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ITERATION OF n ENTIRE FUNCTIONS WITH FINITE ITERATED ORDER

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Abstract: After the works of Banerjee and Adhikary [1] on composition of three entire functions with finite iterated order in this paper we investigate some growth properties of n iterated entire functions of finite iterated order.

Keywords and Phrases: Order, Iterated i-order, Entire function, Composition.
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1. Introduction and Definitions

If $f(z)$ and $g(z)$ be two transcendental entire functions, Clunie [5] showed that $\lim_{r\to\infty}\frac{T_{f\circ g}(r)}{T_f(r)}=\infty$ and $\lim_{r\to\infty}\frac{T_{f\circ g}(r)}{T_g(r)}=\infty$. After this many authors [3, 4, 6, 7, 8, 9, 10, 12] made close investigation on composition of two entire functions with finite order and obtained various results. Recently Jin Tu et.al [11] investigated the composition of entire functions with finite iterated order and proved results on comparative growths of $\log^{[p+q]} T_{f\circ g}(r)\ \ (p,q)\in\mathbb{N}$ with $\log^{[p]} T_f(r)$ and $\log^{[q]} T_g(r)$. In this paper we study some properties on iteration of functions with finite iterated order and extend some earlier results of Banerjee and Adhikary [1] for composition of n entire functions. We first recall the notion of iterated order [7].

Definition 1.1. *The iterated i order $\rho_i(f)$ and iterated i lower order $\mu_i(f)$ of an entire function f are defined by*

$$\rho_i(f)=\limsup_{r\to\infty}\frac{\log^{[i+1]} M_f(r)}{\log r}=\limsup_{r\to\infty}\frac{\log^{[i]} T_f(r)}{\log r},\ (i\in\mathbb{N})$$

and

$$\mu_i(f) = \liminf_{r\to\infty} \frac{\log^{[i+1]} M_f(r)}{\log r} = \liminf_{r\to\infty} \frac{\log^{[i]} T_f(r)}{\log r}, \ (i \in \mathbb{N}),$$

where

$$\log^{[1]}(r) = \log(r), \quad \log^{[i+1]}(r) = \log(\log^{[i]}(r)) \ i \in \mathbb{N}, \ \textit{for all sufficiently large } r.$$

Definition 1.2. *The finiteness degree of the order of an entire function $f(z)$ is defined by*

$$i(f) = \begin{cases} 0 & \textit{for } f \textit{ polynomial,} \\ min\{j \in \mathbb{N} : \rho_j(f) < \infty\} & \textit{for } f \textit{ transcendental,} \\ \infty & \textit{for } f \textit{ with } \rho_j(f) = \infty \textit{ for all } j \in \mathbb{N}. \end{cases}$$

Throughout we assume $f_1, f_2, \cdots, f_{n.}$ etc. are non-constant entire functions and $c_1, c_2, \cdots, c_n$; $d_1, d_2, \cdots, d_n$ etc. are suitable constants.

2. Preliminary Theorems

In this section we presents some results in the form of preliminary theorems which will be needed to prove our main results.

Theorem 2.1. [2] *Let $f_1, f_2, \cdots, f_n$ be n entire functions. Then for all large values of r*

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \geq M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}\cdots\frac{1}{18}M_{f_n}(\frac{r}{2})\cdots))$$

and

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq M_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)).$$

Theorem 2.2. [2] *Let $f_1, f_2, \cdots, f_n$ be n entire functions such that $M_{f_i}(r) > \frac{2+\epsilon}{\epsilon}|f_i(0)|$ for $i = 2, 3, ...n$ and for any $\epsilon > 0$. Then for all large values of r*

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq (1+\epsilon)^{(n-1)} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)).$$

Theorem 2.3. [2] *Let $f_1, f_2, \cdots, f_n$ be n entire functions. Then for all large values of r*

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \geq \frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2}(\cdots\frac{1}{18}M_{f_n}(\frac{r}{8})\cdots)).$$

3. Main Theorems

The following theorems are the main results of this paper.

Theorem 3.1. *Let $f_1, f_2, \cdots, f_n$ be n entire functions having finite iterated orders with $i(f_k) = p_k$ for $k = 1, 2, \cdots, n$ and $\mu_{p_k}(f_k) > 0$ for $k = 1, 2, \cdots, (n-1)$. Then*

$$i(f_1 \circ f_2 \circ \cdots \circ f_n) = p_1 + p_2 + \cdots + p_n$$

and

$$\rho_{[p_1+p_2+\cdots+p_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) = \rho_{p_n}(f_n).$$

Proof. By definition for sufficiently large r and for any given $\epsilon > 0$

$$\begin{aligned} T_{f_1}(r) &< \exp^{[p_1-1]}\{r^{\rho_{p_1}(f_1)+\epsilon}\} \\ M_{f_2}(r) &< \exp^{[p_2]}\{r^{\rho_{p_2}(f_2)+\epsilon}\} \\ &\vdots \\ M_{f_n}(r) &< \exp^{[p_n]}\{r^{\rho_{p_n}(f_n)+\epsilon}\}. \end{aligned}$$

