J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 9, No. 2 (2022), pp. 01-10

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

CUBIC LEVEL ANALOGUE OF RAMANUJAN'S EISENSTEIN SERIES IDENTITIES

Vasuki K. R. and Darshan A.*

Department of Studies in Mathematics,
University of Mysore, Manasagangotri, Mysuru - 570 006, INDIA

E-mail : vasuki_kr@hotmail.com

*Sri. H. D. Devegowda Government First Grade College,
Paduvalahippe, Holenarsipura Tq., Hassan - 573 211, INDIA

E-mail : darshmath@gmail.com

(**Received:** *Apr. 24, 2022* **Accepted:** *May 15, 2022* **Published:** *Jun. 30, 2022)*

Abstract: Let $Q_n = 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^{nk}}{1-q^{nk}}$. On page 51-53 of his lost notebook, Ramanujan recorded very interesting identities which relates Q_1, Q_5, Q_7 with his theta functions. In this article, we establish analogous identities with respect to Q_1 and Q_3.

Keywords and Phrases: Ramanujan's theta functions, Eisenstein series, P-Q theta function identities.

2020 Mathematics Subject Classification: 11F20, 11M36.

1. Introduction

For any complex number a and $q = e^{-\pi\sqrt{n}}$, where n is a positive integer. we define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n).$$

S. Ramanujan defined his theta function $f(a,b)$ by

$$\begin{aligned} f(a,b) &= \sum_{n=-\infty}^{\infty} a^{n(n+1)/2}b^{n(n-1)/2} \\ &= (-a;ab)_\infty(-b;ab)_\infty(ab,ab)_\infty, \quad |ab|<1. \end{aligned}$$

Ramanujan also defined some special cases of $f(a,b)$. For example

$$f(-q)=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{\frac{n(3n+1)}{2}}=(q;q)_\infty. \tag{1.1}$$

Ramanujan's Eisenstein series $Q(q)$ is defined by

$$Q(q)=1+240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}. \tag{1.2}$$

Throughout this article for convenience, We set for any positive integer n,

$$Q_n=Q(q^n), \tag{1.3}$$

and

$$f_n=f(-q^n). \tag{1.4}$$

Ramanujan made many contributions to the theory and applications of Eisenstein series in his notebooks [10] and in his lost notebook [11]. In his lost notebook [11, p. 51-53], Ramanujan stated interesting identities which relates Q_1 and Q_n with products and quotients of f_1 and f_n for $n=5$ and 7. One such identity is

$$Q_5=\frac{f_1^{10}}{f_5^2}+10qf_1^4f_5^4+5q^2\frac{f_5^{10}}{f_1^2}.$$

These identities were studied by different Mathematicians. The first among them was in 1989, by S. Raghavan and S. S. Rangachari [9]. They applied the theory of modular forms and gave the proofs of all the identities. Later, B. C. Berndt, H. H. Chan, J. Sohn and S. H. Son [3] offered two different proofs, both employ Ramanujan's modular equations wherein the second method is more constructive than the first one. Z. G. Liu [8] proved the septic level identities using complex variable theory of elliptic functions, by constructing suitable elliptic integrals which leads to those identities. S. Cooper [5] established the proofs of quintic level identities using the parametrization $k=r(q)r^2(q^2)$, where $r(q)$ denotes Roger-Ramanujan continued fraction. A different proof was given by Cooper and P. C. Toh [7]. As opposed to verification, they in fact derived the quintic and septic level identities by constructing a system non-linear equations which turn out to have polynomial solutions.

The main purpose of this paper is to establish cubic level Eisenstein series identities, expressing Q_1 and Q_3 in terms of f_1 and f_3 in the spirit of Ramanujan which are analogous to quintic and septic level identities of Ramanujan. The method we are going to utilize here, is very elementary and uses only the P-Q theta function identities stated in the Section 2. At the end, we use the cubic level identities to reprove a result of Ramanujan found in his second notebook.

