J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 9, No. 2 (2022), pp. 97-108

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

FRACTIONAL DIFFERENTIAL EQUATIONS OF HYPERGEOMETRIC FUNCTIONS AND LAGUERRE POLYNOMIAL

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(**Received:** *Feb. 13, 2022* **Accepted:** *Jun. 25, 2022* **Published:** *Jun. 30, 2022)*

Abstract: The object of this paper is to find a solution to fractional differential equations of hypergeometric function and Laguerre Polynomials by using Caputo derivatives.

Keywords and Phrases: Caputo derivative, Mittag-Leffler function, Hypergeometric Function, Laguerre Polynomial.

2020 Mathematics Subject Classification: 33E12, 35R11, 26A33, 33CXX.

1. Introduction and Definitions

Caputo Derivative.
The fractional derivative [10] of $f(x)$ in the Caputo sense is defined as

$$D^{\alpha}f(x) = I^{m-\alpha}D^{m}f(x)$$

$$= \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}f^{(m)}(t)dt. \tag{1.1}$$

for $m-1<\alpha\le m, m\in N, x>0$.
For the Caputo derivative, we have $D^{\alpha}C=0$, is constant.

$$D^{\alpha}t^{n} = \begin{cases} 0 & n\le\alpha-1 \\ \frac{\Gamma(n+1)}{\Gamma(n-\alpha+1)}t^{n-\alpha} & n>\alpha-1 \end{cases} \tag{1.2}$$

For more details see [5, 6, 12, 13, 14].

Laguerre Polynomial.
The Laguerre polynomial [15] of degree n is defined by

$$L_n(z) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(n-k)!(k!)^2} z^k, \qquad k \le n, n \in N. \tag{1.3}$$

Hypergeometric Function.
The Hypergeometric function [15], $F(\alpha, \beta; \gamma; z)$ is defined by

$$F[\alpha, \beta; \gamma; z] = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} \frac{z^n}{n!} \qquad |z| < 1, \tag{1.4}$$

where α, β, γ are complex numbers and $\gamma \neq 0, -1, -2, ...$.
The Pochhamer symbol $(\alpha)_n$ where α denotes any number, real or complex, and n any Positive, negative, or 0, is defined by

$$(\alpha)_n = \begin{cases} 1 & if n = 0 \\ \alpha(\alpha+1)...(\alpha+n-1) & if n \ge 1 \end{cases} \tag{1.5}$$

A natural generalization of the hypergeometric function $F(.)$ is the generalized hypergeometric function is called ${}_pF_q$ with defined as

$${}_pF_q[a_1, a_2, ..., a_p; b_1, b_2, ..., b_q; z] = \sum_{n=0}^{\infty} \frac{(a_1)_n ... (a_p)_n}{(b_1)_n ... (b_q)_n} \frac{z^n}{n!}, \tag{1.6}$$

where $(a_i)_n = \frac{\Gamma(a_i+n)}{\Gamma(a_i)}$ and $(a)_n = (a_i)_n$ and p and q are positive integer or 0.

Mittag-Leffler Function
Mittag-Leffler function is defined by the special function

$$E_\eta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\eta k)}, \eta \in C, \Re(\eta) > 0, z \in C. \tag{1.7}$$

and its general form

$$E_{\eta,\mu}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu+\eta k)}, \eta \in C, \Re(\eta) > 0, \Re(\mu) > 0, z \in C. \tag{1.8}$$

with C being the set of Complex numbers are called Mittag-Leffler functions [4, Section 18.1]. The former was introduced by Mittag Leffler [7] in connection with his method of summation of some divergent series.

