Z_{I-1}^{(n)} + Z_{I-2}^{(n)}}{h^2} + e^{U_I^{(n)}} - e^{U_{I-1}^{(n)}},$$

Apply the mean value theorem to obtain

$$Z_i^{(n+1)} = \frac{\Delta t_n}{h^2} Z_{i+1}^{(n)} + (1 - \frac{2\Delta t_n}{h^2}) Z_i^{(n)} + \frac{\Delta t_n}{h^2} Z_{i-1}^{(n)} + e^{\xi_i^{(n)}} Z_i^{(n)}, \quad 1 \le i \le I-2,$$

$$Z_{I-1}^{(n+1)} = \frac{\Delta t_n}{h^2} Z_{I+1}^{(n)} + (1 - \frac{3\Delta t_n}{h^2}) Z_{I-1}^{(n)} + e^{\xi_i^{(n)}} Z_i^{(n)},$$

where $\xi_i^{(n)}$ is an intermediate value between U_i and U_{i+1}. Since $Z_i^{(0)} \ge 0$, $1 \le i \le I-1$, we deduce by induction that $Z_i^{(n)} \ge 0$, $0 \le i \le I$ and the proof is complete. The following lemma is a discrete form of the maximum principle.

Lemma 3.2. *Let $a_h^{(n)}$ be a bounded vector and let $V_h^{(n)}$ a sequence such that*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} \ge 0, \quad 1 \le i \le I, \quad n \ge 0,$$

$$V_i^{(n)} \ge 0, \quad n \ge 0,$$

$$V_i^{(0)} \ge 0, \quad 0 \le i \le I.$$

Then $V_i^{(n)} \geq 0$ *for* $n \geq 0$, $0 \leq i \leq I$ *if* $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$.

Proof. If $V_h^{(n)} \geq 0$ then a routine calculation gives

$$V_i^{(n+1)} \geq \frac{\Delta t_n}{h^2} V_{i+1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty) V_i^{(n)} + \frac{\Delta t_n}{h^2} V_{i-1}^{(n)}, \quad 1 \leq i \leq I-1,$$

$$V_I^{(n+1)} \geq \frac{2\Delta t_n}{h^2} V_{I-1}^{(n)} + (1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty) V_I^{(n)}.$$

Since $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$, we see that $1 - 2\frac{\Delta t_n}{h^2} - \Delta t_n \|a_h^{(n)}\|_\infty$ is nonnegative. Due to the fact that $V_h^{(n)} \geq 0$, we deduce by induction that $V_h^{(n)} \geq 0$ for $n \geq 0$ which ends the proof.

A direct consequence of the above result is the following comparison lemma. Its proof is straightforward.

Lemma 3.3. *Suppose that* $a_h^{(n)}$ *and* $b_h^{(n)}$ *are two vectors such that* $a_h^{(n)}$ *is bounded. Let* $V_h^{(n)}$ *and* $V_h^{(n)}$ *two sequences such that*

$$\delta_t V_i^{(n)} - \delta^2 V_i^{(n)} + a_i^{(n)} V_i^{(n)} + b_h^{(n)} \leq \delta_t W_i^{(n)} - \delta^2 W_i^{(n)} \quad + a_i^{(n)} W_i^{(n)} + b_i^{(n)},$$
$$1 \leq i \leq I, n \geq 0,$$

$$V_0^{(n)} \leq W_0^{(n)}, \quad n \geq 0,$$

$$V_i^{(0)} \leq W_i^{(0)}, \quad 0 \leq i \leq I.$$

Then $V_i^{(n)} \leq W_i^{(n)}$ *for* $n \geq 0$, $0 \leq i \leq I$ *if* $\Delta t_n \leq \frac{h^2}{2+\|a_h^{(n)}\|_\infty h^2}$.

Now, let us give a property of the operator δ_t.

Lemma 3.4. *Let* $U^{(n)} \in R$ *be a sequence such that* $U^{(n)} \geq 0$. *Then we have*

$$\delta_t e^{U^{(n)}} \geq e^{U^{(n)}} \delta_t U^{(n)}, \quad n \geq 0.$$

Proof. From Taylor's expansion, we find that

$$\delta_t e^{U^{(n)}} = e^{U^{(n)}} \delta_t U^{(n)} + \Delta t_n \delta_t (U^{(n)})^2 e^{\theta^{(n)}},$$

where $\theta^{(n)}$ is an intermediate value between $U^{(n)}$ and $U^{(n+1)}$. Use the fact that $U^{(n)} \geq 0$ for $n \geq 0$ to complete the rest of the proof.