2. Preliminary Results

Let f_n be as defined in (1.4), we set

$$R = \frac{f_1}{q^{\frac{1}{12}} f_3}, \qquad S = \frac{f_2}{q^{\frac{1}{6}} f_6},$$

$$U = \frac{f_1}{q^{\frac{1}{24}} f_2}, \qquad V = \frac{f_3}{q^{\frac{1}{8}} f_6},$$

$$M = \frac{f_1}{q^{\frac{5}{24}} f_6}, \quad \text{and} \quad N = \frac{f_2}{q^{\frac{1}{24}} f_3}.$$

The following P-Q theta function identities are the important tools which is used to prove our main results.

$$(RS)^2 + \frac{9}{(RS)^2} = \left(\frac{R}{S}\right)^6 + \left(\frac{S}{R}\right)^6, \tag{2.1}$$

$$(MN)^2 - \frac{9}{(MN)^2} = \left(\frac{M}{N}\right)^3 - 8\left(\frac{N}{M}\right)^3, \tag{2.2}$$

and

$$(UV)^3 + \frac{8}{(UV)^3} = \left(\frac{V}{U}\right)^6 - \left(\frac{U}{V}\right)^6. \tag{2.3}$$

From the above and by the definitions of R, S, U, V, M and N, It follows that

$$(UV)^3 + \frac{8}{(UV)^3} = \left(\frac{S}{R}\right)^6 - \left(\frac{R}{S}\right)^6, \tag{2.4}$$

and

$$(MN)^2 - \frac{9}{(MN)^2} = (RS)^2 - \frac{9}{(RS)^2}. \tag{2.5}$$

Ramanujan recorded (2.1) and (2.2) in the un-organized portion of his notebook [10, p. 325-327] along with many other P-Q theta function identities. Berndt and L. C. Zhang proved most of the identities in [4]. The proofs of all these identities can be found in Chapter 25 of [2]. Ramanujan recorded a modular equation of degree 3 in Chapter 19 [10, Entry 5, xii, p. 230] which is equivalent to (2.3). The proof of (2.3) was given by Berndt [1, p. 231].

Let Q_n and f_n be as defined in (1.3) and (1.4) respectively, then we have

$$\frac{f_1^{16}}{f_2^8} = \frac{1}{15}(-Q_1 + 16Q_2), \tag{2.6}$$

and

$$q\frac{f_2^{16}}{f_1^8} = \frac{1}{240}(Q_1 - Q_2). \tag{2.7}$$

Using modular equations Berndt [1, p. 139] proved the identities (2.6) and (2.7). Cooper [6] gave a different proof of the same, using elliptic function identities. From (2.6) and (2.7) and using the definition of U, we deduce that

$$Q_1 = q^{\frac{1}{2}} f_1^4 f_2^4 \left(U^{12} + \frac{256}{U^{12}}\right). \tag{2.8}$$

By replacing q by q^3 in (2.8) and using the definition of V, we get

$$Q_3 = q^{\frac{3}{2}} f_3^4 f_6^4 \left(V^{12} + \frac{256}{V^{12}}\right). \tag{2.9}$$

3. Cubic Level Identities

Theorem 3.1. *If Q_n and f_n are as defined by (1.3) and (1.4) respectively, then*

$$Q_1 = \left(\frac{f_1^{10}}{f_3^2} + 3^5 q\frac{f_3^{10}}{f_1^2}\right)\left(\frac{f_1^6}{f_3^6} + 3^3 q\frac{f_3^6}{f_1^6}\right)^{\frac{1}{3}}. \tag{3.1}$$

Proof. With some algebraic manipulations, we see that (3.1) is equivalent to

$$Q_1 = q^{\frac{2}{3}} f_1^4 f_3^4 \left(R^6 + \frac{243}{R^6}\right)\left(R^6 + \frac{27}{R^6}\right)^{\frac{1}{3}}.$$

Comparing the above with (2.8), it suffices to prove the following:

$$N^4\left(U^{12} + \frac{256}{U^{12}}\right) = \left(R^6 + \frac{243}{R^6}\right)\left(R^6 + \frac{27}{R^6}\right)^{\frac{1}{3}}.$$

Or equivalently

$$N^{12}\left(U^{12} + \frac{256}{U^{12}}\right)^3 = \left(R^6 + \frac{243}{R^6}\right)^3\left(R^6 + \frac{27}{R^6}\right). \tag{3.2}$$