By Theorem 2.2, we have for sufficiently large r

$$\begin{aligned} &T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \le (1+\epsilon)^{n-1} T_{f_1}(M_{f_2}(\cdots M_{f_n}(r)\cdots)) \\ &\le (1+\epsilon)^{n-1} \exp^{[p_1-1]}\left\{[M_{f_2}(\cdots M_{f_n}(r)\cdots)]^{\rho_{p_1}(f_1)+\epsilon}\right\} \\ &\le (1+\epsilon)^{n-1} \exp^{[p_1]}\left\{c_1 \exp^{[p_2-1]}\left\{[(M_{f_3}(\cdots M_{f_n}(r)\cdots))]^{\rho_{p_2}(f_2)+\epsilon}\right\}\right\} \\ &\vdots \\ &\le (1+\epsilon)^{n-1} \exp^{[p_1]}\left\{c_1 \exp^{[p_2]}\left\{c_2 \exp^{[p_3]}\left\{\cdots\left\{c_{n-1}\exp^{[p_n-1]}\left\{r^{\rho_{p_n}(f_n)+\epsilon}\right\}\right\}\cdots\right\}\right\}\right\} \\ &\le \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{r^{\rho_{p_n}(f_n)+(n-1)\epsilon}\right\}. \end{aligned} \tag{3.1}$$

Hence (3.1) gives

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log r} \le \rho_{p_n}(f_n). \tag{3.2}$$

Again $i(f_n) = p_n$, so

$$\limsup_{r\to\infty} \frac{\log^{[p_n+1]} M_{f_n}(r)}{\log r} = \rho_{p_n}(f_n).$$

If $\rho_{p_n}(f_n) > 0$, then there exists a sequence $\{r_m\} \to \infty$ such that for any ϵ $(0 < \epsilon < \rho_{p_n}(f_n))$ and for sufficiently large r_m, we have

$$M_{f_n}(r_m) \ge \exp^{[p_n]}\left\{r_m^{\rho_{p_n}(f_n)-\epsilon}\right\}. \tag{3.3}$$

If $\{r_m\}$ denotes a sequence tending to infinity not necessarily the same at each occurrence then since $\mu_{p_k}(f_k) > 0$ for $k = 1, 2, \cdots, (n-1)$ so from Theorem 2.3 and using (3.3) we have

$$
\begin{aligned}
&T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r_m) \geq \frac{1}{3} \log M_{f_1}(\frac{1}{9} M_{f_2}(\frac{1}{9} M_{f_3}(\frac{1}{18} M_{f_4} \cdots \frac{1}{18} M_{f_n}(\frac{r_m}{8}) \cdots))) \\
&\geq \frac{1}{3} \exp^{[p_1 - 1]} \left\{ [\frac{1}{9} M_{f_2}(\frac{1}{9} M_{f_3}(\frac{1}{18} M_{f_4} \cdots \frac{1}{18} M_{f_n}(\frac{r_m}{8}) \cdots))]^{\mu_{p_1}(f_1) - \epsilon} \right\} \\
&\geq \frac{1}{3} \exp^{[p_1]} \left\{ d_1 \exp^{[p_2 - 1]} \left\{ [\frac{1}{9} M_{f_3}(\frac{1}{18} M_{f_4} \cdots \frac{1}{18} M_{f_n}(\frac{r_m}{8}) \cdots)]^{\mu_{p_2}(f_2) - \epsilon} \right\} \right\} \\
&\vdots \\
&\geq \frac{1}{3} \exp^{[p_1]} \left\{ d_1 \exp^{[p_2]} \left\{ d_2 \exp^{[p_3]} \left\{ \cdots \left\{ d_{n-1} \exp^{[p_n - 1]} \left\{ (\frac{r_m}{8})^{\rho_{p_n}(f_n) - \epsilon} \right\} \right\} \cdots \right\} \right\} \right\} \\
&\geq \exp^{[p_1 + p_2 + \cdots + p_n - 1]} \left\{ r_m^{\rho_{p_n}(f_n) - (n-1)\epsilon} \right\}. \qquad (3.4)
\end{aligned}
$$

Thus

$$
\limsup_{r \to \infty} \frac{\log^{[p_1 + p_2 + \cdots + p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log r} \geq \rho_{p_n}(f_n). \qquad (3.5)
$$

So from (3.2) and (3.5) we have

$$
\limsup_{r \to \infty} \frac{\log^{[p_1 + p_2 + \cdots + p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log r} = \rho_{p_n}(f_n).
$$

Consequently

$$
i(f_1 \circ f_2 \circ \cdots \circ f_n) = p_1 + p_2 + \cdots + p_n
$$

and

$$
\rho_{[p_1 + p_2 + \cdots + p_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) = \rho_{p_n}(f_n) \quad \text{for} \quad \rho_{p_n}(f_n) > 0.
$$