Our first result on blow-up times is the following theorem.

Theorem 3.1. *Suppose that there exists a positive constant $A \leq 1$ such that*

$$\delta^2\varphi_i + e^{\varphi_i} \geq Aihe^{\varphi_i}, 0 \leq i \leq I. \tag{3.4}$$

Then the solution $U_h^{(n)}$ of (3.1)-(3.3) blows up in a finite time and its numerical blow-up time $T_h^{\Delta t}$ is estimated as follows

$$T_h^{\Delta t} \leq \frac{\tau e^{-\|\varphi_h\|_\infty}}{1 - e^{-\tau'}}.$$

where $\tau' = \min\{\frac{h^2}{3}e^{-\|\varphi_h\|_\infty}, \tau\}$.

Proof. Introduce the vector J_h such that

$$J_i^{(n)} = \delta_t U_i^{(n)} - Aihe^{U_i^{(n)}}, \quad 0 \leq i \leq I.$$

A straightforward computation yields

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = \delta_t(\delta_t U_i^{(n)} - \delta^2 U_i^{(n)}) - Aih\delta_t e^{U_i^{(n)}} + Ah\delta^2(ie^{U_i^{(n)}}), \quad 0 \leq i \leq I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = \delta_t(\delta_t U_I^{(n)} - \delta^2 U_I^{(n)}) - A\delta_t e^{U_I^{(n)}} + A\delta^2(e^{U_I^{(n)}}).$$

Using (3.1), we arrive at

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} = (1 - Aih)\delta_t e^{U_i^{(n)}} + Ah\delta^2(ie^{U_i^{(n)}}), \quad 1 \leq i \leq I-1,$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} = (1 - A)\delta_t e^{U_I^{(n)}} + A\delta^2 e^{U_I^{(n)}}.$$

It follows from Lemmas 2.1, 3.1, 3.2 and 3.4 that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq (1 - Aih)e^{U_i^{(n)}}\delta_t U_i^{(n)} + Ahie^{U_i^{(n)}}\delta^2 U_i^{(n)},$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \geq (1 - A)e^{U_I^{(n)}}\delta_t U_I^{(n)} + Ae^{U_I^{(n)}}\delta^2 U_I^{(n)}.$$

Taking into account (3.1), we deduce that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq e^{U_i^{(n)}}\delta_t U_i^{(n)} - Ahie^{U_i^{(n)}}e^{U_i^{(n)}}, \tag{3.5}$$

$$\delta_t J_I^{(n)} - \delta^2 J_I^{(n)} \geq e^{U_I^{(n)}} \delta_t U_I^{(n)} - A e^{U_I^{(n)}} e^{U_I^{(n)}}, \tag{3.6}$$

which implies that

$$\delta_t J_i^{(n)} - \delta^2 J_i^{(n)} \geq e^{U_i^{(n)}} J_i^{(n)}, 1 \leq i \leq I. \tag{3.7}$$

Obviously, we have $J_0^{(n)} = 0$. From Lemma 3.2 principle, we obtain $J_h^{(0)} \geq 0$. It follows from Lemma 3.2 that $J_h^{(n)} \geq 0$, $0 \leq i \leq I$. Hence, we have

$$\frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} \geq Aihe^{U_i^{(n)}}.0 \leq i \leq I, \tag{3.8}$$

Consequently, we get

$$U_I^{(n+1)} \quad \geq \quad U_I^{(n)} + A\Delta t_n e^{U_I^{(n)}},$$

Since $U_I^{(n)} = \|U_h^{(n)}\|_\infty$, we arrive at

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + A\Delta t_n e^{\|U_h^{(n)}\|_\infty}. \tag{3.9}$$