2. Analysis of the Method

If hypergeometric function (1.4) suggests that the linear term $y(x)$ is decomposed by an infinite series of components:

$$y(x) = F(a,b;c;Ax^{\alpha}) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{(Ax^{\alpha})^n}{n!} \tag{2.1}$$

Then from (1.1), we have

$$D^{\alpha} = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}(D^m t^{n\alpha})dt.$$

$$= \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-m+1)}t^{n\alpha-m}dt.$$

$$= \frac{1}{\Gamma(m-\alpha)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha-m+1)}\int_0^x (x-t)^{m-\alpha-1}t^{n\alpha-m}dt.$$

$$D^{\alpha}y = \sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(\alpha(n-1)+1)}x^{\alpha(n-1)} \tag{2.2}$$

$$D^{2\alpha}y = \sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma(\alpha(n-2)+1)}x^{\alpha(n-2)} \tag{2.3}$$

And if Laguerre's polynomial method [15] of degree n suggests that the linear terms $y(x)$ is decomposed by polynomial series of components:

$$y(x) = L_n(ax^{\alpha}) = \sum_{k=0}^{n}\frac{(-1)^k n!}{(n-k)!(k!)^2}(ax^{\alpha})^k, k \le n, n \in N. \tag{2.4}$$

$$D^{\alpha}y = \frac{1}{\Gamma(m-\alpha)}\int_0^x (x-t)^{m-\alpha-1}\sum_{k=0}^{n}\frac{(-1)^k n! a^k}{(n-k)!(k!)^2}(D^m t^{\alpha k})dt.$$

$$= \frac{1}{\Gamma(m-\alpha)}\sum_{k=0}^{n}\frac{(-1)^k n! a^k}{(n-k)!(k!)^2}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-m+1)}\int_0^x (x-t)^{m-\alpha-1}t^{\alpha k-m}dt. \tag{2.5}$$

If we put $\frac{t}{x} = u$ in the above expression we arrive at

$$D^{\alpha}y = \sum_{k=1}^{n} \frac{(-1)^k n! a^k}{(n-k)!(k!)^2} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-1)+1)} x^{\alpha(k-1)}. \tag{2.6}$$

And

$$D^{2\alpha}y = I^{m-2\alpha} D^m f(x) \tag{2.7}$$

$$= \frac{1}{\Gamma(m-2\alpha)} \int_0^x (x-t)^{m-2\alpha-1} \sum_{k=2}^{n} \frac{(-1)^k n! a^k}{(n-k)!(k!)^2} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-m+1)} t^{\alpha k-m} dt.$$

$$D^{2\alpha}y = \sum_{k=1}^{n} \frac{(-1)^k n! a^k}{(n-k)!(k!)^2} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-2)+1)} x^{\alpha(k-2)}. \tag{2.8}$$

This based on Caupto Derivative.

3. Numerical Applications

Many researchers have studied linear fractional differential equations in different ways with different functions [1, 2, 8, 9, 10, 16, 17]. In this section, we consider some examples that demonstrate the hypergeometric function and Laguerre polynomial for solving linear differential equations with fractional derivatives.

Example 3.1. Consider the following differential equations [11]

$$D^{\alpha}y = A'y. \tag{3.1}$$

With (2.1), (2.2), and (2.3), We have

$$\sum_{n=1}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{A^n}{n!} \frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)} x^{\alpha(n-1)} - A' \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{A^n}{n!} x^{n\alpha} = 0$$

Now Replacing n by $n+1$ in the first summation, we get

$$\sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \frac{A^{n+1}}{(n+1)!} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} x^{\alpha n} - A' \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{A^n}{n!} x^{n\alpha} = 0$$

$$\sum_{n=0}^{\infty} \frac{(a)_{n+1} (b)_{n+1}}{(c)_{n+1}} \frac{A^{n+1}}{(n+1)!} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} x^{\alpha n} - A' \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{A^n}{n!} x^{n\alpha} = 0$$

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \left[\frac{(a+n)(b+n)}{(c+n)} \frac{A^{n+1}}{n+1} \frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)} - A^n A' \right] \frac{x^{n\alpha}}{n!} = 0$$

