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# ON SOME RESULTS ON $(p,q)$-TH RELATIVE ORDER AND $(p,q)$-TH RELATIVE TYPE OF ANALYTIC FUNCTIONS REPRESENTED BY DIRICHLET SERIES IN THE HALF PLANE

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**Abstract:** In this paper we introduce the idea of $(p,q)$-th relative order, $(p,q)$-th relative type and $(p,q)$-th relative weak type of analytic functions represented by Dirichlet series in the half plane with respect to entire Dirichlet series where $p$ and $q$ are positive integers and then study some basic properties of analytic functions represented by Dirichlet series in the half plane using the concepts of $(p,q)$-th relative order, $(p,q)$-th relative type and $(p,q)$-th relative weak type.

**Keywords and Phrases:** Dirichlet series, analytic function, half plane, $(p,q)$-th relative order, $(p,q)$-th relative type, $(p,q)$-th relative weak type, property (H).

**2020 Mathematics Subject Classification:** 30B50, 30D15, 30D99.

## 1. Introduction

Let us consider that the readers are familiar with the fundamental results and the standard notations of the theory of entire functions which are available in [27]. The concept of order $\rho(f)$ of an entire function $f$ is classical in complex analysis.

It is well-known that the growth of entire function which has essential influence on its properties can be studied in terms of order. Generalizing the notion of order, Juneja et al. [8] introduced the definition of $(p, q)$-th order $\rho^{(p,q)}(f)$ of an entire function $f$ where $p$ and $q$ are any two positive integers such that $p \geq q$. For details about $(p, q)$-th order and index-pair $(p, q)$, one may see. [8].

On the other hand, Sheremeta [20] introduced the idea of generalized $\alpha\beta$-order $\rho(\alpha, \beta, f)$ of an entire function $f$ where $\alpha(x)$ is slowly growing function and $\beta(x)$ satisfy the condition $\beta((1 + o(1)x) = (1 + o(1))\beta(x),\ x \to +\infty$. The concept of $(p, q)$-th order matches with the concept of generalized $\alpha\beta$-order when $\alpha(x) = \log^{[p-1]} x = \log(\log^{[p-2]} x)$ with $p \geq 2$ and $\beta(x) = \log^{[q-1]} x = \log(\log^{[q-2]} x)$ with $q \geq 1$, where $\log^{[l]}(x)$ is the $l$-th iterated logarithm and $\log^{[0]}(x) = x$. But the $(1, 1)$-th order is not a particular case of generalized $\alpha\beta$-order since $\alpha(x) = x$ is not a slowly increasing function which is the main disadvantage to say that generalized $\alpha\beta$-order is more generalized concept of $(p, q)$-th order. Considering $f(z) = z$ and $g(z) = z^2$, one can easily verify that $\rho^{(2,1)}(f) = \rho^{(2,1)}(g) = 0$, $\rho^{(2,2)}(f) = \rho^{(2,2)}(g) = 1$ whereas $\rho^{(1,1)}(f) = 1$ and $\rho^{(1,1)}(g) = 2$.

However while most of the scales compare the growth of an entire function with some positive unbounded function, Bernal [3, 4] proposed the notion of the relative order between two entire functions which is a bit different approach to generalize the order of entire function. In the case of relative order, it was then natural for Lahiri et al. [12] to define the $(p, q)$-th relative order of entire functions where $p$ and $q$ are any two positive integers such that $p \geq q$. Sánchez Ruiz et al. [18] gave a more natural definition of $(p, q)$-th relative order of an entire function in light of index-pair ignoring the restriction $p \geq q$. Actually $(p, q)$-th relative order is more generalized concept of relative order as well as $(p, q)$-th order [18].

Keeping this in mind, here in this paper we wish to introduce the idea of $(p, q)$-th relative order, $(p, q)$-th relative type and $(p, q)$-th relative weak type of analytic functions represented by Dirichlet series in the half plane with respect to entire Dirichlet series where $p$ and $q \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers and then study some basic properties of analytic functions represented by Dirichlet series in the half plane using the concepts of $(p, q)$-th relative order, $(p, q)$-th relative type and $(p, q)$-th relative weak type. In fact, this class of function was deeply investigated in [21].

## 2. Preliminaries

Let us define $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ for $x \in [0, \infty)$ and $k \in \mathbb{N}$. Further we assume that throughout the present paper $p$ and $q$

always denote positive integers. Now we consider the Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_n e^{s\lambda_n} \tag{2.1}$$

where $\lambda_n<\lambda_{n+1}(n\geq 1)$, $0\leq\lambda_1$, $\lambda_n\to+\infty$ as $n\to+\infty$, $s=\sigma+it$ ($\sigma$ and $t$ are real variables) and

$$\limsup_{n\to+\infty}\frac{n}{\lambda_n}=D<+\infty. \tag{2.2}$$

If $C$ and $A$ denote respectively the abscissa of convergence and the abscissa of absolute convergence of the series (2.1), then, as (2.2) is satisfied, we get ([13], p. 166) that

$$\limsup_{n\to+\infty}\frac{\log|a_n|^{-1}}{\lambda_n}=A=C. \tag{2.3}$$

If the series represented by (2.1) converges absolutely in the half plane $\sigma<A$ $(-\infty<A<+\infty)$, then it is well known that (13, p. 166) that the series (2.1) characterizes an analytic function in the half-plane $\sigma<A$ $(-\infty<A<+\infty)$. Also the function $M_f(\sigma)$ [6] known as maximum modulus function corresponding to an analytic function $f(s)$ in the half- plane $\sigma<A$ $(-\infty<A<+\infty)$ defined by (2.1), is written as

$$M_f(\sigma)=\underset{-\infty<t<+\infty}{l.u.b.}|f(\sigma+it)|.$$

Moreover it is well know [6] that $\log M_f(\sigma)$ is an increasing convex function of $\sigma$ for $\sigma<A$ $(-\infty<A<\infty)$. Moreover, if $A=\infty$, then $f(s)$, the sum function of (2.1) represents an entire function and during the past decades, several authors {see for example [9, 15, 16, 17, 19]} made closed investigations on the properties of entire Dirichlet series in different directions using the different growth parameters of $M_f(\sigma)$.

Further two entire functions $f(s)$ and $g(s)$ represented by Dirichlet series are said to be asymptotically equivalent in the half plane $\sigma<A$ $(-\infty<A<+\infty)$ if there exists $l$ $(0<l<+\infty)$ such that $\frac{M_f\left(\exp^{[p]}\left(\mu\left(\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\beta}\right)\right)}{M_g\left(\exp^{[p]}\left(\mu\left(\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)\right)\right)^{\beta}}\to l$ as $\sigma\to A$ where $\mu>0$ and $\beta>0$ are any numbers, $p$, $q\in\mathbb{N}$. In this case we write $f(s)\sim g(s)$ and if $f(s)\sim g(s)$, then clearly $g(s)\sim f(s)$.

If $f(s)$ is not an entire function represented by Dirichlet series but represents an analytic function represented by Dirichlet series in the half plane $\sigma<A$ $(-\infty<$

$A < +\infty$), Nandan [14] introduced the notion of order $\rho(f)$ and lower order $\lambda(f)$ of $f(s)$ as

$$\rho(f) = \limsup_{\sigma\to A} \frac{\log^{[2]} M_f(\sigma)}{\log\left(\frac{1}{1-\exp(\sigma-A)}\right)} \text{ and } \lambda(f) = \liminf_{\sigma\to A} \frac{\log^{[2]} M_f(\sigma)}{\log\left(\frac{1}{1-\exp(\sigma-A)}\right)}.$$

Further, if $f(s)$ is of order $\rho(f)$ $(0 < \rho(f) < +\infty)$, Nandan [14] also introduced the definitions of type $\Delta(f)$ and lower type $\overline{\Delta}(f)$ of $f(s)$ which are as follows:

$$\Delta(f) = \limsup_{\sigma\to A} \frac{\log M_f(\sigma)}{\left(\frac{1}{1-\exp(\sigma-A)}\right)^{\rho(f)}} \text{ and } \overline{\Delta}(f) = \liminf_{\sigma\to A} \frac{\log M_f(\sigma)}{\left(\frac{1}{1-\exp(\sigma-A)}\right)^{\rho(f)}}.$$

The above definition of order (respectively lower order) as well as type and lower type does not seem to be feasible if $f(s)$ is of order zero. To over come this situation and in order to study precisely the growth of $f(s)$, Awasthi and Dixit [1, 2] introduced the concepts of logarithmic order (respectively lower logarithmic order) along with logarithmic type ( respectively logarithmic lower type) by increasing $\log^+$ once in the denominator and derived some of the growths of $f(s)$. Therefore the logarithmic order $\rho_{\log}(f)$ (respectively logarithmic lower order $\lambda_{\log}(f)$) and logarithmic type $\Delta_{\log}(f)$ (respectively logarithmic lower type $\overline{\Delta}_{\log}(f)$) of $f(s)$ are define as

$$\begin{aligned} \rho_{\log}(f) &= \limsup_{\sigma\to A} \frac{\log^{[2]} M_f(\sigma)}{\log^{[2]}\left(\frac{1}{1-\exp(\sigma-A)}\right)} \\ \Bigg(\lambda_{\log}(f) &= \liminf_{\sigma\to A} \frac{\log^{[2]} M_f(\sigma)}{\log^{[2]}\left(\frac{1}{1-\exp(\sigma-A)}\right)}\Bigg). \end{aligned}$$

and

$$\begin{aligned} \Delta_{\log}(f) &= \limsup_{\sigma\to A} \frac{\log M_f(\sigma)}{\left(\log\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\rho_{\log}(f)}} \\ \Bigg(\overline{\Delta}_{\log}(f) &= \liminf_{\sigma\to A} \frac{\log M_f(\sigma)}{\left(\log\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\rho_{\log}(f)}}\Bigg). \end{aligned}$$

and following this, a number of papers on the above growth indicators of an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$

have appeared in the literature where growing significance of workers on this topic has been noticed (see for example [23, 24]).

Juneja et al. [9] first introduced the concept of $(p, q)$-th order and $(p, q)$-th lower order of an entire Dirichlet series where $p \geq q + 1 \geq 1$. In the line of Juneja et al. [9], now we shall introduce the definitions of $(p, q)$-th order $\rho^{(p,q)}(f)$ and $(p, q)$ -th lower order $\lambda^{(p,q)}(f)$ of an analytic function $f(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ which is in fact extend the notion of Nandan [14] and Awasthi et al. [1]. In order to keep accordance with the definition of logarithmic order we will give a minor modification to the definition of $(p, q)$-th order introduced by Juneja et. al. [9].

**Definition 1.** *If $f(s)$ be an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$, then the $(p, q)$-th order and $(p, q)$-th lower order of $f$ are respectively defined as:*

$$\rho^{(p,q)}(f) = \limsup_{\sigma \to A} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]}\left(\frac{1}{1-\exp(\sigma - A)}\right)} \quad \text{and}$$

$$\lambda^{(p,q)}(f) = \liminf_{\sigma \to A} \frac{\log^{[p]} M_f(\sigma)}{\log^{[q]}\left(\frac{1}{1-\exp(\sigma - A)}\right)}.$$

These definitions extended the generalized order and generalized lower order of analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$. Clearly $\rho^{(2,1)}(f) = \rho(f)$ (respectively $\lambda^{(2,1)}(f) = \lambda(f)$) and $\rho^{(2,2)}(f) = \rho_{\log}(f)$ (respectively $\lambda^{(2,2)}(f) = \lambda_{\log}(f)$). The above definition avoids the restriction $p > q$ and gives the idea of generalized logarithmic order.