We observe that

$$\begin{aligned} \Delta t_n e^{\|U_h^{(n)}\|_\infty} &= \min\{\frac{h^2}{3}, \tau e^{-\|U_h^{(n)}\|_\infty}\} \times e^{\|U_h^{(n)}\|_\infty} \\ &= \min\{\frac{h^2}{3} e^{\|U_h^{(n)}\|_\infty}, \tau\} \end{aligned}$$

From (3.9), $\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty$ and by induction $\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty = \|\varphi_h\|_\infty$. It follows that

$$\Delta t_n e^{\|U_h^{(n)}\|_\infty} \geq \min\{\frac{h^2}{3} e^{\|\varphi_h\|_\infty}, \tau\} \quad = \tau'.$$

Consequently, we have

$$\|U_h^{(n+1)}\|_\infty \geq \|U_h^{(n)}\|_\infty + \tau'. \tag{3.10}$$

Using a recursion argument, we discover that

$$\|U_h^{(n)}\|_\infty \geq \|U_h^{(0)}\|_\infty + n\tau'. \tag{3.11}$$

Hence, we see that $\|U_h^{(n)}\|_\infty$ goes to infinity as n approaches infinity. Now let us estimate its numerical blow-up time. From the restriction on the time step, we get

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \Sigma_{n=0}^{\infty}\tau e^{-\|U_h^{(n)}\|_\infty}.$$

Due to (3.11), we arrive at

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \Sigma_{n=0}^{\infty}\tau e^{-\|\varphi_h\|_\infty - n\tau'}$$

which implies that

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \tau e^{-\|\varphi_h\|_\infty}\Sigma_{n=0}^{\infty}(e^{-\tau'})^n$$

Since the series on the right hand side the above inequality converges to $\frac{1}{1-e^{-\tau'}}$, we deduce that

$$\Sigma_{n=0}^{\infty}\Delta t_n \leq \frac{\tau e^{-\|\varphi_h\|_\infty}}{1-e^{-\tau'}}.$$

Use the fact that the quantity on the right hand side of the above inequality is finite to complete the rest of the proof.

Remark 3.1. *From (3.11), we get* $\|U_h^{(n)}\|_\infty \geq \|U_h^{(q)}\|_\infty + \tau'(n-q)$. *Hence*

$$T_h^{\Delta t} - T_q \leq \Sigma_{n=q}^{\infty}\Delta t_n \leq \Sigma_{n=q}^{\infty}\tau e^{-\|U_h^{(n)}\|_\infty},$$

which implies that

$$T_h^{\Delta t} - T_q \leq \tau e^{-\|U_h^{(q)}\|_\infty}\Sigma_{n=q}^{\infty}e^{-(n-q)\tau'}.$$

Since the series on the right hand side of the above inequality converges to $\frac{1}{1-e^{-\tau'}}$, *we deduce that*

$$T_h^{\Delta t} - T_q \leq \frac{\tau e^{-\|U_h^{(q)}\|_\infty}}{1-e^{-\tau'}}.$$

Use Taylor's expansion to obtain

$$e^{-\tau'} = 1 - \tau' + o(\tau')$$

which implies that

$$\frac{\tau}{1-e^{-\tau'}} = \frac{\tau}{\tau'(1+o(1))} \leq \frac{2\tau}{\tau'}.$$

Since $\tau' = \min\{\frac{h^2}{3}e^{\|\varphi_h\|_\infty}, \tau\}$, *if we take* $\tau = h^2$, *we get* $\frac{\tau}{\tau'} = \min\{\frac{1}{3}e^{\|\varphi_h\|_\infty}, 1\}$, *which implies that there exists a positive constant* K *such that* $\frac{\tau}{\tau'} \leq K$. *Then, we see that* $\frac{\tau}{1-e^{-\tau'}}$ *is bounded from above by 2K.*

4. Convergence of the Numerical Blow-up Time

In this section, under some conditions, we show that the discrete solution blows up in a finite time and its numerical blow-up time converges to the real one when the mesh size goes to zero. In order to prove this result, we firstly show that the discrete solution approaches the continuous one on any interval $[0, 1] \times [0, T - \tau]$ with $\tau \in (0, T)$ as the parameter h goes to zero.
The result on the convergence of the discrete solution to the theoretical one is stated in the following theorem.