With the coefficients equal to zero and identifying the coefficients, we obtain

$$\frac{(a+n)(b+n)}{(c+n)}\frac{A^{n+1}}{n+1}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}-A^{n}A^{'}=0$$

at $n=0$

$$A\frac{ab}{c}=\frac{A^{'}}{\Gamma(\alpha+1)}$$

at $n=1$

$$A^{2}\frac{a(a+1)b(b+1)}{c(c+1)2}=\frac{A^{'2}}{\Gamma(2\alpha+1)}$$

at $n=2$

$$A^{3}\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)3!}=\frac{A^{'3}}{\Gamma(3\alpha+1)}$$

and so on.
Substituting into (2.1) we get

$$y=1+A^{'}\frac{x^{\alpha}}{\Gamma(\alpha+1)}+A^{'2}\frac{x^{2\alpha}}{\Gamma(2\alpha+1)}+A^{'3}\frac{x^{3\alpha}}{\Gamma(3\alpha+1)}+.... \tag{3.2}$$

Example 3.2. Consider the following differential equations [11]

$$D^{2\alpha}y-y=0. \tag{3.3}$$

With (2.1), (2.3), and (3.3), We get

$$\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)}x^{\alpha(n-2)}-\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

Now Replacing n by $n+2$ in the first summation, we get

$$\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+2)!}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}x^{\alpha n}-\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

$$\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{A^{n+1}}{(n+1)!}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}x^{\alpha n}-\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0$$

$$\sum_{n=0}^{\infty}\left[\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+1)(n+2)}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}-A^{n}\frac{(a)_n(b)_n}{(c)_n}\right]\frac{x^{n\alpha}}{n!}=0$$

With the coefficients equal to zero and identifying the coefficients, we obtain

$$\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+1)(n+2)}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}-A^{n}\frac{(a)_n(b)_n}{(c)_n}$$

at $n=0$

$$(a+1)(b+1)ab\Gamma(2\alpha+1)A^2=c(c+1).2.1$$

at $n=1$

$$(a+1)(b+1)(a+2)(b+2)\Gamma(3\alpha+1)A^3=(c+2)(c+1)3.2.A\Gamma(\alpha+1)$$

at $n=2$

$$(a+3)(b+3)(a+2)(b+2)\Gamma(4\alpha+1)A^3=(c+3)(c+2)4.3.A^2\Gamma(\alpha+1)$$

and so on.
Substituting into (2.1), we get

$$y=1+\frac{ab}{c}Ax^{\alpha}+\frac{1}{\Gamma(2\alpha+1)}x^{2\alpha}+\frac{ab}{c}\frac{\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}Ax^{3\alpha}+.... \tag{3.4}$$

Example 3.3. Consider the following differential equations [11]

$$D^{2\alpha}y+D^{\alpha}y-2y=0. \tag{3.5}$$

With (2.1), (2.2) and (2.3) in (3.5), We get

$$\sum_{n=2}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma((n-2)\alpha+1)}x^{\alpha(n-2)}+\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}\frac{\Gamma(n\alpha+1)}{\Gamma((n-1)\alpha+1)}x^{(n-1)\alpha}$$

$$-2\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0.$$

Now Replacing n by $n+2$ in the first summation, and n by $n+1$ in the second summation, we get

$$\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+2)!}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}x^{\alpha n}$$

$$+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{A^{n+1}}{(n+1)!}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}x^{n\alpha}-2\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{A^n}{n!}x^{n\alpha}=0.$$