If $\rho_{p_n}(f_n) = 0$, then by definition

$$
\limsup_{r \to \infty} \frac{\log^{[p_n]} M_{f_n}(r)}{\log r} = \infty.
$$

So there exists a sequence $\{r_m\} \to \infty$ such that for arbitrary $A > 0$

$$
\frac{\log^{[p_n]} M_{f_n}(r_m)}{\log r_m} \geq A \quad i.e., M_{f_n}(r_m) \geq exp^{[p_n - 1]} \left\{ r_m^A \right\}. \qquad (3.6)
$$

Hence from (3.4) and (3.6) we have

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m) \geq \exp^{[p_1+p_2+\cdots+p_n-2]}\left\{(\frac{r_m}{8})^{A-\epsilon}\right\}.$$

So

$$\frac{\log^{[p_1+p_2+\cdots+p_n-1]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{\log r_m} \geq (A-\epsilon)$$

for a sequence of values $\{r_m\}$ tending to infinity.
Consequently

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n-1]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log r} = \infty.$$

Therefore

$$i(f_1\circ f_2\circ\cdots\circ f_n) = p_1+p_2+\cdots+p_n$$

and

$$\rho_{[p_1+p_2+\cdots+p_n]}(f_1\circ f_2\circ\cdots\circ f_n) = \rho_{p_n}(f_n).$$

Theorem 3.2. *Let $f_1, f_2, \cdots, f_n$ be n entire functions with finite iterated orders with $0 < \rho_{p_1}(f_1) < \infty$ and $0 < \mu_{p_i}(f_i) \leq \rho_{p_i}(f_i) < \infty$ for $i = 2, 3, \cdots, n$. Then*

$$i(f_1\circ f_2\circ\cdots\circ f_n) = p_1+p_2+\cdots+p_n$$

and

$$\mu_{p_n}(f_n) \leq \rho_{[p_1+p_2+\cdots+p_n]}(f_1\circ f_2\circ\cdots\circ f_n) \leq \rho_{p_n}(f_n).$$

Proof. Since $\rho_{p_1}(f_1) > 0$, there exists a sequence $\{R_m\}$ tending to infinity such that for any ϵ $(0 < \epsilon < \rho_{p_1}(f_1))$

$$M_{f_1}(R_m) \geq \exp^{[p_1]}\left\{{R_m}^{\rho_{p_1}(f_1)-\epsilon}\right\}. \quad (3.7)$$

Since $M(r,h)$ is an increasing continuous function, so there exists a sequence $\{r_m\}$ tending to infinity satisfying $R_m = \frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}(\cdots\frac{1}{18}M_{f_n}(\frac{r_m}{2})\cdots))$ such that for sufficiently large r_m and by Theorem 2.1, we have

$$\begin{aligned}
&M_{f_1 \circ f_2 \circ \cdots \circ f_n}(r_m) \geq M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_m}{2})\cdots)))\\
&= M_{f_1}(R_m)\\
&\geq \exp^{[p_1]}\left\{{R_m}^{\rho_{p_1}(f_1)-\epsilon}\right\} \text{ using(3.7)}\\
&\geq \exp^{[p_1]}\left\{\{\frac{1}{9}M_{f_2}(\frac{1}{18}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_m}{2})\cdots))\}^{\rho_{p_1}(f_1)-\epsilon}\right\}\\
&\geq \exp^{[p_1+1]}\left\{c_1 exp^{[p_2-1]}\left\{\frac{1}{18}M_{f_3}(\cdots \frac{1}{18}M_{f_n}(\frac{r_m}{2})\cdots)\right\}^{\mu_{p_2}(f_2)-\epsilon}\right\}\\
&\vdots\\
&\geq \exp^{[p_1+1]}\left\{c_1 \exp^{[p_2]}\left\{c_2 \exp^{[p_3]}\left\{\cdots\left\{c_{n-1}\exp^{[p_n-1]}\left\{(\frac{r_m}{2})^{\mu_{p_n}(f_n)-\epsilon}\right\}\right\}\cdots\right\}\right\}\right\}\\
&\geq \exp^{[p_1+p_2+\cdots+p_n]}\left\{r_m^{\mu_{p_n}(f_n)-(n-1)\epsilon}\right\}. \qquad (3.8)
\end{aligned}$$

Hence

$$\frac{\log^{[p_1+p_2+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{\log r_m} \geq \mu_{p_n}(f_n) - (n-1)\epsilon$$

i.e.,

$$\rho_{[p_1+p_2+\cdots+p_n]}(f_1\circ f_2\circ\cdots\circ f_n) \geq \mu_{p_n}(f_n). \qquad (3.9)$$