For convenience, set

$$(RS)^2 + \frac{9}{(RS)^2} = x. \tag{3.3}$$

This implies that

$$RS + \frac{3}{RS} = \sqrt{x+6}, \tag{3.4}$$

and

$$RS - \frac{3}{RS} = \sqrt{x-6}. \tag{3.5}$$

From [12, p. 55-56], we have at $q = e^{-\frac{2\pi}{\sqrt{3}}}$, $R = \sqrt{3}$ and $S = \frac{3+\sqrt{3}}{\sqrt{2}}$. Thus we have $RS - \frac{3}{RS}$ is positive in some neighborhood of $q = e^{-\frac{2\pi}{\sqrt{3}}}$. Hence we choose positive sign for (3.5). Now cubing (3.4) and (3.5) respectively, we obtain

$$(RS)^3 + \frac{27}{(RS)^3} = (x-3)\sqrt{x+6}, \tag{3.6}$$

and

$$(RS)^3 - \frac{27}{(RS)^3} = (x+3)\sqrt{x-6}. \tag{3.7}$$

Next from (2.1) and (3.3), it follows that

$$\left(\frac{R}{S}\right)^6 + \left(\frac{S}{R}\right)^6 = x. \tag{3.8}$$

Consequently, we find that

$$\left(\frac{R}{S}\right)^3 + \left(\frac{S}{R}\right)^3 = \sqrt{x+2}, \tag{3.9}$$

and

$$\left(\frac{S}{R}\right)^3 - \left(\frac{R}{S}\right)^3 = \sqrt{x-2}. \tag{3.10}$$

Using a similar argument used for $RS - \frac{3}{RS}$, we see that $\left(\frac{S}{R}\right)^3 - \left(\frac{R}{S}\right)^3$ is positive on some neighborhood of $q = e^{-\frac{2\pi}{\sqrt{3}}}$. Hence we choose positive sign for (3.10). Multiplying (3.6) and (3.9), we get

$$R^6 + \frac{27}{S^6} + S^6 + \frac{27}{R^6} = (x-3)\sqrt{x+2}\sqrt{x+6}. \tag{3.11}$$

Similarly multiplying (3.7) and (3.10), we obtain

$$-R^6+\frac{27}{S^6}+S^6-\frac{27}{R^6}=(x+3)\sqrt{x-2}\sqrt{x-6}. \tag{3.12}$$

Now by subtracting (3.12) by (3.11), we find that

$$R^6+\frac{27}{R^6}=\frac{1}{2}[(x-3)\sqrt{x+2}\sqrt{x+6}-(x+3)\sqrt{x-2}\sqrt{x-6}]. \tag{3.13}$$

Likewise multiplying (3.7) and (3.9), then multiplying (3.6) and (3.10) and using the resultant identities, we obtain

$$R^6-\frac{27}{R^6}=\frac{1}{2}[(x+3)\sqrt{x+2}\sqrt{x-6}-(x-3)\sqrt{x-2}\sqrt{x+6}]. \tag{3.14}$$

Subtracting 4 times of (3.14) by 5 times of (3.13), we get

$$\begin{aligned}R^6+\frac{243}{R^6}=\frac{1}{2}[&5(x-3)\sqrt{x+2}\sqrt{x+6}-5(x+3)\sqrt{x-2}\sqrt{x-6}\\&-4(x+3)\sqrt{x+2}\sqrt{x-6}+4(x-3)\sqrt{x-2}\sqrt{x+6}].\end{aligned} \tag{3.15}$$

From (2.4), (3.8) and (2.3), it follows that

$$(UV)^3+\frac{8}{(UV)^3}=\left(\frac{V}{U}\right)^6-\left(\frac{U}{V}\right)^6=\sqrt{x^2-4}. \tag{3.16}$$

Following the same procedure in deducing (3.13) and (3.14) from (3.3) and (3.8), From (3.16) we can establish

$$U^{12}+\frac{64}{U^{12}}=\frac{1}{2}[x(x^2-20)-(x^2-4)\sqrt{x^2-36}], \tag{3.17}$$

and

$$U^{12}-\frac{64}{U^{12}}=\frac{1}{2}[x\sqrt{x^2-4}\sqrt{x^2-36}-(x^2-20)\sqrt{x^2-4}]. \tag{3.18}$$

Subtracting $\frac{3}{2}$ times of (3.18) by $\frac{5}{2}$ times of (3.17), we get

$$\begin{aligned}U^{12}+\frac{256}{U^{12}}=\frac{1}{4}[&5x(x^2-20)-5(x^2-4)\sqrt{x^2-36}\\&-3x\sqrt{x^2-4}\sqrt{x^2-36}+3(x^2-20)\sqrt{x^2-4}].\end{aligned} \tag{3.19}$$

