J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 9, No. 2 (2022), pp. 83-96

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

HEINE'S TRANSFORMATION FORMULA THROUGH q-DIFFERENCE EQUATIONS

Sama Arjika, M. P. Chaudhary* and M. N. Hounkonnou**

Department of Mathematics and Informatics
University of Agadez, P.O. Box: 199, Agadez, NIGER

E-mail : rjkama2008@gmail.com

*International Scientific Research and Welfare Organization
(Albert Einstein Chair Professor of Mathematical Sciences)
New Delhi - 110018, INDIA

E-mail : dr.m.p.chaudhary@gmail.com

**International Chair in Mathematical Physics and Applications
(ICMPA-UNESCO Chair)
University of Abomey-Calavi, 072 BP 50 Cotonou, BENIN REPUBLIC

E-mail : norbert.hounkonou@cipma.uac.bj

(**Received:** *Feb. 13, 2022* **Accepted:** *Apr. 25, 2022* **Published:** *Jun. 30, 2022)*

Abstract: In this paper, we give an extension of the first Heine's transformation formula using q-difference equations. Further, we discussed a Ramanujan's theta function $\psi(q)$ and deduced it as a particular case.

Keywords and Phrases: q-Difference operator; q-Binomial theorem; q-integral identities; q-Difference equations, Ramanujan theta function.

2020 Mathematics Subject Classification: 05A30, 11B65, 33D15, 33D60, 39A13.

1. Introduction

Chen and Liu [12] developed an interesting method of deriving hypergeometric identities by parameter augmentation. This method means that a hypergeometric identity with multiple parameters may be derived from its special case obtained by reducing some parameters to zero. It has more realizations as in [11, 9, 8, 4]. In this short paper, we use the q-operator $\mathbb{T}(a,b,c,d,e,yD_x)$ introduced by Cao *et al.* [4] to give a formal generalization of the *first* Heine's transformation formula by using the properties of q-difference equations. Let us start this study by reviewing some common notation and terminology for basic hypergeometric series.

Throughout this paper, we refer to ([5]-[7]; [14]) for definitions and notations. We also suppose that $0 < q < 1$. For complex numbers a, the q-shifted factorials are defined by

$$(a;q)_0 := 1, \quad (a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \quad \text{and} \quad (a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k), \tag{1.1}$$

where (see, for example, [14] and [23])

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \ (a;q)_{n+m} = (a;q)_n(aq^n;q)_m, \ \left(\frac{q}{a};q\right)_n = \frac{(-a)^{-n}\, q^{\binom{n+1}{2}}(aq^{-n};q)_\infty}{(a;q)_\infty}.$$

We adopt the following notation:

$$(a_1,a_2,\cdots,a_r;q)_m = (a_1;q)_m(a_2;q)_m\cdots(a_r;q)_m \qquad (m \in \mathbb{N} := \{1,2,3,\cdots\}).$$

Also, for m large, we have

$$(a_1,a_2,\cdots,a_r;q)_\infty = (a_1;q)_\infty(a_2;q)_\infty\cdots(a_r;q)_\infty.$$

The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}. \tag{1.2}$$

The basic (or q-) hypergeometric function of the variable z with $\mathfrak{r}$ numerator and $\mathfrak{s}$ denominator parameters is defined as follows (see, for details, the monographs by Slater [23, Chapter 3] and by Srivastava and Karlsson [30, p. 347, Eq. (272)]; see also [24] and [16]):

$${}_{\mathfrak{r}}\Phi_{\mathfrak{s}}\left[\begin{matrix} a_1,a_2,\cdots,a_{\mathfrak{r}}; \\ \\ b_1,b_2,\cdots,b_{\mathfrak{s}}; \end{matrix}\; q;z\right] := \sum_{n=0}^{\infty}\left[(-1)^n\, q^{\binom{n}{2}}\right]^{1+\mathfrak{s}-\mathfrak{r}} \frac{(a_1,a_2,\cdots,a_{\mathfrak{r}};q)_n}{(b_1,b_2,\cdots,b_{\mathfrak{s}};q)_n}\, \frac{z^n}{(q;q)_n},$$

where $q \neq 0$ when $\mathfrak{r} > \mathfrak{s} + 1$. Here, we are mainly concerned with the Cauchy polynomials $p_n(x, y)$ as given below (see [10] and [14]):

$$p_n(x, y) := (x - y)(x - qy) \cdots (x - q^{n-1}y) = \left(\frac{y}{x}; q\right)_n x^n. \tag{1.3}$$