Again for sufficiently large r, from Theorem 2.1, we have

$$\begin{aligned}
&M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq M_{f_1}(M_{f_2}(\cdots M_{f_{n-1}}(M_{f_n}(r))\cdots))\\
&\leq \exp^{[p_1]}[\{M_{f_2}(\cdots M_{f_{n-1}}(M_{f_n}(r))\cdots)\}^{\rho_{p_1}(f_1)+\epsilon}]\\
&\leq \exp^{[p_1+1]}[d_1 \exp^{[p_2-1]}\{M_{f_3}(\cdots M_{f_{n-1}}(M_{f_n}(r))\cdots)\}^{\rho_{p_2}(f_2)+\epsilon}]\\
&\vdots\\
&\leq \exp^{[p_1+1]}[d_1 \exp^{[p_2]}\{d_2 \exp^{[p_3]}\{\cdots\{d_{n-1}\exp^{[p_n-1]}\{(r)^{\rho_{p_n}(f_n)+\epsilon}\}\}\cdots\}\}]\\
&\leq exp^{[p_1+p_2+\cdots+p_n]}\left\{r^{\rho_{p_n}(f_n)+(n-1)\epsilon}\right\}. \qquad (3.10)
\end{aligned}$$

So

$$\frac{\log^{[p_1+p_2+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log r} \leq \rho_{p_n}(f_n) + (n-1)\epsilon$$

i.e.,

$$\rho_{[p_1+p_2+\cdots+p_n]}(f_1\circ f_2\circ\cdots\circ f_n) \leq \rho_{p_n}(f_n). \qquad (3.11)$$

Thus

$$i(f_1 \circ f_2 \circ \cdots \circ f_n) = p_1 + p_2 + \cdots + p_n$$

and from (3.9) and (3.11) we obtain

$$\mu_{p_n}(f_n) \le \rho_{[p_1+p_2+\cdots+p_n]}(f_1 \circ f_2 \circ \cdots \circ f_n) \le \rho_{p_n}(f_n).$$

This completes the proof.

Theorem 3.3. *Let $f_1, f_2, \cdots, f_n$ be n entire functions of iterated orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots, n$ and $\rho_{p_n}(f_n) < \mu_{p_1}(f_1)$. Then* $\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = 0$ *and*

$$\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = 0.$$

Proof. By definition, for sufficiently large r, we get

$$T_{f_1}(r) \ge \exp^{[p_1-1]}\left\{r^{\mu_{p_1}(f_1)-\epsilon}\right\}. \tag{3.12}$$

From (3.1), we have

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \le \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{r^{\rho_{p_n}(f_n)+(n-1)\epsilon}\right\}.$$

Now for sufficiently large r and for any given ϵ, we have from (3.12)

$$\frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} \le \frac{\exp^{[p_1-1]}\left\{r^{\rho_{p_n}(f_n)+(n-1)\epsilon}\right\}}{\exp^{[p_1-1]}\left\{r^{\mu_{p_1}(f_1)-\epsilon}\right\}} \to 0$$

i.e.,

$$\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = 0.$$

Similarly for sufficiently large r

$$\log M_{f_1}(r) \ge \exp^{[p_1-1]}\left\{r^{\mu_{p_1}(f_1)-\epsilon}\right\}. \tag{3.13}$$

Again by (3.10), we obtain

$$M_{f_1\circ f_2\circ\cdots\circ f_n}(r) \le \exp^{[p_1+p_2+\cdots+p_n]}\left\{r^{\rho_{p_n}(f_n)+(n-1)\epsilon}\right\}.$$

Hence from (3.13), we get

$$\frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} \le \frac{\exp^{[p_1-1]}\left\{r^{\rho_{p_n}(f_n)+(n-1)\epsilon}\right\}}{\exp^{[p_1-1]}\left\{r^{\mu_{p_1}(f_1)-\epsilon}\right\}} \to 0$$

i.e.,

$$\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = 0.$$

Example 3.1. The condition $\rho_{p_n}(f_n) < \mu_{p_1}(f_1)$ in Theorem 3.3 is necessary. Which is shown by the following example.
Let $f_1(z) = \exp(z)$, $f_2(z) = \exp^{[2]}(z)$, $\cdots$, $f_n(z) = \exp^{[n]}(z)$ and $p_1 = 1$, $p_2 = 2$, $\cdots$, $p_n = n$.
Then $\rho_{p_n}(f_n)=\rho_n(f_n)=\limsup_{r\to\infty} \frac{\log^{[n+1]} M_{f_n}(r)}{\log r} = \lim_{r\to\infty} \frac{\log r}{\log r} = 1$
and $\mu_{p_1}(f_1) = \mu_1(f_1) = \liminf_{r\to\infty} \frac{\log^{[2]} M_{f_1}(r)}{\log r} = 1$.
But $\lim_{r\to\infty} \frac{\log^{[\frac{n(n+1)}{2}]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \lim_{r\to\infty} \frac{r}{r} = 1 \neq 0$.