However in this connection we just introduce the following definition:

**Definition 2.** *An analytic function $f(s)$ represented by Dirichlet series is said to have index-pair $(p, q)$ in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ if $b < \rho^{(p,q)}(f) < +\infty$ and $\rho^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise.*

Moreover, if $0 < \rho^{(p,q)}(f) < +\infty$, then

$$\begin{cases} \rho^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \rho^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly, for $0 < \lambda^{(p,q)}(f) < +\infty$, one can easily verify that

$$\begin{cases} \lambda^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \lambda^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \quad . \end{cases}$$

An entire function $f(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ of index-pair ($(p, q)$ is said to be of regular $(p, q)$ growth if its $(p, q)$-th order coincides with its $(p, q)$-th lower order, otherwise $f(s)$ is said to be of irregular $(p, q)$ growth.

Now to compare the relative growth of two analytic function represented by Dirichlet series having same non zero finite $(p, q)$-th order in the half plane $\sigma < A$ $(-\infty < A < +\infty)$, one may introduce the definition of $(p, q)$-th type (respectively $(p, q)$-th lower type) in the following manner:

**Definition 3.** *The $(p, q)$-th type and $(p, q)$-th lower type respectively denoted by $\Delta^{(p,q)}(f)$ and $\overline{\Delta}^{(p,q)}(f)$ of an analytic function $f(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with $0 < \rho^{(p,q)}(f) < +\infty$ are defined as:*

$$\Delta^{(p,q)}(f) = \limsup_{\sigma \to A} \frac{\log^{[p-1]} M_f(\sigma)}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)^{\rho^{(p,q)}(f)}} \quad \text{and}$$

$$\overline{\Delta}^{(p,q)}(f) = \liminf_{\sigma \to A} \frac{\log^{[p-1]} M_f(\sigma)}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)^{\rho^{(p,q)}(f)}}.$$

Analogously to determine the relative growth of two analytic functions represented by Dirichlet series having same non zero finite $(p, q)$-th lower order in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ , one may introduce the definition of $(p, q)$-th weak type in the following way:

**Definition 4.** *The $(p, q)$-th weak type $\tau^{(p,q)}(f)$ and the growth indicator $\overline{\tau}^{(p,q)}(f)$ of an analytic function $f(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ such that $0 < \lambda^{(p,q)}(f) < +\infty$ are defined as follows:*

$$\overline{\tau}^{(p,q)}(f) = \limsup_{\sigma \to A} \frac{\log^{[p-1]} M_f(\sigma)}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)^{\lambda^{(p,q)}(f)}} \quad \text{and}$$

$$\tau^{(p,q)}(f) = \liminf_{\sigma \to A} \frac{\log^{[p-1]} M_f(\sigma)}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)^{\lambda^{(p,q)}(f)}}.$$

In this connection we state the following definition which will be needed in the sequel:

**Definition 5.** *A entire function $f(s)$ represented by Dirichlet series is said to have Property (H), if for any $\delta > 1$, $\gamma > 0$ and for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to A*

$$\Big(M_f\Big(\exp^{[p]}\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\beta}\Big)\Big)^2 < M_f\Big(\delta\exp^{[p]}\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\beta}\Big).$$

*where $M_f(\sigma)$ denotes the maximum modulus function.*

The concept of relative order between two entire functions to avoid comparing growth just with $\exp z$ was first initiated in [3, 4]. Considering this idea, now we introduce the definition of relative order and relative lower order of an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function represented by Dirichlet series in the following way:

**Definition 6.** *If $f(s)$ be an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g(s)$ be an entire function represented by Dirichlet series, then the relative order and relative lower order of $f(s)$ with respect to $g(s)$, denoted by $\rho_g(f)$ and $\lambda_g(f)$ respectively are defined by*

$$\rho_g(f) = \limsup_{\sigma\to A}\frac{\log M_g^{-1}(M_f((\sigma))}{\log\left(\frac{1}{1-\exp(\sigma-A)}\right)} \text{ and}$$

$$\lambda_g(f) = \liminf_{\sigma\to A}\frac{\log M_g^{-1}(M_f((\sigma))}{\log\left(\frac{1}{1-\exp(\sigma-A)}\right)}.$$

In order to make some progress in the study of relative order, now we introduce the concepts of $(p, q)$-th relative order $\rho_g^{(p,q)}(f)$ and $(p, q)$-th relative lower order $\lambda_g^{(p,q)}(f)$ of an analytic function $f(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function $g(s)$ which is also represented by Dirichlet series in the following approach:

$$\rho_g^{(p,q)}(f) = \limsup_{\sigma\to A} \frac{\log^{[p]} M_g^{-1}(M_f((\sigma))}{\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)} \text{ and}$$
$$\lambda_g^{(p,q)}(f) = \liminf_{\sigma\to A} \frac{\log^{[p]} M_g^{-1}(M_f((\sigma))}{\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)}.$$

In this connection, we intend to give a definition of relative index-pair of an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function $g(s)$ represented by Dirichlet series which is relevant in the sequel :

**Definition 7.** *Let $f(s)$ be an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g(s)$ be an entire function represented by Dirichlet series. Then the analytic function $f(s)$ is said to have relative index-pair $(p,q)$ with respect to $g(s)$, if $b < \rho_g^{(p,q)}(f) < +\infty$ and $\rho_g^{(p-1,q-1)}(f)$ is not a nonzero finite number, where $b = 1$ if $p = q$ and $b = 0$ otherwise.*

Moreover, if $0 < \rho_g^{(p,q)}(f) < +\infty$, then

$$\begin{cases} \rho_g^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \rho_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \rho_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Similarly, for $0 < \lambda_g^{(p,q)}(f) < +\infty$, one can easily verify that

$$\begin{cases} \lambda_g^{(p-n,q)}(f) = +\infty & \text{for} \quad n < p, \\ \lambda_g^{(p,q-n)}(f) = 0 & \text{for} \quad n < q, \\ \lambda_g^{(p+n,q+n)}(f) = 1 & \text{for} \quad n = 1, 2, \cdots \end{cases}.$$

Further an analytic function $f(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ for which $(p,q)$-th relative order and $(p,q)$-th relative lower order with respect to an entire function $g(s)$ represented by Dirichlet series are the same is called a function of regular relative $(p,q)$ growth with respect to $g(s)$. Otherwise, $f(s)$ is said to be irregular relative $(p,q)$ growth.with respect to $g(s)$.

Now in order to compare the relative growth of two analytic function represented by Dirichlet series having same non zero finite $(p,q)$-th relative order in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function represented by Dirichlet series, one may introduce the concepts of $(p, q)$-th relative type and $(p, q)$-th relative lower type which are as follows:

**Definition 8.** *Let $f(s)$ be an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with $0 < \rho_g^{(p,q)}(f) < +\infty$ where $g(s)$ be an entire function represented by Dirichlet series. Then $(p, q)$-th relative type $\Delta_g^{(p,q)}(f)$ and $(p, q)$-th relative lower type $\overline{\Delta}_g^{(p,q)}(f)$ of $f(s)$ with respect to $g(s)$ are defined as*

$$\Delta_g^{(p,q)}(f) = \limsup_{\sigma \to A} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\rho_g^{(p,q)}(f)}} \text{ and}$$

$$\overline{\Delta}_g^{(p,q)}(f) = \liminf_{\sigma \to A} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\rho_g^{(p,q)}(f)}}.$$

Analogously to determine the relative growth of two analytic function represented by Dirichlet series having same non zero finite $(p, q)$-th relative lower order in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function represented by Dirichlet series, one may introduce the definition of $(p, q)$-th relative weak type in the following way:

**Definition 9.** *Let $f(s)$ be an analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with $0 < \lambda_g^{(p,q)}(f) < +\infty$ where $g(s)$ be an entire function represented by Dirichlet series. Then $(p, q)$-th relative weak type $\tau_g^{(p,q)}(f)$ and the growth indicator $\overline{\tau}_g^{(p,q)}(f)$ of $f(s)$ with respect to $g(s)$ are defined as follows:*

$$\overline{\tau}_g^{(p,q)}(f) = \limsup_{\sigma \to A} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\lambda_g^{(p,q)}(f)}} \text{ and}$$

$$\tau_g^{(p,q)}(f) = \liminf_{\sigma \to A} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\lambda_g^{(p,q)}(f)}}.$$

In this connection we state the following definition which will be needed in the sequel:

**Definition 10.** *A pair of analytic functions $f(s)$ and $g(s)$ represented by Dirichlet* *series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ are mutually said to have Property (X) if for all sufficiently large values of $\sigma$, both*

$$M_{f\cdot g}(\sigma) > M_f(\sigma) \text{ and } M_{f\cdot g}(\sigma) > M_g(\sigma)$$

*hold simultaneously.*

## 3. Results

In this section first we present the following lemmas which will be needed in the sequel.

**Lemma 1.** [11, 25] *Suppose that $f(s)$ be an entire function represented by Dirichlet series, $\alpha > 1$ and $0 < \gamma < \alpha$. Then*

$$M_f(\alpha\sigma) > e^{\gamma\sigma} M_f(\sigma) \text{ for all large } \sigma.$$

**Lemma 2.** *Let $f(s)$ be an entire function represented by Dirichlet series, $\alpha > 1$ and $\beta > 0$. Then for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$*

$$M_f\Big(\alpha \exp^{[p]} \mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)^{\theta}\Big) > \beta M_f\Big(\exp^{[p]} \mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)^{\theta}\Big),$$

*where $\theta > 0$ and $\mu > 0$.*

**Proof.** From Lemma 1, we get for $\alpha > 1$, $0 < \gamma < \alpha$ and for all large $\sigma$ that $\lim\limits_{\sigma\to+\infty} \frac{M_f(\alpha\sigma)}{M_f(\sigma)} = +\infty$ which implies that $M_f(\alpha\sigma) > \beta M_f(\sigma)$ where $\beta > 0$. Therefore Lemma 2 follows by putting $\sigma = \exp^{[p]} \mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\theta}$.

**Lemma 3.** [25] *Let $f(s)$ be an entire function represented by vector valued Dirichlet series, $P$ be a vector valued Dirichlet polynomial, $\alpha \in (0, 1)$, $\beta > 1$ then for all large $\sigma$*

$$M_f(\alpha\sigma) < M_h(\sigma) < M_f(\beta\sigma),$$

*where $h(s) = Pf(s)$.*

**Lemma 4.** *Let $f(s)$ be an entire function represented by Dirichlet series, $P$ be a Dirichlet polynomial, $\alpha \in (0, 1)$, $\beta > 1$ then for $\gamma > 0$, $\mu > 0$ and for $\sigma > \sigma_1(\varepsilon)$*

*sufficiently close to $A$*

$$M_f\Big(\alpha\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\gamma}\Big) < M_h\Big(\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\gamma}\Big)\Big) < M_f\Big(\beta\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\gamma}\Big)\Big),$$

*where $h(s)=Pf(s)$.*

Lemma 4 follows from Lemma 3 simply replacing $\sigma$ by $\exp^{[p]}(\mu(\log^{[q]}(\frac{1}{1-\exp(\sigma-A)}))^{\gamma})$.

Now we present the main results of the paper.