The homogeneous version of the Cauchy identity or the following q-binomial theorem (see, for example, [14], [23] and [30]) is given:

$$\sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k = {}_1\Phi_0 \left[\begin{matrix} a; & \\ & q; z \\ -; & \end{matrix} \right] = \frac{(az;q)_\infty}{(z;q)_\infty} \qquad (|z| < 1). \tag{1.4}$$

Upon further setting $a = 0$, this last relation (1.4) becomes Euler's identity (see, for example, [14]):

$$\sum_{k=0}^{\infty} \frac{z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty} \qquad (|z| < 1), \tag{1.5}$$

or its inverse relation given below [14]:

$$\sum_{k=0}^{\infty} \frac{(-1)^k q^{\binom{k}{2}}}{(q;q)_k} z^k = (z;q)_\infty. \tag{1.6}$$

Based upon the q-binomial theorem (1.4) and Heine's transformations, Srivastava *et al.* [29] established and proved a set of two presumably new theta-function identities (see, for details, [29]).

Many interesting and useful extensions of the familiar basic (or q-) series and basic (or q-)polynomials have been investigated due mainly to their demonstrated applications in a wide variety of fields such as theory of partitions, number theory, combinatorial analysis, finite vector spaces, Lie theory, particle physics, engineering and statistics (see, e.g., [30, pp. 350–351]).

The usual q-difference operator is defined by [11, 26, 22]

$$D_a\big\{f(a)\big\} := \frac{f(a) - f(qa)}{a}, \tag{1.7}$$

and their Leibniz rule is given by (see [21])

$$D_a^n \left\{f(a)g(a)\right\} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-n)} D_a^k \left\{f(a)\right\} D_a^{n-k} \left\{g(q^k a)\right\}. \tag{1.8}$$

Here, and in what follows, D_a^0 is understood as the identity operator. The following important properties of D_a :

$$D_a^k\left\{\frac{1}{(at;q)_\infty}\right\}=\frac{t^k}{(at;q)_\infty},\quad D_a^k\left\{(at;q)_\infty\right\}=(-1)^k q^{\binom{k}{2}}t^k(atq^k;q)_\infty \tag{1.9}$$

$$D_a^k\{a^n\}=\begin{cases}\dfrac{(q,q)_n}{(q,q)_{n-k}}a^{n-k}, & 0\le k\le n-1\\ (q,q)_n, & k=n\\ 0, & k\ge n+1\end{cases} \tag{1.10}$$

are straightforward [20].

Recently, Chen and Liu [11, 12] constructed the following augmentation operator

$$\mathbb{T}(bD_x)=\sum_{n=0}^{\infty}\frac{(bD_x)^n}{(q;q)_n}, \tag{1.11}$$

which is of a great significance for deriving identities by applying their various special cases.

Subsequently, Chen and Gu [9] defined the Cauchy augmentation operator

$$\mathbb{T}(a,bD_x)=\sum_{n=0}^{\infty}\frac{(a;q)_n}{(q;q)_n}(bD_x)^n \tag{1.12}$$

and derived the extensions of the Askey–Wilson integral, the Askey–Roy integral, Sears' two-term summation formula, as well as the q-analogs of Barnes' lemmas.

On the other hand, Fang [13] considered the following finite generalized q-exponential operator with two parameters:

$$\mathbb{T}\left[\begin{matrix}q^{-N},w\\ v\end{matrix}\,\middle|\,q;tD_x\right]=\sum_{n=0}^{N}\frac{(q^{-N},w;q)_n}{(v,q;q)_n}(tD_x)^n \tag{1.13}$$

and derived the finite Heine ${}_2\Phi_1$ transformations and some terminating q-series transformation formulas.

Moreover, Li and Tan [17] constructed a generalized q-exponential operator with three parameters

$$\mathbb{T}\left[\begin{matrix}u,v\\ w\end{matrix}\,\middle|\,q;tD_x\right]=\sum_{n=0}^{\infty}\frac{(u,v;q)_n}{(w,q;q)_n}(tD_x)^n \tag{1.14}$$

and gave two formal extensions of the q-Gauss sum. They derived an extension of q-Chu–Vandermonde sums, a formal extension of the Askey–Wilson integral, a formal extension of Sears' two–term summation formula and some curious q–series identities by the operator technique.