Theorem 3.4. *Let $f_1, f_2, \cdots , f_n$ be n entire functions of iterated orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots , n$ and $\mu_{p_n}(f_n) < \rho_{p_1}(f_1)$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = 0$$

and

$$\liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = 0.$$

Proof. By definition, for sufficiently large r

$$T_{f_1}(r) \geq \exp^{[p_1-1]} \left\{ r^{\mu_{p_1}(f_1)-\epsilon} \right\}. \tag{3.14}$$

Again from (3.1) for a sequence $\{r_m\} \to \infty$ we have

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m) \leq \exp^{[p_1+p_2+\cdots+p_n-1]} \left\{ r_m^{\mu_{p_n}(f_n)+(n-1)\epsilon} \right\}. \tag{3.15}$$

So from (3.14) and (3.15) for a sequence of values of $\{r_m\}$ tending to infinity

$$\frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{T_{f_1}(r_m)} \leq \frac{\exp^{[p_1-1]} \left\{ r_m^{\mu_{p_n}(f_n)+(n-1)\epsilon} \right\}}{\exp^{[p_1-1]} \left\{ {r_m}^{\mu_{p_1}(f_1)-\epsilon} \right\}} \to 0.$$

Therefore

$$\liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = 0.$$

The proof for second part is omitted.

Theorem 3.5. *Let $f_1, f_2, \cdots, f_n$ be n entire functions of iterated orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots, n$ and $0 < \rho_{p_1}(f_1) < \rho_{p_n}(f_n) < \infty$ then*

$$\limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \infty.$$

Proof. By definition, there exists a sequence $\{r_m\} \to \infty$ and for any given $\epsilon(> 0)$, we have

$$T_{f_1}(r_m) \leq \exp^{[p_1-1]}\left\{r_m^{\rho_{p_1}(f_1)+\epsilon}\right\}. \tag{3.16}$$

Again from (3.4)

$$\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m) \geq \exp^{[p_1-1]}\left\{r_m^{\rho_{p_n}(f_n)-(n-1)\epsilon}\right\}.$$

So from (3.16)

$$\frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{T_{f_1}(r_m)} \geq \frac{exp^{[p-1]}\left\{r_m^{\rho_{p_n}(f_n)-(n-1)\epsilon}\right\}}{exp^{[p-1]}\left\{r_m^{\rho_{p_1}(f_1)+\epsilon}\right\}}.$$

Since $\rho_{p_1}(f_1) < \rho_{p_n}(f_n)$, so

$$\limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = \infty.$$

The proof for second part is omitted.

Theorem 3.6. *Let $f_1, f_2, \cdots, f_n$ be n entire functions of non-zero iterated lower orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots, n$ and $\mu_{p_1}(f_1) < \mu_{p_n}(f_n) < \infty$. Then*

$$\limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = \infty$$

and

$$\limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \infty.$$

Proof. For all large r, we get from Theorem 2.3 and (3.3)

$$\begin{aligned}
&T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \geq \frac{1}{3}\log M_{f_1}(\frac{1}{9}M_{f_2}(\frac{1}{9}M_{f_3}(\frac{1}{18}M_{f_4}\cdots\frac{1}{18}M_{f_n}(\frac{r}{8})\cdots)))\\
&\geq \frac{1}{3}\exp^{[p_1-1]}\left\{[\frac{1}{9}M_{f_2}(\frac{1}{9}M_{f_3}(\frac{1}{18}M_{f_4}\cdots\frac{1}{18}M_{f_n}(\frac{r}{8})\cdots))]^{\mu_{p_1}(f_1)-\epsilon}\right\}\\
&\geq \frac{1}{3}\exp^{[p_1]}\left\{d_1\exp^{[p_2-1]}\left\{[\frac{1}{9}M_{f_3}(\frac{1}{18}M_{f_4}\cdots\frac{1}{18}M_{f_n}(\frac{r}{8})\cdots)]^{\mu_{p_2}(f_2)-\epsilon}\right\}\right\}\\
&\vdots\\
&\geq \frac{1}{3}\exp^{[p_1]}\left\{d_1\exp^{[p_2]}\left\{d_2\exp^{[p_3]}\left\{\cdots\left\{d_{n-1}\exp^{[p_n-1]}\left\{(\frac{r}{8})^{\mu_{p_n}(f_n)-\epsilon}\right\}\right\}\cdots\right\}\right\}\right\}\\
&\geq \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{r^{\mu_{p_n}(f_n)-(n-1)\epsilon}\right\}. \qquad (3.17)
\end{aligned}$$

Also from definition there exists a sequence $\{r_m\}\to\infty$ such that for any given $\epsilon(>0)$, we have

$$T_{f_1}(r_m) \leq \exp^{[p_1-1]}\left\{r_m^{\mu_{p_1}(f_1)+\epsilon}\right\}. \qquad (3.18)$$

Combining (3.17) and (3.18) we get for a sequence $\{r_m\}$ tending to infinity, we get

$$\frac{\log^{[p_2+p_3+\cdots+p_n]}T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{T_{f_1}(r_m)} \geq \frac{exp^{[p-1]}\left\{r_m^{\mu_{p_n}(f_n)-(n-1)\epsilon}\right\}}{exp^{[p_1-1]}\left\{r_m^{\mu_{p_1}(f_1)+\epsilon}\right\}} \to \infty$$

i.e., $\limsup_{r\to\infty}\frac{\log^{[p_2+p_3+\cdots+p_n]}T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = \infty$.