Theorem 4.1. *Suppose that the problem (1.1)-(1.3) has a solution* $u \in C^{4,2}([0, 1]\times [0, T])$. *Assume that the initial data at (3.3) verifies*

$$\|\varphi_h - u_h(0)\|_\infty = o(1) as \qquad h \to 0. \tag{4.1}$$

Then the problem (3.1)-(3.3) has a solution $U_h^{(n)}$ *for h sufficiently small,* $0 \leq n \leq J$ *and we have the following estimate*

$$\max_{0\leq n\leq J} \|U_h^{(n)} - u_h(t_n)\|_\infty = O(\|\varphi_h - u_h(0)\|_\infty + h^2 + \Delta t_n), \quad as \quad h \to 0,$$

where J *is such that* $\Sigma_{n=0}^{J-1}\Delta t_n \leq T$ *and* $t_n = \Sigma_{j=0}^{n-1}\Delta t_j$.

Proof. For each h, the problem (3.1)-(3.3) has a solution $U_h^{(n)}$. Let $N \leq J$ be the greatest value of n such that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < 1 \quad \text{for} \quad n < N. \tag{4.2}$$

We know that $N \geq 1$, because of (3.13). The fact that $u \in C^{4,2}$, there exists a positive constant α and that $\|u\| \leq \alpha$. Applying the triangle inequality, we obtain

$$\|U_h^{(n)}\|_\infty \leq \|u_h(t_n)\|_\infty + \|U_h^{(n)} - u_h(t_n)\|_\infty \leq 1 + \alpha. \tag{4.3}$$

As in the proof of Theorem 3.1, using Taylor's expansion, we find that

$$\delta_t u(x_i, t_n) - \delta^2 u(x_i, t_n) - e^{u(x_i,t_n)} = -\frac{h^2}{12}u_{xxxx}(\widetilde{x}, t_n) + \frac{\Delta t_n}{2}u_{tt}(x_i, \widetilde{t_n}).$$

Let $e_h^{(n)} = U_h^{(n)} - u_h(t_n)$ be the error of discretization. From the mean value theorem, we get

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} = e^{\varsigma_i^{(n)}} e_i^{(n)} + \frac{h^2}{12}u_{xxxx}(\widetilde{x}_i, t_n) - \frac{\Delta t_n}{2}u_{tt}(x_i, \widetilde{t_n})$$

where ς_i is an intermediate value between $u(x_i, t_n)$ and $U_i^{(n)}$. Since $u_{xxxx}(x,t)$, $u_{tt}(x,t)$ are bounded, there exists a positive constant M such that

$$\delta_t e_i^{(n)} - \delta^2 e_i^{(n)} \leq e^{\varsigma_i^{(n)}} e_i^{(n)} + M\Delta t_n + Mh^2, 0 \leq i \leq I. \tag{4.4}$$

Let $K = 1 + \alpha$ and introduce the vector $V_h^{(n)}$ defined as follows

$$V_i^{(n)} = e^{(K+1)t_n}(\|\varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n), \quad 0 \leq i \leq I.$$

A straightforward computation gives

$$\begin{aligned} \delta_t V_i^{(n)} - \delta^2 V_i^{(n)} > e^{\varsigma_i^{(n)}} V_i^{(n)} + M\Delta t_n + Mh^2, \quad 0 \leq i \leq I, \\ V_0^{(n)} \geq e_0^{(n)}. \\ V_i^{(0)} \geq e_i^{(0)}, \quad 0 \leq i \leq I. \end{aligned}$$

We observe that $e^{\varsigma_i^{(n)}}$ is bounded from above by e^K. It follows from comparison Lemma 3.3 that $V_h^{(n)} \geq e_h^{(n)}$. By the same way, we also prove that $V_h^{(n)} \geq -e_h^{(n)}$. which implies that

$$\|U_h^{(n)} - u_h(t_n)\|_\infty \leq e^{(K+1)t_n}(\|\varphi_h - u_h(0)\|_\infty) + Mh^2 + M\Delta t_n.$$

Let us show that $N = J$.
Suppose that $N < J$. If we replace n by N in the above inequality and use (3.14), we find that

$$1 \leq \|U_h^{(N)} - u_h(t_N)\|_\infty \leq e^{(K+1)t_N}(\|\varphi_h - u_h(0)\|_\infty + Mh^2 + M\Delta t_n).$$

Since term on the right hand side of the second inequality goes to zero as h tends to zero, we deduce that $1 \leq 0$, which is contradiction and the proof is complete.
Now, we are in a position to prove the mean theorem of this section.