Finally, we recall that Cao *et al.* [4] constructed the following q-operators:

$$\mathbb{T}(a,b,c,d,e,yD_x) = \sum_{n=0}^{\infty} \frac{(a,b,c;q)_n}{(q,d,e;q)_n} (yD_x)^n \tag{1.15}$$

and thereby generalized Arjika's results in [2] by using the q-difference equations (see, for details, [4]).

We remark that the q-operator (1.15) is a particular case of the homogeneous q-difference operator $\mathbb{T}(\mathbf{a},\mathbf{b},cD_x)$ (see [27]) by taking

$$\mathbf{a} = (a,b,c), \quad \mathbf{b} = (d,e) \qquad \text{and} \qquad c = y.$$

Cao *et al.* [4] used the q-operator (1.15) and gave the following results:

Proposition 1. (see [4, Theorems 3]) *Let $f(a,b,c,d,e,x,y)$ be a seven-variable analytic function in a neighborhood of $(a,b,c,d,e,x,y) = (0,0,0,0,0,0,0) \in \mathbb{C}^7$. If $f(a,b,c,d,e,x,y)$ satisfies the following difference equation*:

$$\begin{aligned} x\big\{ & f(a,b,c,d,e,x,y) - f(a,b,c,d,e,x,yq) \\ & - (d+e)q^{-1}\,[f(a,b,c,d,e,x,yq) - f(a,b,c,d,e,x,yq^2)] \\ & + deq^{-2}\,[f(a,b,c,d,e,x,yq^2) - f(a,b,c,d,e,x,yq^3)]\big\} \\ & = y\big\{[f(a,b,c,d,e,x,y) - f(a,b,c,d,e,xq,y)] \\ & - (a+b+c)[f(a,b,c,d,e,x,yq) - f(a,b,c,d,e,xq,yq)] \\ & + (ab+ac+bc)[f(a,b,c,d,e,x,yq^2) - f(a,b,c,d,e,xq,yq^2)] \\ & - abc[f(a,b,c,d,e,x,yq^3) - f(a,b,c,d,e,xq,yq^3)]\big\}, \end{aligned} \tag{1.16}$$

then

$$f(a,b,c,d,e,x,y) = \mathbb{T}(a,b,c,d,e,yD_x)\{f(a,b,c,d,e,x,0)\}. \tag{1.17}$$

Liu [18, 19] initiated the method based upon q-difference equations and deduced several results involving Bailey's ${}_6\psi_6$, q-Mehler formulas for the Rogers-Szegö polynomials and q-integral version of the Sears transformation.

Proposition 2. (see [28]) *The following assertion holds*:

$$\mathbb{T}(r, f, g, v, w, uD_a)\left\{\frac{(as;q)_\infty}{(az,at;q)_\infty}\right\}$$
$$= \frac{(as;q)_\infty}{(az,at;q)_\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(r,f,g;q)_{j+k}\ u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}\frac{\left(\frac{s}{z},at;q\right)_k}{(as,q;q)_k}z^k t^j \tag{1.18}$$

provided that $\max\{|az|,|at|,|ut|\} < 1$.

Based upon the work presented by Srivastava *et al.* [29] on q-binomial theorem (1.4) and Heine's transformations, we will consider the q-operator (1.15) and derive a formal generalization of the *first* Heine's transformation formula [15] by using the q-difference equations and find new analogous or more general identities and their possible applications in theoretical or applied sciences.

2. Main Results

As a prelude to the main results, it is worth mentioning the next q-difference formula:

Theorem 1. *For* $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$, *the following result holds*:

$$\mathbb{T}(r, f, g, v, w, uD_a)\left\{\frac{a^m}{(at;q)_\infty}\right\}$$
$$= \frac{a^m}{(at;q)_\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\begin{bmatrix} m \\ k\end{bmatrix}_q\frac{(r,f,g;q)_{j+k}\ u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}(at;q)_k t^j a^{-k} \tag{2.1}$$

provided that $\max\{|at|,|ut|\} < 1$.