Similarly, we can prove that $\limsup_{r\to\infty}\frac{\log^{[p_2+p_3+\cdots+p_n+1]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \infty$.

Example 3.2. The condition $\mu_{p_1}(f_1) < \mu_{p_n}(f_n)$ in Theorem 3.6 is necessary. This follows from the following example.

Let $f_1(z) = \exp^{[n]}(z)$, $f_2(z) = \exp^{[n-1]}(z)$, $\cdots$, $f_n(z) = \exp(z)$, and $p_1 = n$, $p_2 = n-1$, $\cdots$, $p_{n-1} = 2$, $p_n = 1$. Then we have

$\mu_{p_1}(f_1)=\mu_n(f_1)=\liminf_{r\to\infty}\frac{\log^{[n+1]}M_{f_1}(r)}{\log r} = \lim_{r\to\infty}\frac{\log r}{\log r} = 1$

and $\mu_{p_n}(f_n)=\mu_1(f_n) = \liminf_{r\to\infty}\frac{\log^{[2]}M_{f_n}(r)}{\log r} = 1$.

But $\lim\limits_{r\to\infty}\frac{\log^{[\frac{n(n-1)}{2}+1]}M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \lim\limits_{r\to\infty}\frac{\log^{[\frac{n(n-1)}{2}+1]}(\exp^{[\frac{n(n+1)}{2}]}(r))}{\exp^{[n]}(r)}$

$= \lim\limits_{r\to\infty}\frac{\exp^{[n-1]}(r)}{\exp^{[n-1]}(r)} = 1 \neq \infty.$

Theorem 3.7. *Let $f_1, f_2, \cdots, f_n$ be n entire functions of non-zero iterated lower orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots, n$ and $\rho_{p_1}(f_1) < \mu_{p_n}(f_n) < \infty$. Then*

$$\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = \infty$$

and

$$\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \infty.$$

Proof. By definition, for large r and for any given $\epsilon(> 0)$

$$T_{f_1}(r) \le \exp^{[p_1-1]} \left\{ r^{\rho_{p_1}(f_1)+\epsilon} \right\}. \tag{3.19}$$

Also for all large values of r we get from (3.17)

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \ge \exp^{[p_1+p_2+\cdots+p_n-1]} \left\{ r^{\mu_{p_n}(f_n)-(n-1)\epsilon} \right\}. \tag{3.20}$$

Hence from (3.19) and (3.20) we have

$$\frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} \ge \frac{\exp^{[p_1-1]} \left\{ r^{\mu_{p_n}(f_n)-(n-1)\epsilon} \right\}}{\exp^{[p_1-1]} \left\{ r^{\rho_{p_1}(f_1)+\epsilon} \right\}}.$$

Since $\rho_{p_1}(f_1) < \mu_{p_n}(f_n)$ and $\epsilon(> 0)$ is arbitrary, so

$$\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{T_{f_1}(r)} = \infty.$$

Similarly, we can prove that $\displaystyle\lim_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log M_{f_1}(r)} = \infty.$

Theorem 3.8. *Let $f_1, f_2, \cdots, f_n$ be n entire functions of non-zero iterated lower orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots, n$; $0 < \mu_{p_1}(f_1) \le \rho_{p_1}(f_1) < \infty$ and $0 < \mu_{p_n}(f_n) \le \rho_{p_n}(f_n) < \infty$. Then $\frac{\mu_{p_n}(f_n)}{\rho_{p_1}(f_1)} \le \liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)}$ $\le min \left\{ \frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)} \right\} \le max \left\{ \frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)} \right\}$ $\le \limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \le \frac{\rho_{p_n}(f_n)}{\mu_{p_1}(f_1)}$.*

Proof. We have for sufficiently large r and for any $\epsilon(> 0)$

$$(\mu_{p_1}(f_1) - \epsilon) \log r \le \log^{[p_1]} T_{f_1}(r) \le (\rho_{p_1}(f_1) + \epsilon) \log r. \tag{3.21}$$

Now from (3.4) we can easily say that

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m) \geq \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{r_m^{\mu_{p_n}(f_n)-(n+1)\epsilon}\right\}.$$

So from above for all large r and any $\epsilon(>0)$ we have from (3.2)

$$\{\mu_{p_n}(f_n)-(n-1)\epsilon\}\log r \leq \log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq (\rho_{p_n}(f_n)+\epsilon)\log r. \tag{3.22}$$