**Theorem 1.** *Let $f(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma<A$ $(-\infty<A<+\infty)$ and $g(s)$, $h(s)$ be any two entire functions represented by Dirichlet series. If $g(s)\sim h(s)$ then $\rho_h^{(p,q)}(f)=\rho_g^{(p,q)}(f)$ and $\lambda_h^{(p,q)}(f)=\lambda_g^{(p,q)}(f)$.*

**Proof.** Since $g(s)\sim h(s)$, for any $l$ $(0<l<+\infty)$, there exist an arbitrary $\varepsilon>0$, $\beta>0$ and $\sigma>\sigma_1(\varepsilon)$ sufficiently close to $A$ for which get that

$$M_g\Big(\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\beta}\Big)\Big) < (l+\varepsilon)M_h\Big(\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\beta}\Big)\Big).$$

Now for $\alpha>\max\{1,(l+\varepsilon)\}$, we get by Lemma 2 for any arbitrary $\varepsilon>0$ and $\sigma>\sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_g\Big(\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\beta}\Big)\Big) < M_h\Big(\alpha\exp^{[p]}\Big(\mu\Big(\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\beta}\Big)\Big). \tag{3.1}$$

Therefore in view of above we get for any arbitrary $\varepsilon>0$ and $\sigma>\sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_f(\sigma) \le M_g\Big(\exp^{[p]}\Big((\rho_g^{(p,q)}(f)+\varepsilon)\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)\Big)$$

$$i.e.,\ M_f(\sigma) \le M_h\Big(\alpha\exp^{[p]}\Big((\rho_g^{(p,q)}(f)+\varepsilon)\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)\Big)$$

$$i.e., \ \frac{\log^{[p]} M_h^{-1}(M_f(\sigma)) + O(1)}{\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)} \leq (\rho_g^{(p,q)}(f) + \varepsilon).$$

Hence we get from above that $\rho_h^{(p,q)}(f) \leq \rho_g^{(p,q)}(f)$. The reverse inequality is clear because $h(s) \sim g(s)$ and so $\rho_g^{(p,q)}(f) = \rho_h^{(p,q)}(f)$. In a similar manner, $\lambda_h^{(p,q)}(f) = \lambda_g^{(p,q)}(f)$. This proves the theorem.

**Theorem 2.** *Let $f(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g(s)$, $h(s)$ be any two entire functions represented by Dirichlet series. If $g(s) \sim h(s)$ then $\Delta_h^{(p,q)}(f) = \Delta_g^{(p,q)}(f)$ and $\overline{\Delta}_h^{(p,q)}(f) = \overline{\Delta}_g^{(p,q)}(f)$.*

**Proof.** In view of Theorem 1 and (3.1), we get for $\alpha > \max\{1, (l+\varepsilon)\}$, any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$\begin{aligned} & M_f(\sigma) \\ \leq \ & M_g\Big(\exp^{[p-1]}\Big((\Delta_g^{(p,q)}(f)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_h^{(p,q)}(f)}\Big)\Big) \\ & i.e., \ M_f(\sigma) \\ \leq \ & M_h\Big(\exp^{[p-1]}\Big(\alpha(\Delta_g^{(p,q)}(f)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_h^{(p,q)}(f)}\Big)\Big) \end{aligned}$$

$$i.e., \ \frac{\log^{[p-1]} M_h^{-1}(M_f(\sigma))}{\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)^{\rho_h^{(p,q)}(f)}} \leq \alpha(\Delta_g^{(p,q)}(f) + \varepsilon).$$

Now letting $\alpha \to 1^+$, we get from above that $\Delta_h^{(p,q)}(f) \leq \Delta_g^{(p,q)}(f)$. The reverse inequality is clear because $h(s) \sim g(s)$ and so $\Delta_g^{(p,q)}(f) = \Delta_h^{(p,q)}(f)$. In a similar manner, $\overline{\Delta}_h^{(p,q)}(f) = \overline{\Delta}_g^{(p,q)}(f)$. Hence the theorem follows.

In the line of Theorem 2 and in view of Theorem 1, one may prove the following theorem:

**Theorem 3.** *Let $f(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g(s)$, $h(s)$ be any two entire functions represented by Dirichlet series. If $g(s) \sim h(s)$ then $\tau_h^{(p,q)}(f) = \tau_g^{(p,q)}(f)$ and $\overline{\tau}_h^{(p,q)}(f) = \overline{\tau}_g^{(p,q)}(f)$.*

Now we state the following three theorems which can easily be carried out from the definitions of $(p,q)$-th relative growth indicators of analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function represented by Dirichlet series and with the help of Theorem 1, Theorem 2 and Theorem 3 and therefore their proofs are omitted.

**Theorem 4.** *Let $f(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g(s)$, $h(s)$ be any two entire functions represented by Dirichlet series such that $g(s) \sim h(s)$. Also let $0 < \lambda_g^{(p,q)}(f) \leq \rho_g^{(p,q)}(f) < \infty$ and $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$. Then*

$$\liminf_{\sigma\to A} \frac{\log^{[p]} M_g^{-1}(M_f(\sigma))}{\log^{[p]} M_h^{-1}(M_f(\sigma))} \leq 1 \leq \limsup_{\sigma\to A} \frac{\log^{[p]} M_g^{-1}(M_f(\sigma))}{\log^{[p]} M_h^{-1}(M_f(\sigma))}.$$

**Theorem 5.** *Let $f(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g(s)$, $h(s)$ be any two entire functions represented by Dirichlet series such that $g(s) \sim h(s)$. Also let either $0 < \Delta_g^{(p,q)}(f) < \infty$ $(0 < \Delta_h^{(p,q)}(f) < \infty)$ or $0 < \overline{\tau}_g^{(p,q)}(f) < \infty$ $(0 < \overline{\tau}_h^{(p,q)}(f) < \infty)$. Then*

$$\liminf_{\sigma\to A} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\log^{[p-1]} M_h^{-1}(M_f(\sigma))} \leq 1 \leq \limsup_{\sigma\to A} \frac{\log^{[p-1]} M_g^{-1}(M_f(\sigma))}{\log^{[p-1]} M_h^{-1}(M_f(\sigma))}.$$

**Theorem 6.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$ be any entire function represented by Dirichlet series. Also let at least $f_1(s)$ or $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$. Then*

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}.$$

*The equality holds when $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least $f_j(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ where $i,j = 1,2$ and $i \neq j$.*

**Proof.** If $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) = 0$ then the result is obvious. So we suppose that $\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) > 0$ . We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k)$ is finite for $k = 1,2$. Further let $\max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\} = \Delta$ and $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$. Now there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we have from the definition of $\lambda_{g_1}^{(p,q)}(f_1)$ and for any arbitrary $\varepsilon > 0$ that

$$M_{f_1}(\sigma) \leq M_{g_1}\Big(\exp^{[p]}\Big((\lambda_{g_1}^{(p,q)}(f_1) + \varepsilon)\log^{[q]}\Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)\Big)$$

$$i.e.,\ M_{f_1}(\sigma) \leq M_{g_1}\Big(\exp^{[p]}\Big((\Delta + \varepsilon)\log^{[q]}\Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)\Big). \tag{3.2}$$

Also for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we obtain from the definition of $\rho_{g_1}^{(p,q)}(f_2)(= \lambda_{g_1}^{(p,q)}(f_2))$ that

$$M_{f_2}(\sigma) \leq M_{g_1}\Big( \exp^{[p]} \Big( (\lambda_{g_1}^{(p,q)}(f_2) + \varepsilon) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big) \tag{3.3}$$

$$i.e., \ M_{f_2}(\sigma) \leq M_{g_1}\Big( \exp^{[p]} \Big( (\Delta + \varepsilon) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big). \tag{3.4}$$

Again there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$ for which we obtain from (3.2) and (3.4) that

$$i.e., \ M_{f_1 \pm f_2}(\sigma) < 2M_{g_1}\Big( \exp^{[p]} \Big( (\Delta + \varepsilon) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big). \tag{3.5}$$

Therefore there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we obtain from (3.5) with the help of Lemma 2 , and for any $\beta > 1$ that

$$M_{f_1 \pm f_2}(\sigma) < M_{g_1}\Big( \beta \exp^{[p]} \Big( (\Delta + \varepsilon) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big)$$

$$i.e., \ \frac{1}{\beta} M_{g_1}^{-1}(M_{f_1 \pm f_2}(\sigma)) < \exp^{[p]} \Big( (\Delta + \varepsilon) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big)$$

$$i.e., \ \frac{\log^{[p]} M_{g_1}^{-1}(M_{f_1 \pm f_2}(\sigma)) + O(1)}{\log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big)} < (\Delta + \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}.$$

Similarly, if we consider that $f_1(s)$ is of regular relative $(p, q)$ growth with respect to $g_1(s)$ or both $f_1(s)$ and $f_2(s)$ are of regular relative $(p, q)$ growth with respect to $g_1(s)$, then one can easily verify that

$$\lambda_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}. \tag{3.6}$$

Further without loss of any generality, let $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$, $f_1(s)$ is of regular relative $(p, q)$ growth with respect to $g_1(s)$ and $f(s) = f_1(s) \pm f_2(s)$. Then in view of (3.6) we get that $\lambda_{g_1}^{(p,q)}(f) \leq \lambda_{g_1}^{(p,q)}(f_2)$. $As, f_2(s) = \pm(f(s) - f_1(s))$ and in this case we obtain that $\lambda_{g_1}^{(p,q)}(f_2) \leq \max\ \{\lambda_{g_1}^{(p,q)}(f),\ \lambda_{g_1}^{(p,q)}(f_1)\}$. As we assume that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$, therefore we have $\lambda_{g_1}^{(p,q)}(f_2) \leq \lambda_{g_1}^{(p,q)}(f)$ and hence $\lambda_{g_1}^{(p,q)}(f)$

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 6.