Proof. By the means of the definition (1.15) of the operator $\mathbb{T}(r, f, g, v, w, uD_a)$ and the Leibniz rule (1.8), the left-hand side of (2.1) equals

$$\sum_{n=0}^{\infty}\frac{(r,f,g;q)_n u^n}{(v,w,q;q)_n}D_a^n\left\{\frac{a^m}{(at;q)_\infty}\right\}$$
$$= \sum_{n=0}^{\infty}\frac{(r,f,g;q)_n u^n}{(v,w,q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n \\ k\end{bmatrix}_q q^{k(k-n)}\ D_a^k\{a^m\}\,D_a^{n-k}\left\{\frac{1}{(atq^k;q)_\infty}\right\}. \tag{2.2}$$

Using the q-identities (1.9) and (1.10), respectively, we get:

$$\begin{aligned}
&\mathbb{T}(r,f,g,v,w,uD_a)\left\{\frac{a^m}{(at;q)_\infty}\right\}\\
&\quad=\sum_{n=0}^{\infty}\frac{(r,f,g;q)_n u^n}{(v,w,q;q)_n}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q q^{k(k-n)}\frac{(q;q)_m\, a^{m-k}}{(q;q)_{m-k}}\frac{(tq^k)^{n-k}}{(atq^k;q)_\infty}\\
&\quad=\frac{(q;q)_m\, a^m}{(at;q)_\infty}\sum_{k=m}^{\infty}\frac{(at;q)_k\, a^{-k}}{(q;q)_{m-k}(q;q)_k}\sum_{k=n}^{\infty}\frac{(r,f,g;q)_n\, u^n\, t^{n-k}}{(v,w;q)_n(q;q)_{n-k}}.
\end{aligned}\tag{2.3}$$

Substituting $n-k=n$, the right-hand side of (2.3) takes the form:

$$\frac{a^m}{(at;q)_\infty}\sum_{k=0}^{m}\sum_{j=0}^{\infty}\begin{bmatrix} m\\ k\end{bmatrix}_q \frac{(r,f,g;q)_{j+k}\ u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}(at;q)_k t^j a^{-k}.\tag{2.4}$$

Summarizing the above calculations (2.2)-(2.4), we get the formula (2.1) of Theorem 1.

We remark that, when $m=0$, Theorem 1 reduces to the concluding result of Srivastava *et al.* [4].

Corollary 1. *It is asserted that*

$$\mathbb{T}(r,f,g,v,w,uD_s)\left\{\frac{1}{(xs;q)_\infty}\right\}=\frac{1}{(xs;q)_\infty}{}_3\Phi_2\left[\begin{matrix} r,f,g; \\ & q;xu \\ v,w; \end{matrix}\right]\tag{2.5}$$

provided that $\max\{|xs|,|xu|\}<1$.

Let us now recall the following *first* Heine's transformation formula (see, for example, [1], [14], [23] and [30]):

$$\sum_{n=0}^{\infty}\frac{(a,b;q)_n\, z^n}{(c,q;q)_n}=\frac{(a,bz;q)_\infty}{(c,z;q)_\infty}\sum_{n=0}^{\infty}\frac{(\frac{c}{a},z;q)_n}{(bz,q;q)_n}a^n.\tag{2.6}$$

In Theorem 2 below, we give a formal generalization of the *first* Heine's transformation formula (2.6) by using the properties of the q-difference equations. Before stating our main theorem, let us provide Heine's [15] proof of the transformation formula by using the q-Binomial theorem as follows:

$$\sum_{k=0}^{\infty}\frac{(a;q)_k(b;q)_k}{(q;q)_k(c;q)_k}z^k=\frac{(b;q)_\infty}{(c;q)_\infty}\sum_{k=0}^{\infty}\frac{(a;q)_k}{(q;q)_k}z^k\frac{(cq^k;q)_\infty}{(bq^k;q)_\infty}$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} z^k \sum_{j=0}^{\infty} \frac{(\frac{c}{b};q)_j}{(q;q)_j} (bq^k)^j$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{j=0}^{\infty} \frac{(\frac{c}{b};q)_j}{(q;q)_j} b^j \sum_{k=0}^{\infty} \frac{(a;q)_k}{(q;q)_k} (zq^j)^k$$

$$= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{j=0}^{\infty} \frac{(\frac{c}{b};q)_j}{(q;q)_j} b^j \frac{(azq^j;q)_\infty}{(zq^j;q)_\infty}$$

$$= \frac{(b;q)_\infty (az;q)_\infty}{(c;q)_\infty (z;q)_\infty} \sum_{j=0}^{\infty} \frac{(\frac{c}{b};q)_j (z;q)_j}{(q;q)_j (az;q)_j} b^j.$$