Theorem 4.2. *Suppose that the problem (1.1)-(1.3) has a solution u which blows up in a finite time T_0 and $u \in C^{4,2}([0,1] \times [0,T_0])$. Assume that the initial data at (3.6) satisfies*

$$\|\varphi_h - u_h(0)\|_\infty = 0(1) as \qquad h \to 0. \tag{4.5}$$

Under the assumption of Theorem 3.1, the problem (3.1)-(3.3) has a solution $U_h^{(n)}$ which blows up in a finite time $T_h^{\Delta t}$ and the following relation holds

$$\lim_{h \to 0} T_h^{\Delta t} = T_0.$$

Proof. We know from Remark that $\frac{\tau}{1-e^{-\tau'}}$, is bounded. Letting $\varepsilon > 0$ there exists a constant $R > 0$ such that

$$\frac{\tau e^{-x}}{1-e^{-\tau'}} < \frac{\varepsilon}{2} \quad \text{for} \quad x \in [R, +\infty). \tag{4.6}$$

Since u blows up at the time T_0, there exists $T_1 \in (T_0 - \frac{\varepsilon}{2}, T_0)$ such that

$$\|u(x,t)\|_\infty \geq 2R \quad \text{for} \quad t \in [T_1, T_0].$$

Let $T_2 = \frac{T_1+T_2}{2}$ and q be a positive integer such that $T_q = \sum_{n=0}^{q-1} \Delta t_n \in [T_1, T_2]$ for h small enough. We have $\sup_{t\in[0,T_2]} \|u_h(t_n)\|_\infty < +\infty$. It follows from Theorem 3.1 that the problem (3.1)-(3.3) has a solution $U_h^{(n)}$ which obeys

$$\|U_h^{(n)} - u_h(t_n)\|_\infty < R \quad \text{for} \quad n \leq q.$$

which implies that

$$\|U_h^{(q)}\|_\infty \;\; \geq \;\; \|u_h(t_q)\|_\infty - \|U_h^{(q)} - u_h(t_q)\|_\infty \geq R.$$

From Theorem 3.1 $U_h^{(n)}$ blows up at the time $T_h^{\Delta t}$. It follows from remark 3.1 and (3.17) that

$$|T_h^{\Delta t} - t_q| \;\; \leq \;\; \frac{\tau e^{-\|U_h^{(q)}\|_\infty}}{1-e^{-\tau'}} \leq \frac{\varepsilon}{2}$$

because $\|U_h^{(q)}\|_\infty \geq R$. We deduce that

$$|T_0 - T_h^{\Delta t}| \leq |T_0 - t_q| + |t_q - T_h^{\Delta t}| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \leq \varepsilon.$$

and the proof is complete.

Remark 4.1. *Consider the classical solution u of the B V P*

$$\begin{aligned} u_t(x,t) = u_{xx}(x,t) + e^{u(x,t)}, x \in (-1,1), \qquad & t \in (0,T), \\ u(-1,t) = 0, u(1,t) = 0, \qquad & t \in (0,T), \\ u(x,0) = u_0(x), x \in [-1,1], \qquad & \end{aligned} \tag{4.7}$$

where $u_0(x)$ is a positive and symmetric function in [-1,1] and $u^{'}(x) \geq 0$ for $x \in (-1,0)$. Since $u_0(x)$ is symmetric in [-1,1], from the maximum principle u is also *symmetric in [-1,1]. We observe that $u_x(0,t)=0$ because $u(x,t)=u(-x,t)$. Consider now the solution v of the boundary value problem below*

$$\begin{aligned} v_t(x,t) = v_{xx}(x,t) + e^{v(x,t)}, &\quad x \in (-1,0), \quad t \in (0,T), \\ v(-1,t) = 0, &\quad v_x(0,t) = 0, \quad t \in (0,T), \\ v(x,0) = v_0(x), &\qquad in \qquad [-1,0], \end{aligned}$$

Since u is symmetric, we have

$$\max_{-1\le x\le 1} |u(x,t)| = \max_{-1\le x\le 0} |u(x,t)| = \max_{-1\le x\le 0} |v(x,t)|.$$

Hence, to get an approximation of the blow-up time of the solution u, it suffices to obtain the one of the classical solution v which has been the subject of investigation of present paper.