Again following the same procedure and using the identity (2.5), we can deduce that

$$N^{12} = \frac{1}{256}[(x^2-9)(x^2-20)\sqrt{x^2-36} - x(x^2-27)\sqrt{x^2-4}\sqrt{x^2-36} + x(x^2-20)(x^2-27) - (x^2-9)(x^2-36)\sqrt{x^2-4}]. \quad (3.20)$$

Finally multiplying the cube of (3.19) with (3.20), we obtain

$$\begin{aligned} & N^{12}\left(U^{12}+\frac{256}{U^{12}}\right)^3 \\ &= -\frac{1}{2}[-365x^8 + \sqrt{x+6}\sqrt{x-6}(364x^7 - 13832x^5 + 127404x^3 - 291600x) \\ &+ \sqrt{x+6}\sqrt{x-6}\sqrt{x-2}\sqrt{x+2}(365x^6 - 13140x^4 + 102195x^2 - 121500) \\ &+ \sqrt{x-2}\sqrt{x+2}(-364x^7 + 19656x^5 - 278100x^3 + 874800x) \\ &+ 20440x^6 - 319005x^4 + 1428840x^2 - 1458000]. \end{aligned} \quad (3.21)$$

And multiplying (3.13) with the cube of (3.15), we obtain

$$\begin{aligned} & \left(R^6+\frac{243}{R^6}\right)^3\left(R^6+\frac{27}{R^6}\right) \\ &= -\frac{1}{2}[-365x^8 + \sqrt{x+6}\sqrt{x-6}(364x^7 - 13832x^5 + 127404x^3 - 291600x) \\ &+ \sqrt{x+6}\sqrt{x-6}\sqrt{x-2}\sqrt{x+2}(365x^6 - 13140x^4 + 102195x^2 - 121500) \\ &+ \sqrt{x-2}\sqrt{x+2}(-364x^7 + 19656x^5 - 278100x^3 + 874800x) \\ &+ 20440x^6 - 319005x^4 + 1428840x^2 - 1458000]. \end{aligned} \quad (3.22)$$

From (3.21) and (3.22), we obtain the required result.

Theorem 3.2. *If Q_n and f_n are as defined by (1.3) and (1.4) respectively, then the following identity holds*

$$Q_3 = \left(\frac{f_1^{10}}{f_3^2} + 3q\frac{f_3^{10}}{f_1^2}\right)\left(\frac{f_1^6}{f_3^6} + 3^3 q\frac{f_3^6}{f_1^6}\right)^{\frac{1}{3}}. \quad (3.23)$$

Proof. We see that (3.23) is equivalent to

$$Q_3 = q^{\frac{2}{3}} f_1^4 f_3^4 \left(R^6 + \frac{3}{R^6}\right)\left(R^6 + \frac{27}{R^6}\right)^{\frac{1}{3}}. \quad (3.24)$$

Comprising (2.9) and (3.24), To prove (3.23), it is enough to prove the following:

$$\left(V^{12}+\frac{256}{V^{12}}\right)^3=M^{12}\left(R^6+\frac{3}{R^6}\right)^3\left(R^6+\frac{27}{R^6}\right). \tag{3.25}$$

Adding $\frac{5}{9}$ times of (3.13) and $\frac{4}{9}$ times of (3.14), we obtain

$$\begin{aligned}R^6+\frac{3}{R^6}=\frac{1}{18}[&5(x-3)\sqrt{x+2}\sqrt{x+6}-5(x+3)\sqrt{x-2}\sqrt{x-6}\\&+4(x+3)\sqrt{x+2}\sqrt{x-6}-4(x-3)\sqrt{x-2}\sqrt{x+6}].\end{aligned} \tag{3.26}$$

Following the same procedure in deducing (3.13) and (3.14) from (3.3) and (3.8), we can establish

$$V^{12}+\frac{64}{V^{12}}=\frac{1}{2}[x(x^2-20)+(x^2-4)\sqrt{x^2-36}], \tag{3.27}$$

and

$$V^{12}-\frac{64}{V^{12}}=\frac{1}{2}[x\sqrt{x^2-4}\sqrt{x^2-36}+(x^2-20)\sqrt{x^2-4}]. \tag{3.28}$$