Theorem 2. *For $|z| < 1$ and $|b| < 1$, the following result holds*:

$$\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(a,b;q)_n\, z^n}{(c,q;q)_n} \frac{(r,f,g;q)_{j+k}\, u^{j+k}}{(q;q)_j (v,w;q)_{j+k}} \frac{\left(\frac{c}{a},d;q\right)_k q^{nk}}{(cq^n,q;q)_k} a^k d^j$$
$$= \frac{(a,bz;q)_\infty}{(c,z;q)_\infty} \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(\frac{c}{a},z;q)_n\, a^n}{(bz,q;q)_n} \frac{(r,f,g;q)_{j+k}\, u^{j+k}}{(q;q)_j (v,w;q)_{j+k}} (d;q)_k d^j \quad (2.7)$$

provided that each of both sides of (2.7) exists.

Proof of Theorem 2. Upon first setting $a \to ax$ and $c \to cx$ in (2.6) and then multiplying both sides of the resulting equation by $\frac{1}{(dx;q)_\infty}$, we get

$$\sum_{n=0}^{\infty} \frac{(b;q)_n\, z^n}{(q;q)_n} \cdot \frac{(cxq^n;q)_\infty}{(axq^n,dx;q)_\infty} = \frac{(bz;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} \frac{(\frac{c}{a},z;q)_n\, a^n}{(bz,q;q)_n} \cdot \frac{x^n}{(dx;q)_\infty}. \quad (2.8)$$

Eq. (2.7) can be written equivalently as follows:

$$\sum_{n=0}^{\infty} \frac{(b;q)_n\, z^n}{(q;q)_n} \cdot \frac{(cxq^n;q)_\infty}{(axq^n,dx;q)_\infty} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(r,f,g;q)_{j+k}\, u^{j+k}}{(q;q)_j (v,w;q)_{j+k}} \frac{\left(\frac{c}{a},dx;q\right)_k q^{nk}}{(q;q)_k (cxq^n;q)_k} a^k d^j$$
$$= \frac{(bz;q)_\infty}{(z;q)_\infty} \sum_{n=0}^{\infty} \frac{(\frac{c}{a},z;q)_n\, a^n}{(bz,q;q)_n} \cdot \frac{x^n}{(dx;q)_\infty} \sum_{k=0}^{n}\sum_{j=0}^{\infty} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(r,f,g;q)_{j+k}\, u^{j+k}}{(q;q)_j (v,w;q)_{j+k}} (dx;q)_k d^j x^{-k}. \quad (2.9)$$

If we use $F(r,f,g,v,w,x,u)$ to denote the right-hand side of (2.9), it is easy to verify that $F(r,f,g,v,w,x,u)$ satisfies (1.16). By applying (1.17), we thus find that

$$\begin{aligned}F(r,f,g,v,w,x,u) &= \mathbb{T}(r,f,g,v,w,uD_x)\Big\{F(r,f,g,v,w,x,0)\Big\}\\ &= \mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{(bz;q)_\infty}{(z;q)_\infty}\sum_{n=0}^{\infty}\frac{(\frac{c}{a},z;q)_n\,a^n}{(bz,q;q)_n}\cdot\frac{x^n}{(dx;q)_\infty}\right\} \text{ by (2.8)}\\ &= \mathbb{T}(r,f,g,v,w,uD_x)\left\{\sum_{n=0}^{\infty}\frac{(b;q)_n\,z^n}{(q;q)_n}\cdot\frac{(cxq^n;q)_\infty}{(axq^n,dx;q)_\infty}\right\}\\ &= \sum_{n=0}^{\infty}\frac{(b;q)_n\,z^n}{(q;q)_n}\cdot\mathbb{T}(r,f,g,v,w,uD_x)\left\{\frac{(cxq^n;q)_\infty}{(axq^n,dx;q)_\infty}\right\}.\end{aligned}$$

The proof of the formula (2.7) of Theorem 2 can now be completed by making use of the relation (1.18) and $x=1$.