5. Numerical Results

In this section, we present some numerical approximations to the blow-up time for the solution of problem (1.1)–(1.3) in the case where $U_0(x) = 2sin(\frac{\pi x}{2})$. Firstly, we consider the explicit scheme in (3.1)-(3.3). Secondly, we use the following implicit scheme

$$\begin{aligned} \frac{U_i^{(n+1)} - U_i^{(n)}}{\Delta t_n} &= \delta^2 U_i^{(n+1)} + \gamma e^{U_i^{(n)}}, && 1 \le i \le I, \\ U_0^{(n)} &= 0, \\ U_i^{(0)} &= \phi_i, && 0 \le i \le I, \end{aligned} \tag{5.1}$$

where

$$\Delta t_n = \tau e^{-\|U_h^{(n)}\|\infty}, \quad \tau = h^2.$$

In both cases, we take $\varphi_i = 2\varepsilon \sin(\frac{i\pi h}{2})$, $0 \le i \le I$. For the above implicit scheme, the positivity of the discrete solution $U_h^{(n)}$ is guaranteed using standard methods (see [12]). In the tables 1 and 2, in rows, we present the numerical blow-up times, the numbers of iterations, the CPU times and the orders of the approximations corresponding to meshes of 16, 32, 64, 128, 256, 512. We take for the numerical blow-up time $T^n = \sum_{j=0}^{n-1} \Delta t_j$ which is computed at the first time when

$$\Delta t_n = |T^{n+1} - T^n| \le 10^{-16}.$$

The order(s) of the method is computed from

$$s = \frac{\log((T_{4h} - T_{2h})/(T_{2h} - T_h))}{\log(2)}.$$

Numerical experiments for $\gamma f(U_k^{(n)}) = \gamma e^{U_k^{(n)}}$, $\varphi_i = 0$.
First case $\gamma = 10$. 0.111001642852658

Table 1: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.111998076818558	216	0.015	-
32	0.111001642852658	824	0.031	-
64	0.110341491751344	3152	0.156	1.994
128	0.110085396737967	12037	5.296	1.998
256	0.110021357450119	45874	112.062	1.999
512	0.110005346658750	174408	6115.546	2.000

Table 2: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.111017021553261	209	0.046	-
32	0.111965902335882	801	0.296	-
64	0.110489092886515	3059	4.796	2.034
128	0.110122131175136	11667	22.125	2.008
256	0.110030530697220	44397	311.093	2.002
512	0.110007639322847	168500	8170.125	2.000

Numerical experiments for $\gamma f(U_k^{(n)}) = \gamma e^{U_k^{(n)}}$, $\varphi_i = 0$.
Second case $\gamma = 5$.

Table 3: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.286437651041976	12	0.018	-
32	0.293873861534947	724	0.056	-
64	0.293925491751344	2320	0.205	1.994
128	0.294301396737967	13007	4.289	1.998
256	0.294411357450119	35002	113.073	1.999
512	0.295010000534660	132504	7002.526	2.000

Table 4: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.289800556424729	12	0.018	-
32	0.293873861534252	724	0.056	-
64	0.293873549175135	2320	0.205	1.994
128	0.2989301296679678	13007	4.289	1.998
256	0.2990223474502095	35002	113.073	1.999
512	0.2995010000062520	132504	7002.526	2.000

Numerical experiments for $\gamma f(U_k^{(n)}) = \gamma e^{U_k^{(n)}}$, $\varphi_i = 0$.
Third case $\gamma = 1.5$.