**Theorem 7.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$ be any entire function represented by Dirichlet series. Also let $f_1(s)$ and $f_2(s)$ be analytic functions represented by Dirichlet series with relative index-pair $(p, q)$ with respect to $g_1(s)$. Then*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \max\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\}.$$

*The equality holds when $\rho_{g_1}^{(p,q)}(f_1) \neq, \rho_{g_1}^{(p,q)}(f_2)$.*

**Theorem 8.** *Let us consider $f_1(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any two entire functions represented by Dirichlet series. Also let $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists. Then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) \geq \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}.$$

*The equality holds when $\lambda_{g_1}^{(p,q)}(f_1) \neq \lambda_{g_2}^{(p,q)}(f_1)$.*

**Proof.** If $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) = +\infty$, then the result is obvious. So we suppose that $\lambda_{g_1 \pm g_2}^{(p,q)}(f_1) < +\infty$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1$, 2. Further let $\Psi = \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\}$. Now for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we have from the definition of $\lambda_{g_k}^{(p,q)}(f_1)$ where $k = 1$, 2 that

$$M_{g_k}\left(\exp^{[p]}\left((\lambda_{g_k}^{(p,q)}(f_1) - \varepsilon)\log^{[q]}\left(\frac{1}{1 - \exp(\sigma - A)}\right)\right)\right) \leq M_{f_1} \tag{3.7}$$

$$i.e,\ M_{g_k}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\left(\frac{1}{1 - \exp(\sigma - A)}\right)\right)\right) \leq M_{f_1}(\sigma).$$

Therefore for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we obtain from above and first part of Lemma 2 and $\beta > 1$ that

$$\begin{aligned} M_{g_1 \pm g_2}&\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\left(\frac{1}{1 - \exp(\sigma - A)}\right)\right)\right) \\ &< M_{g_1}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\left(\frac{1}{1 - \exp(\sigma - A)}\right)\right)\right) \\ &+ M_{g_2}\left(\exp^{[p]}\left((\Psi - \varepsilon)\log^{[q]}\left(\frac{1}{1 - \exp(\sigma - A)}\right)\right)\right) \end{aligned}$$

$$i.e.,\ M_{g_1 \pm g_2}\Big(\exp^{[p]}\Big((\Psi-\varepsilon)\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)\Big) < 2M_{f_1}(\sigma)$$

$$i.e.,\ M_{g_1 \pm g_2}\Big(\frac{1}{\beta}\exp^{[p]}\Big((\Psi-\varepsilon)\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)\Big) < M_{f_1}(\sigma)$$

$$i.e.,\ \exp^{[p]}(\Big((\Psi-\varepsilon)\log^{[q]}(\Big(\frac{1}{1-\exp(\sigma-A)}\Big)) < \beta M^{-1}_{g_1 \pm g_2}(M_{f_1}(\sigma))$$

$$i.e.,\ \frac{\log^{[p]} M^{-1}_{g_1 \pm g_2}(M_{f_1}(\sigma)) + O(1)}{\log^{[q]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)} > \Psi - \varepsilon.$$

Since $\varepsilon > 0$ are arbitrary, we get from above that

$$\lambda^{(p,q)}_{g_1 \pm g_2}(f_1) \geq \Psi = \min\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\}. \tag{3.8}$$

Now without loss of any generality, we may consider that $\lambda^{(p,q)}_{g_1}(f_1) < \lambda_{g_2}(p,q)(f_1)$ and $g(s) = g_1(s) \pm g_2(s)$. Then in view of (3.8) we get that $\lambda^{(p,q)}_{g}(f_1) \geq \lambda^{(p,q)}_{g_1}(f_1)$. Further, $g_1(s) = (g(s) \pm g_2(s))$ and in this case we obtain that $\lambda^{(p,q)}_{g_1}(f_1) \geq \min\{\lambda^{(p,q)}_{g}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\}$. As we assume that $\lambda^{(p,q)}_{g_1}(f_1) < \lambda^{(p,q)}_{g_2}(f_1)$, therefore we have $\lambda^{(p,q)}_{g_1}(f_1) \geq \lambda^{(p,q)}_{g}(f_1)$ and hence $\lambda^{(p,q)}_{g}(f_1) = \lambda^{(p,q)}_{g_1}(f_1) = \min\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\}$. Therefore, $\lambda^{(p,q)}_{g_1 \pm g_2}(f_1) = \lambda^{(p,q)}_{g_i}(f_1) \mid i = 1, 2$ provided $\lambda^{(p,q)}_{g_1}(f_1) \neq \lambda^{(p,q)}_{g_2}(f_1)$. Thus the theorem follows.

**Theorem 9.** *Let us consider $f_1(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any two entire functions represented by Dirichlet series. Also let the relative index-pair of $f_1(s)$ with respect to $g_1(s)$ and $g_2(s)$ is $(p,q)$. Also let $f_1(s)$ is of regular relative $(p,q)$ growth with respect to at least any one of $g_1(s)$ or $g_2(s)$. Then*

$$\rho^{(p,q)}_{g_1 \pm g_2}(f_1) \geq \min\left\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\right\}.$$

*The equality holds when $\rho^{(p,q)}_{g_i}(f_1) < \rho^{(p,q)}_{g_j}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_j(s)$ where $i, j = 1, 2$ and $i \neq j$.*

We omit the proof of Theorem 9 as it can easily be carried out in the line of Theorem 8.

**Remark 1.** *Using the results of Theorem 6 to Theorem 9 and in view of the proofs of Theorem 13 and Theorem 14 of [5], one can easily deduce the similar conclusion of Theorem 13 and Theorem 14 of [5] under somewhat different conditions for any two analytic functions $f_1(s)$ and $f_2(s)$ represented by Dirichlet series in the half* *plane $\sigma < A$ $(-\infty < A < +\infty)$ and any two entire functions $g_1(s)$ and $g_2(s)$ represented by Dirichlet series.*

**Theorem 10.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$ be any entire function represented by Dirichlet series. Also let at least $f_1(s)$ or $f_2(s)$ is of regular relative $(p, q)$ growth with respect to $g_1(s)$. Then*

$$\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}$$

*when $g_1(s)$ satisfy the Property (H). The equality holds when $f_1(s)$ and $f_2(s)$ satisfy Property (X).*

**Proof.** Suppose that $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) > 0$. Otherwise if $\lambda_{g_1}^{(p,q)}(f_1 \cdot f_2) = 0$ then the result is obvious. Let us consider that $f_2(s)$ is of regular relative $(p, q)$ growth with respect to $g_1(s)$. Also suppose that $\max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\} = \Delta$. We can clearly assume that $\lambda_{g_1}^{(p,q)}(f_k)$ is finite for $k = 1, 2$. Now there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which it follows from the definition of $\rho_{g_1}^{(p,q)}(f_1)$ and for any arbitrary $\frac{\varepsilon}{2} > 0$ that

$$M_{f_1}(\sigma) \leq M_{g_1}\Big( \exp^{[p]} \Big( (\lambda_{g_1}^{(p,q)}(f_1) + \frac{\varepsilon}{2}) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big)$$

$$i.e., \ M_{f_1}(\sigma) \leq M_{g_1}\Big( \exp^{[p]} \Big( \Delta + \frac{\varepsilon}{2} \Big) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big). \tag{3.9}$$

Also for any arbitrary $\frac{\varepsilon}{2} > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we obtain from the definition of $\rho_{g_1}^{(p,q)}(f_2)$ $(= \lambda_{g_1}^{(p,q)}(f_2))$ that

$$M_{f_2}(\sigma) \leq M_{g_1}\Big( \exp^{[p]} \Big( (\lambda_{g_1}^{(p,q)}(f_2) + \frac{\varepsilon}{2}) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big)$$

$$i.e., \ M_{f_2}(\sigma) \leq M_{g_1}\Big( \exp^{[p]} \Big( \Big( \Delta + \frac{\varepsilon}{2} \Big) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big) \Big). \tag{3.10}$$

Observe that

$$\frac{\Delta + \varepsilon}{\Delta + \frac{\varepsilon}{2}} > 1.$$

Therefore we consider the expression $\dfrac{\exp^{[p]} \Big( (\Delta + \varepsilon) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big)}{\exp^{[p]} \Big( (\Delta + \frac{\varepsilon}{2}) \log^{[q]} \Big( \frac{1}{1 - \exp(\sigma - A)} \Big) \Big)}$ for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$. Thus for any $\delta > 1$, it follows from the above expression for

$\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, say $\sigma \geq \sigma_1 \geq \sigma_0$ that

$$\frac{\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\left(\frac{1}{1-\exp(\sigma_0-A)}\right)\right)}{\exp^{[p]}\left((\Delta+\frac{\varepsilon}{2})\log^{[q]}\left(\frac{1}{1-\exp(\sigma_0-A)}\right)\right)} = \delta. \tag{3.11}$$

Now there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we have from (3.9), (3.10) that

$$M_{f_1\cdot f_2}(\sigma) < \left(M_{g_1}\left(\exp^{[p]}\left(\left(\Delta+\frac{\varepsilon}{2}\right)\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)\right)\right)^2.$$

Now we obtain from above for a sequence values of $\sigma$ tending to infinity that

$$M_{f_1\cdot f_2}(\sigma) < M_{g_1}\left(\delta\left(\exp^{[p]}\left(\left(\Delta+\frac{\varepsilon}{2}\right)\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)\right)\right),$$

since $g_1(s)$ has the Property (H) and $\delta > 1$. Therefore in view of (3.11), it follows for a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$ that

$$M_{f_1\cdot f_2}(\sigma) < M_{g_1}\left(\exp^{[p]}\left((\Delta+\varepsilon)\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)\right). \tag{3.12}$$

Since $\varepsilon > 0$ is arbitrary, we obtain from above that

$$\lambda_{g_1}^{(p,q)}(f_1\cdot f_2) \leq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}.$$

Similarly, if we consider that $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ or both $f_1(s)$ and $f_2(s)$ are of regular relative $(p,q)$ growth with respect to $g_1(s)$, then also one can easily verify that

$$\lambda_{g_1}^{(p,q)}(f_1\cdot f_2) \leq \Delta = \max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}. \tag{3.13}$$

Now let $f_1(s)$ and $f_2(s)$ are satisfy Property (X), then of course we have $M_{f_1\cdot f_2}(\sigma) > M_{f_1}(\sigma)$ and $M_{f_1\cdot f_2}(\sigma) > M_{f_2}(\sigma)$ for all sufficiently large values of $\sigma$. Therefore from the definition of relative $(p,q)$-th lower order, we get that $\lambda_{g_1}^{(p,q)}(f_1\cdot f_2) \geq \lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_1}^{(p,q)}(f_1\cdot f_2) \geq \lambda_{g_1}^{(p,q)}(f_2)$. Hence the theorem follows.

Now we state the following theorem without its proof as it can easily be carried out in the line of Theorem 10.

**Theorem 11.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$*

*be any entire function represented by Dirichlet series. Also let $f_1(s)$ and $f_2(s)$ be analytic functions represented by Dirichlet series with relative index-pair $(p,q)$ with respect to entire $g_1(s)$. Then*

$$\rho_{g_1}^{(p,q)}(f_1 \cdot f_2) \leq \max\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_1}^{(p,q)}(f_2)\},$$

*when $g_1(s)$ satisfy the Property (H). The equality holds when $f_1(s)$ and $f_2(s)$ satisfy Property (X).*

**Theorem 12.** *Let us consider $f_1(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any two entire functions represented by Dirichlet series. Also let $\lambda_{g_1}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_1)$ exists. Then*

$$\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) \geq \min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\},$$

*when $g_1 \cdot g_2(s)$ satisfies the Property (H). The equality holds when $g_1(s)$ and $g_2(s)$ satisfy Property (X).*

**Proof.** Suppose that $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) < +\infty$. Otherwise if $\lambda_{g_1 \cdot g_2}^{(p,q)}(f_1) = +\infty$, then the result is obvious. Also suppose that $\min\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_1)\} = \Psi$. We can clearly assume that $\lambda_{g_k}^{(p,q)}(f_1)$ is finite for $k = 1, 2$. Now for any arbitrary $\varepsilon > 0$, with $\varepsilon < \Psi$, we obtain for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_{g_k}\left(\exp^{[p]}\left(\left(\lambda_{g_k}^{(p,q)}(f_1) - \frac{\varepsilon}{2}\right)\log^{[q]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)\right) \leq M_{f_1}(\sigma)$$

$$i.e.,\ M_{g_k}\left(\exp^{[p]}\left(\left(\Psi - \frac{\varepsilon}{2}\right)\log^{[q]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)\right) \leq M_{f_1}(\sigma). \tag{3.14}$$