Corollary 2. *It is asserted that*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(a;q)_n\,z^n}{(q;q)_n}\,\frac{(r,f,g;q)_{j+k}\;u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}\,\frac{(d;q)_k\,q^{nk}}{(q;q)_k}a^kd^j\\ &\quad=\frac{(a;q)_\infty}{(z;q)_\infty}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(z;q)_n\,a^n}{(q;q)_n}\;\frac{(r,f,g;q)_{j+k}\;u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}(d;q)_kd^j, \qquad (2.10)\end{aligned}$$

$|z|<1$ *and* $|b|<1$.

Remark 1. *For $u=0$, the formula (2.7) reduces to (2.6). For $b=c=0$, it gives (2.10), while for $b=c=u=0$, it yields the q-binomial theorem*

$$\sum_{n=0}^{\infty}\frac{(a;q)_n\,z^n}{(q;q)_n}=\frac{(az;q)_\infty}{(z;q)_\infty}\quad(|z|<1). \qquad (2.11)$$

As a special case of Theorem 2, if we take $z\to\frac{qz}{b}$ and then letting $b\to\infty$ and $c\to 0$ in (2.6), we have the following corollary.

Corollary 3. *The following relation holds*:

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^n\,q^{\frac{n(n+1)}{2}}(a;q)_n\,z^n}{(q;q)_n}\;\frac{(r,f,g;q)_{j+k}\;u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}\,\frac{(d;q)_k}{(q;q)_k}(aq^n)^kd^j\\ &\quad=(a,qz;q)_\infty\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{j=0}^{\infty}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{a^n}{(qz,q;q)_n}\;\frac{(r,f,g;q)_{j+k}\;u^{j+k}}{(q;q)_j(v,w;q)_{j+k}}(d;q)_kd^j \qquad (2.12)\end{aligned}$$

provided that each of both sides of (2.12) exists.

Remark 2.

1. *For* $u = 0$, $z = -z$ *and* $|q| < 1$, *the expression (2.12) reduces to the concluding remarks of [29, Eq. (12)]:*

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} q^{\frac{n(n+1)}{2}} z^n = (a, -qz; q)_\infty \sum_{n=0}^{\infty} \frac{a^n}{(-qz, q; q)_n}. \tag{2.13}$$

2. *As an application, for* $u = 0$, $a = q$ *and* $z = -1$, *the expression (2.12) reduces to a Ramanujan theta function [3, pp. 36–37, Entry 22]:*

$$\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = (q, -q; q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q^2; q^2)_n} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty} = \psi(q). \tag{2.14}$$

3. Concluding Remarks

In a recently published review-cum-expository review article, in addition to applying the q-analysis to Geometric Function Theory of Complex Analysis, Srivastava [25] pointed out the fact that the results for the q-analogues can easily (and possibly trivially) be translated into the corresponding results for the (p, q)-analogues (with $0 < |q| < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being *redundant*. Such exposition and observation by Srivastava [25, p. 340] might be also applied to our present results with $|q| < 1$.

Acknowledgements

The research work of the second author (M.P.Chaudhary) is supported through a major research project of National Board of Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE), Government of India by its sanction letter Ref. No. 02011/12/2020 NBHM(R.P.)/R & D II/7867, dated 19th October 2020.

References

[1] Andrews, G. E., The Theory of Partitions, Cambridge University Press, Cambridge, London and New York, 1998.

[2] Arjika, S., q-difference equations for homogeneous q-difference operators and their applications, J. Differ. Equ. Appl., 26 (2020), 987-999.

[3] Berndt, B. C., Ramanujan's Notebooks- Part III, Springer-Verlag, Berlin, Heidelberg and New York, 1991.

[4] Cao, J., Xu, B. and Arjika, S., A note on generalized q-difference equations for general Al-Salam-Carlitz polynomials, Adv. Differ. Equa., (2020), Article ID 668.

[5] Chaudhary, M. P., Certain Aspects of Special Functions and Integral Operators, Lambert Academic Publishing, Germany, 2014.

[6] Chaudhary, M. P., On modular relations for the Roger-Ramanujan type identities, Pacific J. Appl. Math., 7 (2016), 177-184.

[7] Chaudhary, M. P. and Jorge Luis Cimadevilla Villacorta, Representations of certain theta function identities in terms of combinatorial partition identities, Far East J. Math. Sci., 102 (2017), 1605-1601.

[8] Chen, W. Y. C., Saad, H. L. and Sun, L. H., The bivariate Rogers–Szegö polynomials, J. Phys. A, 40 (2007), 6071-6084.