From (3.21) and (3.22) we obtain for sufficiently large values of r

$$\frac{\rho_{p_n}(f_n)+\epsilon}{\mu_{p_1}(f_1)-\epsilon} \geq \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \geq \frac{\mu_{p_n}(f_n)-(n-1)\epsilon}{\rho_{p_1}(f_1)+\epsilon}. \tag{3.23}$$

Since $\epsilon > 0$ is arbitrary, we get from (3.23)

$$\liminf_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \geq \frac{\mu_{p_n}(f_n)}{\rho_{p_1}(f_1)}.$$

and

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_1}(f_1)}.$$

Also by definition, there exist two sequences $\{r_m\}$ and $\{R_m\}$ tending to infinity such that

$$\log^{[p_1]} T_{f_1}(r_m) \geq (\rho_{p_1}(f_1)-\epsilon)\log r_m, \quad \log^{[p_1]} T_{f_1}(R_m) \leq (\mu_{p_1}(f_1)+\epsilon)\log R_m. \tag{3.24}$$

From (3.1) it can be easily seen that for a sequence of values tending to infinity

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{r^{\mu_{p_n}(f_n)+(n-1)\epsilon}\right\}.$$

So from above and (3.4) there exists two sequences $\{r_m'\}$ and $\{R_m'\}$ tending to infinity such that

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m') \leq \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{{r_m'}^{\mu_{p_n}(f_n)+(n-1)\epsilon}\right\}.$$

and

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(R_m') \geq \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{{R_m'}^{\rho_{p_n}(f_n)-(n-1)\epsilon}\right\}$$

So

$$\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m') \leq \{\mu_{p_n}(f_n)+(n-1)\epsilon\}\log r_m' \tag{3.25}$$

and

$$\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(R_m^{'}) \geq \{\rho_{p_n}(f_n) - (n-1)\epsilon\} \log R_m^{'}. \tag{3.26}$$

From (3.22) and (3.24), we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r_m)}{\log^{[p_1]} T_{f_1}(r_m)} \leq \frac{(\rho_{p_n}(f_n) + \epsilon) \log r_m}{(\rho_{p_1}(f_1) - \epsilon) \log r_m}$$

i.e.,

$$\liminf_{r \to \infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p_1]} T_1(r)} \leq \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}.$$

From (3.21) and (3.25), we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r_m^{'})}{\log^{[p_1]} T_{f_1}(r_m^{'})} \leq \frac{\{\mu_{p_n}(f_n) + (n-1)\epsilon\} \log r_m^{'}}{(\mu_{p_1}(f_1) - \epsilon) \log r_m^{'}}$$

i.e.,

$$\liminf_{r \to \infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p]} T(r, f)} \leq \frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}.$$

Hence

$$\liminf_{r \to \infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p]} T(r, f)} \leq min \left\{ \frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)} \right\}.$$

Again from (3.22) and (3.24), we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(R_m)}{\log^{[p_1]} T_{f_1}(R_m)} \geq \frac{\{\mu_{p_n}(f_n) - (n-1)\epsilon\} \log R_m}{(\mu_{p_1}(f_1) + \epsilon) \log R_m}$$

i.e.,

$$\limsup_{r \to \infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(r)}{\log^{[p]} T_{f_1}(r)} \geq \frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}.$$

Also from (3.21) and (3.26), we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1 \circ f_2 \circ \cdots \circ f_n}(R_m^{'})}{\log^{[p_1]} T_{f_1}(R_m^{'})} \geq \frac{\{\rho_{p_n}(f_n) - (n-1)\epsilon\} \log R_m^{'}}{(\rho_{p_1}(f_1) + \epsilon) \log R_m^{'}}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \geq \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}.$$

Hence

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \geq max\left\{\frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}\right\}.$$

Therefore $\frac{\mu_{p_n}(f_n)}{\rho_{p_1}(f_1)} \leq \liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \leq min\left\{\frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}\right\}$
$\leq max\left\{\frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_1}(f_1)}$.
This completes the proof.

Corollary 3.1. *Let f_1, f_2, $\cdots$, f_n satisfy the hypotheses of Theorem 3.8, then*
$\frac{\mu_{p_n}(f_n)}{\rho_{p_1}(f_1)} \leq \liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1^{(k)}}(r)} \leq min\left\{\frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}\right\}$
$\leq max\left\{\frac{\mu_{p_n}(f_n)}{\mu_{p_1}(f_1)}, \frac{\rho_{p_n}(f_n)}{\rho_{p_1}(f_1)}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_1]} T_{f_1^{(k)}}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_1}(f_1)}$.
for k=1,2,..