Observe that

$$\frac{\Psi - \frac{\varepsilon}{2}}{\Psi - \varepsilon} > 1.$$

Now we consider the expression $\frac{\left(\exp^{[p]}\left(\left(\Psi - \frac{\varepsilon}{2}\right)\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)\right)}{\left(\exp^{[p]}\left((\Psi-\varepsilon)\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)\right)\right)}$ for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$. Thus for any $\delta > 1$, it follows from the above expression for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, say $\sigma \geq \sigma_1 \geq \sigma_0$ that

$$\frac{\left(\exp^{[p]}\left(\left(\Psi - \frac{\varepsilon}{2}\right)\log^{[q]}\left(\frac{1}{1-\exp(\sigma_0-A)}\right)\right)\right)}{\left(\exp^{[p]}\left((\Psi-\varepsilon)\log^{[q]}\left(\frac{1}{1-\exp(\sigma_0-A)}\right)\right)\right)} = \delta. \tag{3.15}$$

Therefore we get from (3.14) for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_{g_1 \cdot g_2}\Big( \exp^{[p]} \Big( \Big(\Psi - \frac{\varepsilon}{2}\Big) \log^{[q]} \Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)\Big) < (M_{f_1}(\sigma))^2. \qquad (3.16)$$

Also we obtain from above for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$i.e., \ \Big(M_{g_1 \cdot g_2}\Big( \exp^{[p]} \Big( \Big(\Psi - \frac{\varepsilon}{2}\Big) \log^{[q]} \Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)\Big)\Big)^{\frac{1}{2}} < M_{f_1}(\sigma)$$

$$i.e., \ M_{g_1 \cdot g_2}\Big(\frac{1}{\delta}\Big( \exp^{[p]} \Big( \Big(\Psi - \frac{\varepsilon}{2}\Big) \log^{[q]} \Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)\Big)\Big) < M_{f_1}(\sigma),$$

since $g_1 \cdot g_2(s)$ has the Property (H) and $\delta > 1$. Therefore in view of (3.15), it follows from above for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_{g_1 \cdot g_2}\Big( \exp^{[p]} \Big( (\Psi - \varepsilon) \log^{[q]} \Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)\Big) < M_{f_1}(\sigma)$$

$$i.e, \ M^{-1}_{g_1 \cdot g_2}(M_{f_1}(\sigma)) > \exp^{[p]} \Big( (\Psi - \varepsilon) \log^{[q]} \Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)$$

$$i.e, \ \frac{\log^{[p]} M^{-1}_{g_1 \cdot g_2}(M_{f_1}(\sigma))}{\log^{[q]} \Big(\frac{1}{1-\exp(\sigma - A)}\Big)} > (\Psi - \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, we get from above that

$$\lambda^{(p,q)}_{g_1 \cdot g_2}(f_1) \geq \Psi = \min\{\lambda^{(p,q)}_{g_1}(f_1), \lambda^{(p,q)}_{g_2}(f_1)\}. \qquad (3.17)$$

Now let $g_1(s)$ and $g_2(s)$ are satisfy Property (X), then of course we have $M_{g_1 \cdot g_2}(\sigma) > M_{g_1}(\sigma)$ and $M_{g_1 \cdot g_2}(\sigma) > M_{g_2}(\sigma)$ for all sufficiently large values of $\sigma$. Therefore for all sufficiently large values of $\sigma$ , we obtain that $M^{-1}_{g_1 \cdot g_2}(\sigma) < M^{1}_{g_1}(\sigma)$ and $M^{-1}_{g_1 \cdot g_2}(\sigma) < M^{-1}_{g_2}(\sigma)$. So $\lambda^{(p,q)}_{g_1 \cdot g_2}(f_1) \leq \lambda^{(p,q)}_{g_1}(f_1)$ and $\lambda^{(p,q)}_{g_1 \cdot g_2}(f_1) \leq \lambda^{(p,q)}_{g_2}(f_1)$. Hence the theorem follows.

**Theorem 13.** *Let us consider $f_1(s)$ be any analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any two entire functions represented by Dirichlet series. Also let the relative index-pair of $f_1(s)$ with respect to $g_1(s)$ and $g_2(s)$ is $(p, q)$. Also let $f_1(s)$ is of regular relative $(p, q)$ growth with respect to at least any one of $g_1(s)$ or $g_2(s)$. Then*

$$\rho^{(p,q)}_{g_1 \cdot g_2}(f_1) \geq \min\{\rho^{(p,q)}_{g_1}(f_1), \rho^{(p,q)}_{g_2}(f_1)\},$$

*when $g_1 \cdot g_2(s)$ satisfy the Property (H). The equality holds when $g_1(s)$ and $g_2(s)$ satisfy Property (X).*

We omit the proof of Theorem 13 as it can easily be carried out in the line of Theorem 12.

**Remark 2.** *Using the results of Theorem 10 to Theorem 13 and in view of the proofs of Theorem 19 and Theorem 20 of [5], one can easily deduce the similar conclusion of Theorem 19 and Theorem 20 of [5] under somewhat different conditions for any two analytic functions $f_1(s)$ and $f_2(s)$ represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and any two entire functions $g_1(s)$ and $g_2(s)$ represented by Dirichlet series.*

**Theorem 14.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any entire functions represented by Dirichlet series. Also let $\rho_{g_1}^{(p,q)}(f_1)$, $\rho_{g_1}^{(p,q)}(f_2)$, $\rho_{g_2}^{(p,q)}(f_1)$ and $\rho_{g_2}^{(p,q)}(f_2)$ are all non zero and finite.*
***(A)*** *If $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ for $i$ , $j = 1,2$ and $i \neq j$, then*

$$\Delta_{g_1}^{(p,q)}(f_1 \pm f_2) = \Delta$$

***(B)*** *If $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_j(s)$ for $i$ , $j = 1,2$ and $i \neq j$, then*

$$\Delta_{g_1 \pm g_2}^{(p,q)}(f_1) = \Delta_{g_i}^{(p,q)}(f_1) \text{ and } \overline{\Delta}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_i}^{(p,q)}(f_1) \mid i = 1,2.$$

***(C)*** *Assume the functions $f_1(s)$, $f_2(s)$, $g_1(s)$ and $g_2(s)$ satisfy the following conditions:*
*(i) $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_j(s)$ for $i = 1$, $2$, $j = 1,2$ and $i \neq j$;*
*(ii) $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ with at least $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_j(s)$ for $i = 1$, $2$, $j = 1,2$ and $i \neq j$;*
*(iii) $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ and $\rho_{g_2}^{(p,q)}(f_i) > \rho_{g_2}^{(p,q)}(f_j)$ holds simultaneously for $i = 1,2$; $j = 1,2$ and $i \neq j$;*
*(iv)$\rho_{g_m}^{(p,q)}(f_l) = \max[\min\{\rho_{g_1}^{(p,q)}(f_1), \rho_{g_2}^{(p,q)}(f_1)\}, \min\{\rho_{g_1}^{(p,q)}(f_2), \rho_{g_2}^{(p,q)}(f_2)\}] \mid l = m = 1,2$;*
*then we have*

$$\Delta_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \Delta_{g_m}^{(p,q)}(f_l) \mid l = m = 1,2$$

*and*

$$\overline{\Delta}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \overline{\Delta}_{g_m}^{(p,q)}(f_l) \mid l = m = 1,2.$$

**Proof.** From the definition of relative $(p,q)$-th type and relative $(p,q)$-th lower type of analytic function represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ with respect to an entire function represented by Dirichlet series, we have for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_{f_k}(\sigma) \le M_{g_l}\Big(\exp^{[p-1]}\Big((\Delta_{g_l}^{(p,q)}(f_k)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_l}^{(p,q)}(f_k)}\Big)\Big), \tag{3.18}$$

$$M_{f_k}(\sigma) \ge M_{g_l}\Big(\exp^{[p-1]}\Big((\overline{\Delta}_{g_l}^{(p,q)}(f_k)-\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_l}^{(p,q)}(f_k)}\Big)\Big) \tag{3.19}$$

$$i.e., \; M_{g_l}(\sigma) \le M_{f_k}\Bigg(\log\Bigg(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\overline{\Delta}_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\Big)}\Bigg)+A\Bigg), \tag{3.20}$$

where $k = 1,\ 2$ and $l = 1, 2$.

Also there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we obtain that

$$M_{f_k}(\sigma) \ge M_{g_l}\Big(\exp^{[p-1]}\Big((\Delta_{g_l}^{(p,q)}(f_k)-\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_l}^{(p,q)}(f_k)}\Big)\Big) \tag{3.21}$$

$$i.e., \; M_{g_l}(\sigma) \le M_{f_k}\Bigg(\log\Bigg(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\Delta_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\rho_{g_l}^{(p,q)}(f_k)}}\Big)}\Bigg)+A\Bigg), \tag{3.22}$$

and

$$M_{f_k}(\sigma) \le$$

$$M_{g_l}\Big(\exp^{[p-1]}\Big((\overline{\Delta}_{g_l}^{(p,q)}(f_k)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_l}^{(p,q)}(f_k)}\Big)\Big), \tag{3.23}$$

where $\varepsilon > 0$ is any arbitrary positive number, $k = 1,\ 2$ and $l = 1, 2$.

**Case I.** Suppose that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Hence for any arbitrary $\varepsilon > 0$, we get in view of (3.18) and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, that

$$M_{f_1 \pm f_2}(\sigma) \leq$$

$$M_{g_1}\Big(\exp^{[p-1]}\Big((\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_1}^{(p,q)}(f_1)}\Big)\Big)(1+A_1) \tag{3.24}$$

where $\varpi_1 = \dfrac{M_{g_1}\Big(\exp^{[p-1]}\Big((\Delta_{g_1}^{(p,q)}(f_2)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_1}^{(p,q)}(f_2)}\Big)\Big)}{M_{g_1}\Big(\exp^{[p-1]}\Big((\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_1}^{(p,q)}(f_1)}\Big)\Big)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we can make the term $\varpi_1$ sufficiently small. Hence for any $\alpha > 1$, it follows for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, Lemma 2 and (3.24) that

$$M_{f_1 \pm f_2}(\sigma) \leq$$

$$M_{g_1}\Big(\exp^{[p-1]}\Big((\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_1}^{(p,q)}(f_1)}\Big)\Big)(1+\varepsilon_1)$$

$$i.e.,\ M_{f_1 \pm f_2}(\sigma) \leq \qquad M_{g_1}\Big(\exp^{[p-1]}\Big(\alpha(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon$$

Therefore in view of Theorem 7 and $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we get from above for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$\frac{\log^{[p-1]} M_{g_1}^{-1} M_{f_1 \pm f_2}(\sigma)}{\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)}} \leq \alpha(\Delta_{g_1}^{(p,q)}(f_1)+\varepsilon).$$

Hence making $\alpha \to 1+$, we obtain from above for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$\limsup_{\sigma \to \infty} \frac{\log^{[p-1]} M_{g_1}^{-1} M_{f_1 \pm f_2}(\sigma)}{\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{g_1}^{(p,q)}(f_1 \pm f_2)}} \leq \Delta_{g_1}^{(p,q)}(f_1)$$

$$i.e.,\ \Delta_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta_{g_1}^{(p,q)}(f_1). \tag{3.25}$$

Now we may consider that $f(s) = f_1(s) \pm f_2(s)$. Since $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Then $\Delta_{g_1}^{(p,q)}(f) = \Delta_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \Delta_{g_1}^{(p,q)}(f_1)$. Further, let $f_1(s) = (f(s) \pm f_2(s))$. Therefore in view of Theorem 7 and $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we obtain that $\rho_{g_1}^{(p,q)}(f) > \rho_{g_1}^{(p,q)}(f_2)$ holds. Hence in view of ( 3.25) $\Delta_{g_1}^{(p,q)}(f_1) \leq \Delta_{g_1}^{(p,q)}(f) = \Delta_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\Delta_{g_1}^{(p,q)}(f) = \Delta_{g_1}^{(p,q)}(f_1) \Rightarrow \Delta_{g_1}^{(p,q)}(f_1 \pm f_2) = \Delta_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2$, then one can easily verify that $\Delta_{g_1}^{(p,q)}(f_1 \pm f_2) = \Delta_{g_1}^{(p,q)}(f_2)$.