[9] Chen, V. Y. B. and Gu, N. S. S., The Cauchy operator for basic hypergeometric series, Adv. Appl. Math., 41 (2008), 177-196.

[10] Chen, W. Y. C., Fu, A. M. and Zhang, B., The homogeneous q-difference operator, Adv. Appl. Math., 31 (2003), 659-668.

[11] Chen, W. Y. C. and Liu, Z.-G., Parameter augmenting for basic hypergeometric series II, J. Combin. Theory Ser. A, 80 (1997), 175-195.

[12] Chen, W. Y. C. and Liu, Z.-G., Parameter augmentation for basic hypergeometric series I, In: B. E. Sagan and R. P. Stanley (Editors), Mathematical Essays in Honor of Gian-Carlo Rota, Birkäuser, Basel and New York, (1998), 111-129.

[13] Fang, J. P., Some applications of q-differential operator, J. Korean Math. Soc., 47 (2010), 223-233.

[14] Gasper, G. and Rahman, M., Basic Hypergeometric Series, (with a Foreword by Richard Askey), Encyclopedia of Mathematics and Its Applications, Vol. 35, Cambridge University Press, Cambridge, New York, Port Chester, Melbourne and Sydney, (1990); Second edition, Encyclopedia of Mathematics and Its Applications, Vol. 96, Cambridge University Press, Cambridge, London and New York, 2004.

[15] Heine, E., Über die Reihe $1 + \frac{(q^\alpha-1)(q^\beta-1)}{(q-1)(q^\gamma-1)} x + \frac{(q^\alpha-1)(q^{\alpha+1}-1)(q^\beta-1)(q^{\beta+1}-1)}{(q-1)(q^2-1)(q^\gamma-1)(q^{\gamma+1}-1)} x^2 + \cdots$. (Aus einem Schreiben an Lejeune Dirichlet), J. Reine Angrew. Math., 32 (1846), 210-212.

[16] Koekock, R. and Swarttouw, R. F., The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue, Report No. 98-17, Delft University of Technology, Delft, The Netherlands, 1998.

[17] Li, N. N. and Tan, W., Two generalized q-exponential operators and their applications, Adv. Differ. Equ., (2016), Article ID 53, 1-14.

[18] Liu, Z.-G., Two q-difference equations and q-operator identities, J. Differ. Equ. Appl., 16 (2010), 1293-1307.

[19] Liu, Z.-G., An extension of the non-terminating ${}_6\phi_5$-summation and the Askey-Wilson polynomials, J. Differ. Equ. Appl., 17 (2011), 1401-1411.

[20] Liu, Z.-G., Two q-difference equations and q-operator identities, J. Differ. Equ. Appl., 16 (2010), 1293-1307.

[21] Roman, S., The theory of the umbral calculus I, J. Math. Anal. Appl., 87 (1982), 58-115.

[22] Saad, H. L. and Sukhi, A. A., Another homogeneous q-difference operator, Appl. Math. Comput., 215 (2010), 4332-4339.

[23] Slater, L. J., Generalized Hypergeometric Functions, Cambridge University Press, Cambridge, London and New York, 1966.

[24] Srivastava, H. M., Certain q-polynomial expansions for functions of several variables I and II, IMA J. Appl. Math., 30 (1983), 315-323.; ibid. 33 (1984), 205-209.

[25] Srivastava, H. M., Operators of basic (or q-) calculus and fractional q-calculus and their applications in geometric function theory of complex analysis, Iran. J. Sci. Technol. Trans. A: Sci., 44 (2020), 327-344.

[26] Srivastava, H. M., and Abdlhusein, M. A., New forms of the Cauchy operator and some of their applications, Russian J. Math. Phys., 23 (2016), 124-134.

[27] Srivastava, H. M., and Arjika, S., Generating functions for some families of the generalized Al-Salam-Carlitz q-polynomials, Adv. Differ. Equ., (2020), Article ID 498, 1-17.

[28] Srivastava, H. M., Cao, J. and Arjika, S., A Note on Generalized q-Difference Equations and Their Applications Involving q-Hypergeometric Functions, Symmetry, 12 (2020), 1816.

[29] Srivastava, H. M., Chaudhary, M. P. and Wakene, F. K., A family of theta-function identities based upon q-binomial theorem and Heine's transformations, Montes Taurus J. Pure Appl. Math., 2 (2020), 1-6.

[30] Srivastava, H. M. and Karlsson, P. W., Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.