Corollary 3.2. *We can obtain the similar result if we replace $T_{f_1\circ f_2\circ\cdots\circ f_n}(r)$ and $T_{f_1}(r)$ with $\log M_{f_1\circ f_2\circ\cdots\circ f_n}(r)$ and $\log M_{f_1}(r)$ respectively in Theorem 3.8.*

Theorem 3.9. *Let $f_1, f_2, \cdots, f_n$ be n entire functions of non-zero iterated lower orders with $i(f_k) = p_k$, for $k = 1, 2, \cdots, n$; $0 < \mu_{p_1}(f_1) \leq \rho_{p_1}(f_1) < \infty$ and $0 < \mu_{p_n}(f_n) \leq \rho_{p_n}(f_n) < \infty$. Then*
$\frac{\mu_{p_n}(f_n)}{\rho_{p_n}(f_n)} \leq \liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq 1$
$\leq \limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n]} T_{f_n\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_n}(f_n)}$
and
$\frac{\mu_{p_n}(f_n)}{\rho_{p_n}(f_n)} \leq \liminf_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} \log M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n+1]} \log M_{f_n}(r)} \leq 1$
$\leq \limsup_{r\to\infty} \frac{\log^{[p_2+p_3+\cdots+p_n+1]} \log M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n+1]} \log M_{f_n}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_n}(f_n)}$.
Proof. For sufficiently large r and for any $\epsilon > 0$, we have

$$\log^{[p_n]} T_{f_n}(r) \leq (\rho_{p_n}(f_n) + \epsilon) \log r. \tag{3.27}$$

Again for sufficiently large r we have from (3.4)

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m) \geq \exp^{[p_1+p_2+\cdots+p_n-1]}\left\{r_m^{\mu_{p_n}(f_n)-(n-1)\epsilon}\right\}. \tag{3.28}$$

So from (3.27) and (3.28) we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{\log^{[p_n]} T_{f_n}(r_m)} \geq \frac{\{\mu_{p_n}(f_n)-(n-1)\epsilon\}\log r_m}{(\rho_{p_n}(f_n)+\epsilon)\log r_m}.$$

As $\epsilon > 0$ is any arbitrary, so

$$\liminf_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \geq \frac{\mu_{p_n}(f_n)}{\rho_{p_n}(f_n)}.$$

Again by definition, there exists a sequence $\{r_m\}$ tending to infinity such that

$$\log^{[p_n]} T_{f_n}(r_m) \geq (\rho_{p_n}(f_n)-\epsilon)\log r_m. \tag{3.29}$$

From (3.2) for any given $\epsilon > 0$ and sufficiently large r, we get

$$\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r) \leq (\rho_{p_n}(f_n)+\epsilon)\log r. \tag{3.30}$$

Also

$$\log^{[p_n]} T_{f_n}(r) \leq (\rho_{p_n}(f_n)+\epsilon)\log r, \log^{[p_n]} T_{f_n}(r) \geq (\mu_{p_n}(f_n)-\epsilon)\log r. \tag{3.31}$$

From (3.29) and (3.30), we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r_m)}{\log^{[p_n]} T_{f_n}(r_m)} \leq \frac{(\rho_{p_n}(f_n)+\epsilon)\log r_m}{(\rho_{p_n}(f_n)-\epsilon)\log r_m}$$

i.e.,

$$\liminf_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq 1.$$

Again from (3.31) we get

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq \frac{(\rho_{p_n}(f_n)+\epsilon)\log r}{(\mu_{p_n}(f_n)-\epsilon)\log r}$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_n}(f_n)}.$$

Also for a sequence $\{R_m\}$ tending to infinity, we have from (3.4)

$$T_{f_1\circ f_2\circ\cdots\circ f_n}(R_m) \geq exp^{[p_1+p_2+\cdots+p_n-1]}\left\{R_m^{\rho_{p_n}(f_n)-(n-1)\epsilon}\right\}. \tag{3.32}$$

From (3.31) and (3.32), we obtain

$$\frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(R_m)}{\log^{[p_n]} T_{f_n}(R_m)} \geq \frac{\{\rho_{p_n}(f_n)-(n-1)\epsilon\}\log R_m}{(\rho_{p_n}(f_n)+\epsilon)\log R_m}.$$

i.e.,

$$\limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \geq 1.$$

Combining all we have $\frac{\mu_{p_n}(f_n)}{\rho_{p_n}(f_n)} \leq \liminf_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq 1$
$\leq \limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n]} T_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n]} T_{f_n}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_n}(f_n)}$.
Similarly we can show that $\frac{\mu_{p_n}(f_n)}{\rho_{p_n}(f_n)} \leq \liminf_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n+1]} M_{f_n}(r)} \leq 1$
$\leq \limsup_{r\to\infty} \frac{\log^{[p_1+p_2+\cdots+p_n+1]} M_{f_1\circ f_2\circ\cdots\circ f_n}(r)}{\log^{[p_n+1]} M_{f_n}(r)} \leq \frac{\rho_{p_n}(f_n)}{\mu_{p_n}(f_n)}$.

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