**Case II.** Let us consider that $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold. Now there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we get from (3.18) and (3.23) and for any arbitrary $\varepsilon > 0$ that

$M_{f_1 \pm f_2}(\sigma_n) \leq$

$$M_{g_1}\left(\exp^{[p-1]}\left((\overline{\Delta}_{g_1}^{(p,q)}(f_1) + \varepsilon)\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma_n - A)}\right)\right)^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)(1+B) \tag{3.26}$$

where $\varpi_2 = \dfrac{M_{g_1}\left(\exp^{[p-1]}\left((\Delta_{g_1}^{(p,q)}(f_2)+\varepsilon)\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma_n - A)}\right)\right)^{\rho_{g_1}^{(p,q)}(f_2)}\right)\right)}{M_{g_1}\left(\exp^{[p-1]}\left((\overline{\Delta}_{g_1}^{(p,q)}(f_1)+\varepsilon)\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma_n - A)}\right)\right)^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$, we can make the term $\varpi_2$ sufficiently small by taking $n$ sufficiently large and therefore using the similar technique for as executed in the proof of Case I we get from (3.26) that $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_1)$ when $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_1}^{(p,q)}(f_2)$ hold.

Likewise, if we consider $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_1}^{(p,q)}(f_2)$, then one can easily verify that $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\Delta}_{g_1}^{(p,q)}(f_2)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

**Case III.** Let us consider that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_2(s)$. Therefore there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we obtain from (3.20) and (3.22) and for any arbitrary $\varepsilon > 0$ that

$$M_{g_1 \pm g_2}(\sigma_n) \leq M_{f_1}\left(\log\left(1 - \frac{1}{\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)}\right) + A\right)(1+C), \tag{3.27}$$

where $$\varpi_3 = \frac{M_{f_1}\left(\log\left(1-\frac{1}{\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{(\Delta_{g_2}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\right)}\right)+A\right)}{M_{f_1}\left(\log\left(1-\frac{1}{\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)}\right)+A\right)}.$$

Since $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we can make the term $\varpi_3$ sufficiently small by taking $n$ sufficiently large. Hence in view of Lemma 2 and Theorem 9, we get from (3.27) for a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$ that

$$M_{g_1\pm g_2}(\sigma_n) < M_{f_1}\left(\log\left(1-\frac{1}{\exp^{[q-1]}\left(\left(\frac{\log^{[p-1]}\sigma_n}{(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon)}\right)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\right)}\right)+A\right)$$

$$i.e.,\ M_{g_1\pm g_2}\left(\exp^{[p-1]}\left((\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon)\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma_n-A)}\right)\right)^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right) < M_{f_1}(\sigma_n)(1+\varepsilon_1)$$

$$i.e.,\ M_{g_1\pm g_2}\left(\frac{1}{\alpha}\exp^{[p-1]}\left((\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon)\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma_n-A)}\right)\right)^{\rho_{g_1}^{(p,q)}(f_1)}\right)\right) < M_{f_1}(\sigma_n),$$

where $\alpha > 1$. Hence, making $\alpha \to 1+$, we obtain from above for a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$ that

$$(\Delta_{g_1}^{(p,q)}(f_1)-\varepsilon)\left(\log^{[q-1]}\left(\frac{1}{1-\exp(\sigma_n-A)}\right)\right)^{\rho_{g_1\pm g_2}^{(p,q)}(f_1)} < \log^{[p-1]} M_{g_1\pm g_2}^{-1}(M_{f_1}(\sigma_n)).$$

Since $\varepsilon > 0$ is arbitrary, we get that

$$\Delta_{g_1\pm g_2}^{(p,q)}(f_1) \geq \Delta_{g_1}^{(p,q)}(f_1). \tag{3.28}$$

Now we may consider that $g(s) = g_1(s) \pm g_2(s)$. Also $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least $f_1(s)$ is of regular relative $(p, q)$ growth with respect to $g_2(s)$. Then $\Delta_{g}^{(p,q)}(f_1)$ $= \Delta_{g_1\pm g_2}^{(p,q)}(f_1) \geq \Delta_{g_1}^{(p,q)}(f_1)$. Further let $g_1(s) = (g(s) \pm g_2(s))$. Therefore in view of Theorem 9 and $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, we obtain that $\rho_{g}^{(p.q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ as at least $f_1(s)$ is of regular relative $(p, q)$ growth with respect to $g_2(s)$. Hence in view of (3.28), $\Delta_{g_1}^{(p,q)}(f_1) \geq \Delta_{g}^{(p,q)}(f_1) = \Delta_{g_1\pm g_2}^{(p,q)}(f_1)$. Therefore $\Delta_{g}^{(p,q)}(f_1) = \Delta_{g_1}^{(p,q)}(f_1) \Rightarrow$ $\Delta_{g_1\pm g_2}^{(p,q)}(f_1) = \Delta_{g_1}^{(p,q)}(f_1)$.

Similarly if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$, then $\Delta_{g_1\pm g_2}^{(p,q)}(f_1) = \Delta_{g_2}^{(p,q)}(f_1)$.

**Case IV.** In this case suppose that $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_2(s)$. Therefore for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we get from (3.20) that

$$M_{g_1\pm g_2}(\sigma) \leq M_{f_1}\Bigg(\log\Bigg(1-\frac{1}{\exp^{[q-1]}\Bigg(\Bigg(\frac{\log^{[p-1]}\sigma}{(\overline{\Delta}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Bigg)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Bigg)}\Bigg)+A\Bigg)(1+D), \quad (3.29)$$

where $\varpi_4 = \dfrac{M_{f_1}\Bigg(\log\Bigg(1-\frac{1}{\exp^{[q-1]}\Bigg(\Bigg(\frac{\log^{[p-1]}\sigma}{(\overline{\Delta}_{g_2}^{(p,q)}(f_1)-\varepsilon)}\Bigg)^{\frac{1}{\rho_{g_2}^{(p,q)}(f_1)}}\Bigg)}\Bigg)+A\Bigg)}{M_{f_1}\Bigg(\log\Bigg(1-\frac{1}{\exp^{[q-1]}\Bigg(\Bigg(\frac{\log^{[p-1]}\sigma}{(\overline{\Delta}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Bigg)^{\frac{1}{\rho_{g_1}^{(p,q)}(f_1)}}\Bigg)}\Bigg)+A\Bigg)}$ and in view of $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$, and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we can make the term $\varpi_4$ sufficiently small and therefore using the similar technique for as executed in the proof of Case III we get from (3.29) that $\overline{\Delta}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_1}^{(p,q)}(f_1)$ where $\rho_{g_1}^{(p,q)}(f_1) < \rho_{g_2}^{(p,q)}(f_1)$ and at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_2(s)$. Likewise if we consider $\rho_{g_1}^{(p,q)}(f_1) > \rho_{g_2}^{(p,q)}(f_1)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$, then $\overline{\Delta}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\Delta}_{g_2}^{(p,q)}(f_1)$. Thus combining Case III and Case IV, we obtain the second part of the theorem.

The third part of the theorem is a natural consequence of Remark 2 and the first part and second part of the theorem. Hence its proof is omitted.

**Theorem 15.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any entire functions represented by Dirichlet series. Also let $\lambda_{g_1}^{(p,q)}(f_1)$, $\lambda_{g_1}^{(p,q)}(f_2)$, $\lambda_{g_2}^{(p,q)}(f_1)$ and $\lambda_{g_2}^{(p,q)}(f_2)$ are all non zero and finite.*
***(A)*** *If $\lambda_{g_1}^{(p,q)}(f_i) > \lambda_{g_1}^{(p,q)}(f_j)$ with at least $f_j(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ for $i$ , $j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_i) \text{ and } \overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_i) \mid i = 1,2.$$

***(B)*** *If $\lambda_{g_i}^{(p,q)}(f_1) < \lambda_{g_j}^{(p,q)}(f_1)$ for $i$ , $j = 1,2$ and $i \neq j$, then*

$$\tau_{g_1\pm g_2}^{(p,q)}(f_1) = \tau_{g_i}^{(p,q)}(f_1) \text{ and } \overline{\tau}_{g_1\pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_i}^{(p,q)}(f_1) \mid i = 1,2.$$

***(C)*** *Assume the functions* $f_1(s)$, $f_2(s)$, $g_1(s)$ *and* $g_2(s)$ *satisfy the following conditions:*
*(i)* $\rho_{g_1}^{(p,q)}(f_i) > \rho_{g_1}^{(p,q)}(f_j)$ *with at least* $f_j(s)$ *is of regular relative* $(p, q)$ *growth with respect to* $g_1(s)$ *for* $i$ , $j = 1, 2$ *and* $i \neq j$*;*
*(ii)* $\rho_{g_2}^{(p,q)}(f_i) > \rho_{g_2}^{(p,q)}(f_j)$ *with at least* $f_j(s)$ *is of regular relative* $(p, q)$ *growth with respect to* $g_2(s)$ *for* $i$ , $j = 1, 2$ *and* $i \neq j$*;*
*(iii)* $\rho_{g_i}^{(p,q)}(f_1) < \rho_{g_j}^{(p,q)}(f_1)$ *and* $\rho_{g_i}^{(p,q)}(f_2) < \rho_{g_j}^{(p,q)}(f_2)$ *holds simultaneously for* $i$, $j = 1, 2$ *and* $i \neq j$*;*
*(iv)* $\lambda_{g_m}^{(p,q)}(f_l) = \min[\max\{\lambda_{g_1}^{(p,q)}(f_1), \lambda_{g_1}^{(p,q)}(f_2)\}, \max\{\lambda_{g_2}^{(p,q)}(f_1), \lambda_{g_2}^{(p,q)}(f_2)\}] \mid l = m = 1, 2$*;*
*then we have*

$$\tau_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \tau_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2$$

*and*

$$\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_m}^{(p,q)}(f_l) \mid l = m = 1, 2.$$

**Proof.** For any arbitrary positive number $\varepsilon(> 0)$, we have for $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_{f_k}(\sigma) \leq$$

$$M_{g_l}\Big(\exp^{[p-1]}\Big((\overline{\tau}_{g_l}^{(p,q)}(f_k) + \varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1 - \exp(\sigma - A)}\Big)\Big)^{\lambda_{g_l}^{(p,q)}(f_k)}\Big)\Big), \quad (3.30)$$

$$M_{f_k}(\sigma) \geq$$

$$M_{g_l}\Big(\exp^{[p-1]}\Big((\tau_{g_l}^{(p,q)}(f_k) - \varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1 - \exp(\sigma - A)}\Big)\Big)^{\lambda_{g_l}^{(p,q)}(f_k)}\Big)\Big) \quad (3.31)$$

$$\text{i.e., } M_{g_l}(\sigma) \leq M_{f_k}\Big(\log\Big(1 - \frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\tau_{g_l}^{(p,q)}(f_k) - \varepsilon)}\Big)^{\frac{1}{\lambda_{g_l}^{(p,q)}(f_k)}}\Big)}\Big) + A\Big), \quad (3.32)$$

and for a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$ we obtain that

$$M_{f_k}(\sigma) \geq$$

$$M_{g_l}\Big(\exp^{[p-1]}\Big((\overline{\tau}_{g_l}^{(p,q)}(f_k)-\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\lambda_{g_l}^{(p,q)}(f_k)}\Big)\Big) \tag{3.33}$$

$$i.e.,\ M_{g_l}(\sigma)\le M_{f_k}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\overline{\tau}_{g_l}^{(p,q)}(f_k)-\varepsilon)}\Big)^{\frac{1}{\lambda_{g_l}^{(p,q)}(f_k)}}\Big)}\Big)+A\Big), \tag{3.34}$$

and

$M_{f_k}(\sigma)\le$

$$M_{g_l}\Big(\exp^{[p-1]}\Big((\tau_{g_l}^{(p,q)}(f_k)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\lambda_{g_l}^{(p,q)}(f_k)}\Big)\Big), \tag{3.35}$$

where $k=1,2$ and $l=1,2$.