Table 5: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	$CPU\ time$	s
16	0.308909002938009	20	0.062	-
32	0.307530002938009	76	0.075	-
64	0.307189002009179	286	0.562	2.278
128	0.307103009879189	1087	4.046	2.083
256	0.307097560999088	4119	30.015	2.021
512	0.307099494501844	15568	751.875	2.005

Table 6: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	*CPU time*	s
16	0.314007692874664	20	0.015	-
32	0.308007750763581	76	0.062	-
64	0.307007801488999	286	0.125	2.278
128	0.307182230005350	1087	0.296	2.083
256	0.307101756130886	4119	4.921	2.021
512	0.307096494578945	15568	279.343	2.005

Numerical experiments for $\gamma f(U_k^{(n)}) = \gamma e^{U_k^{(n)}}$, $\varphi_i = 2\sin(\frac{i\pi h}{2})$.
Fourth case: $\gamma = 1$.

Table 7: Numerical blow-up times, numbers of iterations, CPU times (seconds), and orders of the approximations obtained with the explicit Euler method

I	T^n	n	*CPU time*	s
16	0.337145469287464	20	0.015	-
32	0.335773076358199	76	0.062	-
64	0.334189880148892	286	0.125	2.278
128	0.334044230005350	1087	0.296	2.083
256	0.333975756130886	4119	4.921	2.021
512	0.333955649457894	15568	279.343	2.005

Table 8: Numerical blow-up times, numbers of iterations, CPU times (seconds) and orders of the approximations obtained with the implicit Euler method

I	T^n	n	*CPU time*	s
16	0.334213453972435	20	0.062	-
32	0.334444504907909	76	0.075	-
64	0.334584879608179	286	0.562	2.278
128	0.334668298791891	1087	4.046	2.083
256	0.334665756099908	4119	30.015	2.021
512	0.334664649450184	15568	751.875	2.005

6. Conclusion

In this research, we have proposed two algorithms for the numerical solution of semi-linear heat equations. The numerical blow-up solutions are computed for semi-linear heat equations with Dirichlet boundary conditions. Explicit and implicit Euler finite difference schemes with a special time-steps formula are presented and analyzed in order to solve the proposed problem and estimate the blow-up times. The numerical result obtained by the proposed methods is analyzed, simulated and presented in the form of tables and figures. Numerical examples show that the proposed methods are successfully implemented with good efficiency and high order of convergence.

In the following, we also give some plots to illustrate our analysis. In Figures 1 to 12, we can appreciate that the discrete solution blows up globally. Let us notice that, theoretically, we know that the continuous solution blows up globally under the assumptions given in the introduction of the present paper.

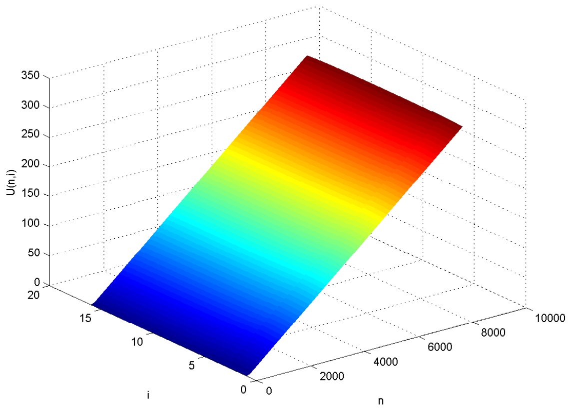

Figure 1: Evolution of the discrete solution, source $\gamma e^{u(a,t)}$, $\gamma = 1$, $\varphi_i = 2\sin(\frac{i\pi h}{2})$

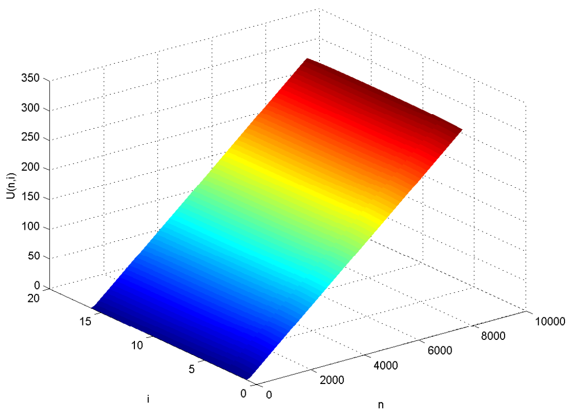

Figure 2: Evolution of the discrete solution source $\gamma e^{u(a,t)}$, $\gamma = 1$, $\varphi_i = 2\sin(\frac{i\pi h}{2})$