With the coefficients equal to zero and identifying the coefficients, we obtain

$$\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}}\frac{A^{n+2}}{(n+2)(n+1)}\frac{\Gamma((n+2)\alpha+1)}{\Gamma(n\alpha+1)}+\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}}\frac{A^{n+1}}{(n+1)}\frac{\Gamma((n+1)\alpha+1)}{\Gamma(n\alpha+1)}$$

$$-2\frac{(a)_n(b)_n}{(c)_n}A^n=0$$

at $n=0$

$$a(a+1)b(b+1)\Gamma(2\alpha+1)A^2+2ab(c+1)\Gamma(\alpha+1)-4(c+1)c=0$$

at $n=1$

$$(a+1)(a+2)(b+1)(b+2)\Gamma(3\alpha+1)A^3+3(a+1)(b+1)(c+2)\Gamma(2\alpha+1)A^2$$

$$-2.6(c+2)(c+1)\Gamma(\alpha+1)A=0$$

$n=2$

$$(a+2)(a+3)(b+2)(b+3)\Gamma(4\alpha+1)A^4+4(a+2)(b+2)(c+3)\Gamma(3\alpha+1)A^3$$

$$-24(c+3)(c+2)\Gamma(2\alpha+1)A^2=0$$

and so on.
Substituting into (2.1) we find that

$$y=1+\frac{ab}{c}Ax^\alpha+\frac{2c-ab\Gamma(\alpha+1)A}{c\Gamma(2\alpha+1)}x^{2\alpha}+\frac{3ab\Gamma(\alpha+1)A-2c}{c\Gamma(3\alpha+1)}x^{3\alpha}+.... \qquad (3.6)$$

Example 3.4. Consider the following differential equations [11]

$$D^\alpha y=Ay. \qquad (3.7)$$

With (2.4) and (2.6), We get

$$\sum_{k=1}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-1)+1)}x^{\alpha(k-1)}-A\sum_{k=0}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}x^{\alpha k}=0.$$

Now Replacing k by $k+1$ in the first summation, we get

$$\sum_{k=0}^{n}\frac{(-1)^{k+1}n!a^{k+1}}{(n-k-1)!((k+1)!)^2}\frac{\Gamma((k+1)\alpha+1)}{\Gamma(\alpha k+1)}x^{\alpha k}-A\sum_{k=0}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}x^{\alpha k}=0.$$

With the coefficients equal to zero and identifying the coefficients, we obtain

$$\frac{-a\Gamma(\alpha(k+1)+1)}{((k+1)!)^2\Gamma(\alpha k+1)} - A\frac{1}{(n-k)!(k!)^2} = 0$$

at $k = 0$

$$a = -\frac{A}{n\Gamma(\alpha+1)}$$

at $k = 1$

$$a^2 = \frac{2^2A^2}{n(n-1)\Gamma(2\alpha+1)}$$

at $k = 2$

$$a^3 = \frac{2^23^2A^3}{n(n-1)(n-2)\Gamma(3\alpha+1)}$$

and so on.
Substituting into (2.4), we get

$$y = 1 + A\frac{x^\alpha}{\Gamma(\alpha+1)} + A^2\frac{x^{2\alpha}}{\Gamma(2\alpha+1)} + ... + A^n\frac{x^{n\alpha}}{\Gamma(n\alpha+1)}. \tag{3.8}$$

Example 3.5. Consider the following differential equations [11]

$$D^{2\alpha}y - y = 0. \tag{3.9}$$

With Equation (2.4), (2.8), we have

$$\sum_{k=2}^{n}\frac{(-1)^kn!a^k}{(n-k)!(k!)^2}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-2)+1)}x^{\alpha(k-2)} - \sum_{k=0}^{n}\frac{(-1)^kn!a^k}{(n-k)!(k!)^2}x^{\alpha k} = 0.$$

Now Replacing k by $k+2$ in the first summation, we get

$$\sum_{k=0}^{n}\frac{(-1)^{k+2}n!a^{k+2}}{(n-k-2)!((k+2)!)^2}\frac{\Gamma(\alpha(k+2)+1)}{\Gamma(\alpha k+1)}x^{\alpha k}$$

$$-\sum_{k=0}^{n}\frac{(-1)^kn!a^k}{(n-k)(n-k-1)(n-k-2)!(k!)^2}x^{\alpha k} = 0.$$