**Case I.** Let $\lambda_{g_1}^{(p,q)}(f_1)>\lambda_{g_1}^{(p,q)}(f_2)$ with at least $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$. Now there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we get from (3.30) and (3.35) and for any arbitrary $\varepsilon>0$ that

$M_{f_1\pm f_2}(\sigma_n)\le$

$$M_{g_1}\Big(\exp^{[p-1]}\Big((\tau_{g_1}^{(p,q)}(f_1)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma_n-A)}\Big)\Big)^{\lambda_{g_1}^{(p,q)}(f_1)}\Big)\Big)(1+E), \tag{3.36}$$

where $\varpi_5=\dfrac{M_{g_1}\Big(\exp^{[p-1]}\Big((\overline{\tau}_{g_1}^{(p,q)}(f_2)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma_n-A)}\Big)\Big)^{\lambda_{g_1}^{(p,q)}(f_2)}\Big)\Big)}{M_{g_1}\Big(\exp^{[p-1]}\Big((\tau_{g_1}^{(p,q)}(f_1)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma_n-A)}\Big)\Big)^{\lambda_{g_1}^{(p,q)}(f_1)}\Big)\Big)}$ and in view of $\lambda_{g_1}^{(p,q)}(f_1)>\lambda_{g_1}^{(p,q)}(f_2)$, we can make the term $\varpi_5$ sufficiently small by taking $n$ sufficiently large. Therefore with the help of Lemma 2, Theorem 6 and using the similar technique of Case I of Theorem 14, we get from (3.36) that

$$\tau_{g_1}^{(p,q)}(f_1\pm f_2)\le\tau_{g_1}^{(p,q)}(f_1)\ . \tag{3.37}$$

Further, we may consider that $f(s) = f_1(s) \pm f_2(s)$. Also suppose that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$. Then $\tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2) \leq \tau_{g_1}^{(p,q)}(f_1)$. Now let $f_1(s) = (f(s) \pm f_2(s))$. Therefore in view of Theorem 6, $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$, we obtain that $\lambda_{g_1}^{(p,q)}(f) > \lambda_{g_1}^{(p,q)}(f_2)$ holds. Hence in view of (3.37), $\tau_{g_1}^{(p,q)}(f_1) \leq \tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1 \pm f_2)$. Therefore $\tau_{g_1}^{(p,q)}(f) = \tau_{g_1}^{(p,q)}(f_1) \Rightarrow \tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_1)$.

Similarly, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ then one can easily verify that $\tau_{g_1}^{(p,q)}(f_1 \pm f_2) = \tau_{g_1}^{(p,q)}(f_2)$.

**Case II.** Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ with at least $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$. Also let $\varepsilon(>0)$ be arbitrary. Hence for any arbitrary $\varepsilon > 0$, we get in view of (3.30) and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, that

$$M_{f_1 \pm f_2}(\sigma) \leq$$

$$M_{g_1}\Big(\exp^{[p-1]}\Big((\overline{\tau}_{g_1}^{(p,q)}(f_1) + \varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma - A)}\Big)\Big)^{\lambda_{g_1}^{(p,q)}(f_1)}\Big)\Big)(1+F), \tag{3.38}$$

where $\varpi_6 = \dfrac{M_{g_1}\Big(\exp^{[p-1]}\Big((\overline{\tau}_{g_1}^{(p,q)}(f_2)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\lambda_{g_i}^{(p,q)}(f_2)}\Big)\Big)}{M_{g_1}\Big(\exp^{[p-1]}\Big((\overline{\tau}_{g_1}^{(p,q)}(f_1)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\lambda_{g_1}^{(p,q)}(f_1)}\Big)\Big)}$ and in view of $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$, and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we can make the term $\varpi_6$ sufficiently small and therefore for similar reasoning of Case I we get from (3.38) that $\overline{\tau}_{g_1}^{(p_i,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ when $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_1}^{(p,q)}(f_2)$ and at least $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$.

Likewise, if we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_1}^{(p,q)}(f_2)$ with at least $f_1(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ then one can easily verify that $\overline{\tau}_{g_1}^{(p,q)}(f_1 \pm f_2) = \overline{\tau}_{g_1}^{(p,q)}(f_2)$.

Thus combining Case I and Case II, we obtain the first part of the theorem.

**Case III.** Let us consider that $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$. Now for any arbitrary $\varepsilon > 0$, we get in view of (3.32) and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, that

$$M_{g_1 \pm g_2}(\sigma) \leq$$

$$M_{f_1}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\tau_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\tau_{g_1}^{(p,q)}(f_1)}}\Big)}\Big)+A\Big)(1+G), \tag{3.39}$$

where $\varpi_7 = \dfrac{M_{f_1}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\tau_{g_2}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\tau_{g_2}^{(p,q)}(f_1)}}\Big)}\Big)+A\Big)}{M_{f_1}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma}{(\tau_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\tau_{g_1}^{(p,q)}(f_1)}}\Big)}\Big)+A\Big)}$, and as $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$, we can make the term $\varpi_7$ sufficiently small. Now with the help of Lemma 2 and Theorem 8 and using the similar technique of Case III of Theorem 14, we get from (3.39) that

$$\tau_{g_1\pm g_2}^{(p,q)}(f_1) \geq \tau_{g_1}^{(p,q)}(f_1)\ . \tag{3.40}$$

Further, we may consider that $g(s) = g_1(s) \pm g_2(s)$. As $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$, so $\tau_g^{(p,q)}(f_1) = \tau_{g_1\pm g_2}^{(p,q)}(f_1) \geq \tau_{g_1}^{(p,q)}(f_1)$. Also let $g_1(s) = (g(s) \pm g_2(s))$. Therefore in view of Theorem 8 and $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ we obtain that $\lambda_g^{(p.q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$ holds. Hence in view of (3.40) $\tau_{g_1}^{(p,q)}(f_1) \geq \tau_g^{(p,q)}(f_1) = \tau_{g_1\pm g_2}^{(p,q)}(f_1)$. Therefore $\tau_g^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1) \Rightarrow \tau_{g_1\pm g_2}^{(p,q)}(f_1) = \tau_{g_1}^{(p,q)}(f_1)$.

Likewise, if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\tau_{g_1\pm g_2}^{(p,q)}(f_1) = \tau_{g_2}^{(p,q)}(f_1$.

**Case IV.** In this case further we consider $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$. Hence there exists a sequence $\{\sigma_n\}$ of values of $\sigma$ tending to $A$, for which we obtain from (3.32) and (3.34) and for any arbitrary $\varepsilon > 0$ that

$$M_{g_1\pm g_2}(\sigma_n) \leq$$

$$M_{f_1}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma_n}{(\overline{\tau}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\Big)}\Big)+A\Big)(1+H), \tag{3.41}$$

where $\varpi_8 = \dfrac{M_{f_1}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma_n}{(\tau_{g_2}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\lambda_{g_2}^{(p,q)}(f_1)}}\Big)}\Big)+A\Big)}{M_{f_1}\Big(\log\Big(1-\frac{1}{\exp^{[q-1]}\Big(\Big(\frac{\log^{[p-1]}\sigma_n}{(\overline{\tau}_{g_1}^{(p,q)}(f_1)-\varepsilon)}\Big)^{\frac{1}{\lambda_{g_1}^{(p,q)}(f_1)}}\Big)}\Big)+A\Big)}$, and in view of $\lambda_{g_1}^{(p,q)}(f_1) <$

$\lambda_{g_2}^{(p,q)}(f_1)$, we can make the term $\varpi_8$ sufficiently small by taking $n$ sufficiently large and therefore using the similar technique as executed in the proof of Case IV of Theorem 14, we get from (3.41) that $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_1}^{(p,q)}(f_1)$ when $\lambda_{g_1}^{(p,q)}(f_1) < \lambda_{g_2}^{(p,q)}(f_1)$.

Similarly, if we consider that $\lambda_{g_1}^{(p,q)}(f_1) > \lambda_{g_2}^{(p,q)}(f_1)$, then one can easily verify that $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) = \overline{\tau}_{g_2}^{(p,q)}(f_1)$.

Thus combining Case III and Case IV, we obtain the second part of the theorem.

The proof of the third part of the Theorem is omitted as it can be carried out in view of Remark 2 and the above cases.

In the next four theorems we reconsider the equalities in Theorem 7 to Theorem 9 under somewhat different conditions.

**Theorem 16.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any entire functions represented by Dirichlet series.*
***(A)*** *The following condition is assumed to be satisfied:*
*(i) Either $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_1}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_1}^{(p,q)}(f_2)$ holds, then*

$$\rho_{g_1}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2).$$

***(B)*** *The following conditions are assumed to be satisfied:*
*(i) Either $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_2}^{(p,q)}(f_1)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_1)$ holds;*
*(ii) $f_1(s)$ is of regular relative $(p,q)$ growth with respect to at least any one of $g_1(s)$ or $g_2(s)$, then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_1).$$

**Theorem 17.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any entire functions represented by Dirichlet series. Also let $p$ and $q \in \mathbb{N}$.*
***(A)*** *The following conditions are assumed to be satisfied:*
*(i) $(f_1(s) \pm f_2(s))$ is of regular relative $(p,q)$ growth with respect to at least any one of $g_1(s)$ or $g_2(s)$;*
*(ii) Either $\Delta_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \Delta_{g_2}^{(p,q)}(f_1 \pm f_2)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1 \pm f_2) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_1 \pm f_2)$;*
*(iii) Either $\Delta_{g_1}^{(p,q)}(f_1) \neq \Delta_{g_1}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_1}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_1}^{(p,q)}(f_2)$;*
*(iv) Either $\Delta_{g_2}^{(p,q)}(f_1) \neq \Delta_{g_2}^{(p,q)}(f_2)$ or $\overline{\Delta}_{g_2}^{(p,q)}(f_1) \neq \overline{\Delta}_{g_2}^{(p,q)}(f_2)$; then*

$$\rho_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \rho_{g_1}^{(p,q)}(f_1) = \rho_{g_1}^{(p,q)}(f_2) = \rho_{g_2}^{(p,q)}(f_1) = \rho_{g_2}^{(p,q)}(f_2).$$

***(B)*** *The following conditions are assumed to be satisfied:*
*(i) $f_1(s)$ and $f_2(s)$ are of regular relative $(p,q)$ growth with respect to at least any*