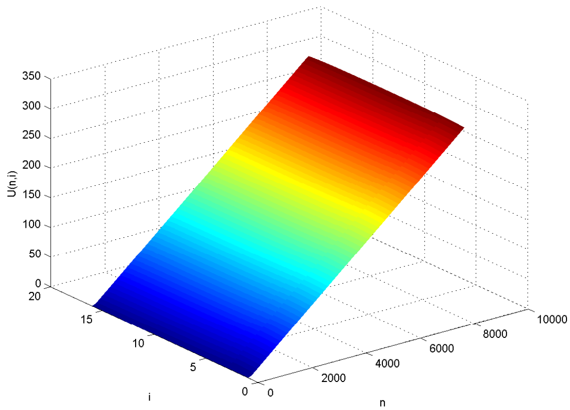

Figure 3: Evolution of the discrete solution, source $\gamma e^{u(a,t)}$, $\gamma = 10$, $\varphi_i = 0$,

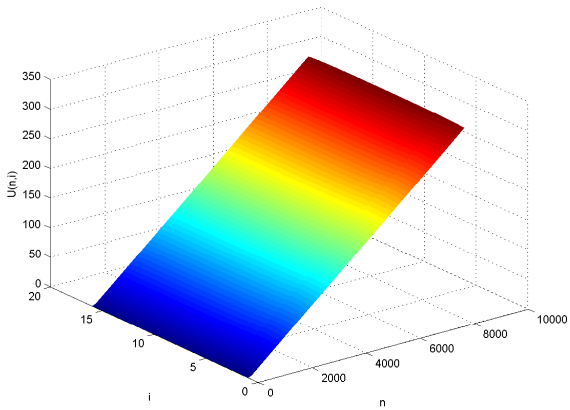

Figure 4: Evolution of the discrete solution source $\gamma e^{u(a,t)}$, $\gamma = 10$, $\varphi_i = 0$,

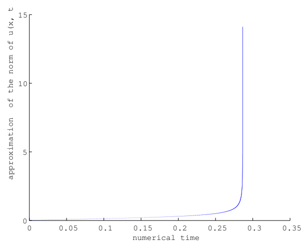

Figure 5: Evolution of the discrete solution, source $\gamma e^{u(a,t)}$, $\gamma = 1$, $\varphi_i = 2\sin(\frac{i\pi h}{2})$

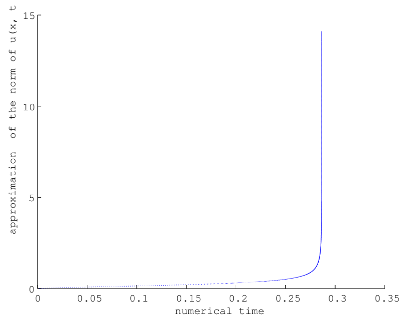

Figure 6: Evolution of the discrete solution source $\gamma e^{u(a,t)}$, $\gamma = 1$, $\varphi_i = 2\sin(\frac{i\pi h}{2})$

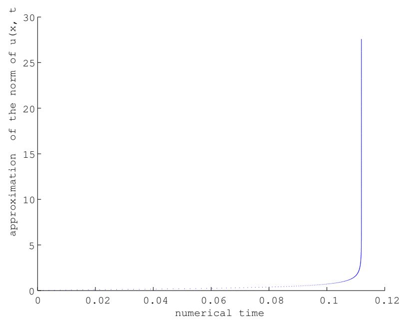

Figure 7: Evolution of the discrete solution, source $\gamma e^{u(a,t)}$, $\gamma = 10$, $\varphi_i = 0$

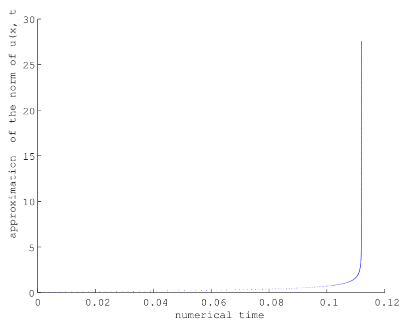

Figure 8: Evolution of the discrete solution source $\gamma e^{u(a,t)}$, $\gamma = 10$, $\varphi_i = 0$

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