With the coefficients equal to zero and identifying the coefficients, we obtain

$$\frac{a^{k+2}}{((k+2)!)^2}\frac{\Gamma(\alpha(k+2)+1)}{\Gamma(\alpha k+1)} - \frac{a^k}{(n-k)(n-k-1)(k!)^2} = 0$$

at $k = 0$

$$a^2 n(n-1)\Gamma(2\alpha+1) = 1^2 2^2$$

at $k = 1$

$$a^3(n-1)(n-2)\Gamma(3\alpha+1) = a2^2 3^2\Gamma(\alpha+1)$$

at $k = 2$

$$a^4(n-2)(n-3)\Gamma(4\alpha+1) = a^2 3^2 4^2\Gamma(2\alpha+1)$$

and so on.
From (2.4), we get

$$y = 1 - anx^\alpha + \frac{x^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{an\Gamma(\alpha+1)}{\Gamma(3\alpha+1)}x^{3\alpha} + ... + nterms. \tag{3.10}$$

Example 3.6. Consider the following differential equations [11]

$$D^{2\alpha}y + D^\alpha y - 2y = 0. \tag{3.11}$$

With Equation (2.4), (2.6) and (2.8) in (3.11) We get

$$\sum_{k=2}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-2)+1)}x^{\alpha(k-2)} + \sum_{k=1}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}\frac{\Gamma(\alpha k+1)}{\Gamma(\alpha(k-1)+1)}$$

$$x^{\alpha(k-1)} - 2\sum_{k=0}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}x^{\alpha k} = 0.$$

Now Replacing k by $k+2$ in the first summation and k by $k+1$ in the second summation, we get

$$\sum_{k=0}^{n}\frac{(-1)^{k+2}n!a^{k+2}}{(n-k-2)!((k+2)!)^2}\frac{\Gamma(\alpha(k+2)+1)}{\Gamma(\alpha k+1)}x^{\alpha k}$$

$$+\sum_{k=0}^{n}\frac{(-1)^{k+1}n!a^{k+1}}{(n-k-1)!((k+1)!)^2}\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}x^{\alpha k}$$

$$-2\sum_{k=0}^{n}\frac{(-1)^k n!a^k}{(n-k)!(k!)^2}x^{\alpha k} = 0.$$

With the coefficients equal to zero and identifying the coefficients, we obtain

$$\frac{a^{k+2}}{((k+2)!)^2}\frac{\Gamma(\alpha(k+2)+1)}{\Gamma(\alpha k+1)} - \frac{a^{k+1}}{(n-k-1)((k+1)!)^2}\frac{\Gamma(\alpha(k+1)+1)}{\Gamma(\alpha k+1)}$$

$$-2\frac{a^k}{(n-k)(n-k-1)(k!)^2} = 0$$

at $k = 0$

$$a^2 n(n-1)\Gamma(2\alpha+1) - an2^2\Gamma(\alpha+1) - 2a^0 1^2 2^2 = 0$$

at $k = 1$

$$a^3(n-1)(n-2)\Gamma(3\alpha+1) - a^2 3^2(n-1)\Gamma(2\alpha+1) - 2a2^2 3^2\Gamma(\alpha+1) = 0$$

at $k = 2$

$$a^4(n-2)(n-3)\Gamma(4\alpha+1) - a^3 4^2(n-2)\Gamma(3\alpha+1) - 2a^2 3^2 4^2\Gamma(2\alpha+1) = 0$$

and so on.
From (2.4), we get

$$y = 1 - anx^\alpha + n(n-1)\left(\frac{8+4an\Gamma(\alpha+1)}{n(n-1)\Gamma(2\alpha+1)}\right)\frac{x^{2\alpha}}{(2!)^2}$$

$$-n(n-1)(n-2)\left(\frac{72+108an\Gamma(\alpha+1)}{n(n-1)(n-2)\Gamma(3\alpha+1)}\right)\frac{x^{3\alpha}}{(3!)^2} + ... + nterms \qquad (3.12)$$

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