*one of $g_1(s)$ or $g_2(s)$;*
*(ii) Either $\Delta^{(p,q)}_{g_1\pm g_2}(f_1) \neq \Delta^{(p,q)}_{g_1\pm g_2}(f_2)$ or $\overline{\Delta}^{(p,q)}_{g_1\pm g_2}(f_1) \neq \overline{\Delta}^{(p,q)}_{g_1\pm g_2}(f_2)$;*
*(iii) Either $\Delta^{(p,q)}_{g_1}(f_1) \neq \Delta^{(p,q)}_{g_2}(f_1)$ or $\overline{\Delta}^{(p,q)}_{g_1}(f_1) \neq \overline{\Delta}^{(p,q)}_{g_2}(f_1)$;*
*(iv) Either $\Delta^{(p,q)}_{g_1}(f_2) \neq \Delta^{(p,q)}_{g_2}(f_2)$ or $\overline{\Delta}^{(p,q)}_{g_1}(f_2) \neq \overline{\Delta}^{(p,q)}_{g_2}(f_2)$; then*

$$\rho^{(p,q)}_{g_1\pm g_2}(f_1 \pm f_2) = \rho^{(p,q)}_{g_1}(f_1) = \rho^{(p,q)}_{g_1}(f_2) = \rho^{(p,q)}_{g_2}(f_1) = \rho^{(p,q)}_{g_2}(f_2).$$

**Theorem 18.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any entire functions represented by Dirichlet series.*
***(A)*** *The following conditions are assumed to be satisfied:*
*(i) At least any one of $f_1(s)$ or $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ where $p$ and $q \in \mathbb{N}$;*
*(ii) Either $\tau^{(p,q)}_{g_1}(f_1) \neq \tau^{(p,q)}_{g_1}(f_2)$ or $\overline{\tau}^{(p,q)}_{g_1}(f_1) \neq \overline{\tau}^{(p,q)}_{g_1}(f_2)$ holds, then*

$$\lambda^{(p,q)}_{g_1}(f_1 \pm f_2) = \lambda^{(p,q)}_{g_1}(f_1) = \lambda^{(p,q)}_{g_1}(f_2).$$

***(B)*** *The following conditions are assumed to be satisfied:*
*(i) $f_1(s)$, $g_1(s)$ and $g_2(s)$ be any three entire functions such that $\lambda^{(p,q)}_{g_1}(f_1)$ and $\lambda^{(p,q)}_{g_2}(f_1)$ exists where $p$ and $q \in \mathbb{N}$;*
*(ii) Either $\tau^{(p,q)}_{g_1}(f_1) \neq \tau^{(p,q)}_{g_2}(f_1)$ or $\overline{\tau}^{(p,q)}_{g_1}(f_1) \neq \overline{\tau}^{(p,q)}_{g_2}(f_1)$ holds, then*

$$\lambda^{(p,q)}_{g_1\pm g_2}(f_1) = \lambda^{(p,q)}_{g_1}(f_1) = \lambda^{(p,q)}_{g_2}(f_1).$$

**Theorem 19.** *Let us consider $f_1(s)$, $f_2(s)$ be any two analytic functions represented by Dirichlet series in the half plane $\sigma < A$ $(-\infty < A < +\infty)$ and $g_1(s)$, $g_2(s)$ be any entire functions represented by Dirichlet series.*
***(A)*** *The following conditions are assumed to be satisfied:*
*(i) At least any one of $f_1(s)$ or $f_2(s)$ is of regular relative $(p,q)$ growth with respect to $g_1(s)$ and $g_2(s)$ where $p$ and $q \in \mathbb{N}$*
*(iii) Either $\tau^{(p,q)}_{g_1}(f_1) \neq \tau^{(p,q)}_{g_1}(f_2)$ or $\overline{\tau}^{(p,q)}_{g_1}(f_1) \neq \overline{\tau}^{(p,q)}_{g_1}(f_2)$;*
*(iv) Either $\tau^{(p,q)}_{g_2}(f_1) \neq \tau^{(p,q)}_{g_2}(f_2)$ or $\overline{\tau}^{(p,q)}_{g_2}(f_1) \neq \overline{\tau}^{(p,q)}_{g_2}(f_2)$; then*

$$\lambda^{(p,q)}_{g_1\pm g_2}(f_1 \pm f_2) = \lambda^{(p,q)}_{g_1}(f_1) = \lambda^{(p,q)}_{g_1}(f_2) = \lambda^{(p,q)}_{g_2}(f_1) = \lambda^{(p,q)}_{g_2}(f_2).$$

***(B)*** *The following conditions are assumed to be satisfied:*
*(i) At least any one of $f_1(s)$ or $f_2(s)$ are of regular relative $(p,q)$ growth with respect to $g_1(s) \pm g_2(s)$ where $p$ and $q \in \mathbb{N}$;*

*(ii) Either* $\tau_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \tau_{g_1 \pm g_2}^{(p,q)}(f_2)$ *or* $\overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_1) \neq \overline{\tau}_{g_1 \pm g_2}^{(p,q)}(f_2)$ *holds;*
*(iii) Either* $\tau_{g_1}^{(p,q)}(f_1) \neq \tau_{g_2}^{(p,q)}(f_1)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_1) \neq \overline{\tau}_{g_2}^{(p,q)}(f_1)$ *holds;*
*(iv) Either* $\tau_{g_1}^{(p,q)}(f_2) \neq \tau_{g_2}^{(p,q)}(f_2)$ *or* $\overline{\tau}_{g_1}^{(p,q)}(f_2) \neq \overline{\tau}_{g_2}^{(p,q)}(f_2)$ *holds, then*

$$\lambda_{g_1 \pm g_2}^{(p,q)}(f_1 \pm f_2) = \lambda_{g_1}^{(p,q)}(f_1) = \lambda_{g_1}^{(p,q)}(f_2) = \lambda_{g_2}^{(p,q)}(f_1) = \lambda_{g_2}^{(p,q)}(f_2).$$

Theorem 16, Theorem 17, Theorem 18 and Theorem 19 one can be prove using the similar arguments adopted in the proofs of Theorem 23, Theorem 24, Theorem 25 and Theorem 26 of [5] respectively. We omit the details.

**Remark 3.** *Using the results of Theorem 10 to Theorem 13 and in view of the proofs of Theorem 27 and Theorem 28 of [5], one can easily deduce the similar conclusion of Theorem 27 and Theorem 28 of [5] under somewhat different conditions for any two analytic functions* $f_1(s)$ *and* $f_2(s)$ *represented by Dirichlet series in the half plane* $\sigma < A$ $(-\infty < A < +\infty)$ *and any two entire functions* $g_1(s)$ *and* $g_2(s)$ *represented by Dirichlet series.*

**Theorem 20.** *Let* $f(s)$ *be any analytic function represented by Dirichlet series in the half plane* $\sigma < A$ $(-\infty < A < +\infty)$, $g(s)$ *be an entire function represented by Dirichlet series and* $P$ *be a Dirichlet polynomial. Then* $\rho_g^{(p,q)}(f) = \rho_{Pg}^{(p,q)}(f)$ *and* $\lambda_g^{(p,q)}(f) = \lambda_{Pg}^{(p,q)}(f)$.

**Proof.** In view of the first part of Lemma 4, it follows for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_f(\sigma) \leq M_g\Big(\exp^{[p]}\left((\rho_g^{(p,q)}(f) + \varepsilon)\log^{[q]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)\Big)$$

$$i.e.,\ M_f(\sigma) \leq M_{Pg}\Big(\frac{1}{\alpha}\exp^{[p]}\left((\rho_g^{(p,q)}(f) + \varepsilon)\log^{[q]}\left(\frac{1}{1-\exp(\sigma - A)}\right)\right)\Big)$$

$$i.e.,\ \frac{\log^{[p]} M_{Pg}^{-1}(M_f(\sigma)) + O(1)}{\log^{[q]}\left(\frac{1}{1-\exp(\sigma-A)}\right)} \leq (\rho_g^{(p,q)}(f) + \varepsilon).$$

Hence we get from above that $\rho_{Pg}^{(p,q)}(f) \leq \rho_g^{(p,q)}(f)$. The reverse inequality can also be carried out using the second part of Lemma 4 and therefore $\rho_g^{(p,q)}(f) = \rho_{Pg}^{(p,q)}(f)$. In a similar manner, $\lambda_g^{(p,q)}(f) = \lambda_{Pg}^{(p,q)}(f)$. Hence the theorem follows.

**Theorem 21.** *Let* $f(s)$ *be any analytic function represented by Dirichlet series in the half plane* $\sigma < A$ $(-\infty < A < +\infty)$, $g(s)$ *be an entire function represented* *by Dirichlet series and $P$ be a Dirichlet polynomial. Then* $\Delta_g^{(p,q)}(f) = \Delta_{Pg}^{(p,q)}(f)$, $\overline{\Delta}_g^{(p,q)}(f) = \overline{\Delta}_{Pg}^{(p,q)}(f)$, $\tau_g^{(p,q)}(f) = \tau_{Pg}^{(p,q)}(f)$ *and* $\overline{\tau}_g^{(p,q)}(f) = \overline{\tau}_{Pg}^{(p,q)}(f)$.

**Proof.** In view of Theorem 20 and the first part of Lemma 4, we get for any arbitrary $\varepsilon > 0$ and $\sigma > \sigma_1(\varepsilon)$ sufficiently close to $A$ that

$$M_f(\sigma) \le M_g\Big(\exp^{[p-1]}\Big((\Delta_g^{(p,q)}(f)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_g^{(p,q)}(f)}\Big)\Big)$$

$$i.e.,\ M_f(\sigma) \le M_{Pg}\Big(\frac{1}{\alpha}\exp^{[p-1]}\Big((\Delta_g^{(p,q)}(f)+\varepsilon)\Big(\log^{[q-1]}\Big(\frac{1}{1-\exp(\sigma-A)}\Big)\Big)^{\rho_{Pg}^{(p,q)}(f)}\Big)\Big).$$

Now letting $\alpha \to 1^-$, we get from above that $\Delta_{Pg}^{(p,q)}(f) \le \Delta_g^{(p,q)}(f)$. The reverse inequality can also be carried out using the second part of Lemma 4 and therefore $\Delta_g^{(p,q)}(f) = \Delta_{Pg}^{(p,q)}(f)$. In a similar manner, $\overline{\Delta}_g^{(p,q)}(f) = \overline{\Delta}_{Pg}^{(p,q)}(f)$.

Since in view of Theorem 20 $\lambda_g^{(p,q)}(f) = \lambda_{Pg}^{(p,q)}(f)$, therefore using the same technique of above one can easily verify that $\tau_g^{(p,q)}(f) = \tau_{Pg}^{(p,q)}(f)$ and $\overline{\tau}_g^{(p,q)}(f) = \overline{\tau}_{Pg}^{(p,q)}(f)$. Hence the theorem follows.

## 4. Conclusion

The main aim of the paper is to extend and modify the notion of order and type (respectively weak type) to relative order and relative type (respectively relative weak type) of higher dimensions in case of analytic functions represented by Dirichlet series in the half plane and in this connection we have established some theorems. Moreover the notion of relative order, relative type and relative weak type of higher dimensions may also be developed using the concepts [7, 22, 26] of analytic functions represented by vector valued Dirichlet series of in the half plane.

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