Now subtracting $\frac{5}{2}$ times of (3.28) from $\frac{3}{2}$ times of (3.27), we can establish

$$\begin{aligned}V^{12}+\frac{256}{V^{12}}=\frac{1}{4}[&5x(x^2-20)+5(x^2-4)\sqrt{x^2-36}\\&-3x\sqrt{x^2-4}\sqrt{x^2-36}-3(x^2-20)\sqrt{x^2-4}].\end{aligned} \tag{3.29}$$

Again following a similar steps and using the identity (2.5), we can deduce that

$$\begin{aligned}M^{12}=\frac{1}{4}[&(x^2-9)(x^2-20)\sqrt{x^2-36}+x(x^2-27)\sqrt{x^2-4}\sqrt{x^2-36}\\&+x(x^2-20)(x^2-27)+(x^2-9)(x^2-36)\sqrt{x^2-4}].\end{aligned} \tag{3.30}$$

Finally cubing (3.29), we get

$$\begin{aligned}&\left(V^{12}+\frac{256}{V^{12}}\right)^3\\&=-\frac{1}{4}[-65x^9+\sqrt{x-2}\sqrt{x+2}(63x^8-3843x^6+67284x^4-323280x^2+216000)\\&\quad+\sqrt{x+6}(\sqrt{x-6}(-65x^8+2925x^6-35100x^4+115760x^2-72000)\\&\quad+\sqrt{x-6}\sqrt{x-2}\sqrt{x+2}(63x^7-2709x^5+28728x^3-61200x))\\&\quad+4095x^7-77220x^5+463120x^3-734400x].\end{aligned} \tag{3.31}$$

And multiplying the cube of (3.26) with (3.13) and (3.30), we obtain that

$$\begin{aligned}
&M^{12}\left(R^6+\frac{3}{R^6}\right)^3\left(R^6+\frac{27}{R^6}\right)\\
=&-\frac{1}{4}[-65x^9+\sqrt{x-2}\sqrt{x+2}(63x^8-3843x^6+67284x^4-323280x^2+216000)\\
&+\sqrt{x+6}(\sqrt{x-6}(-65x^8+2925x^6-35100x^4+115760x^2-72000)\\
&+\sqrt{x-6}\sqrt{x-2}\sqrt{x+2}(63x^7-2709x^5+28728x^3-61200x))\\
&+4095x^7-77220x^5+463120x^3-734400x].
\end{aligned} \tag{3.32}$$

From (3.31) and (3.32), we get the desired result.

From (3.1) and (3.23), one can see that

$$\left\{1+24\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}+216\sum_{k=1}^{\infty}\frac{k^3q^{3k}}{1-q^{3k}}\right\}^{\frac{1}{2}}=\left\{\frac{f_1^{12}+27qf_3^{12}}{f_1^3f_3^3}\right\}^{\frac{2}{3}}.$$

This identity was recorded by Ramanujan in Chapter 21 of his second notebook [10, Entry 3, i, p. 253]. A different proof of the same, was given by Berndt in [1, p. 460-463].

References

[1] Berndt B. C., Ramanujans Notebooks, Part III, Springer Science & Business Media, 1991.

[2] Berndt B. C., Ramanujans Notebooks, Part IV, Springer Science & Business Media, 1994.

[3] Berndt B. C., Chan H. H., Sohn J. and Son S. H., Eisenstein series in Ramanujan's Lost Notebook, The Ramanujan Journal, 4(1) (2000), 81-114.

[4] Berndt B. C. and Zhang L. C., Ramanujan's Identities for Eta-functions, Mathematische Annalen, 292(1) (1992), 561-573.

[5] Cooper S., On Ramanujans Function $k(q) = r(q)r^2(q^2)$, The Ramanujan Journal, 20(3) (2009), 311-328.

[6] Cooper S. Ramanujan's Theta Functions, Springer, 2017.

[7] Cooper S. and Toh P. C., Quintic and Septic Eisenstein Series, The Ramanujan Journal, 19(2) (2009), 163-181.

[8] Liu Z. G., Some Eisenstein Series Identities related to Modular Equations of the seventh order, Pacific Journal of Mathematics, 209(1) (2003), 103-130.

[9] Raghavan S. and Rangachari S. S., On Ramanujan's Elliptic Integrals and Modular Identities, Number theory and related topics, (1989), 119-149.

[10] Ramanujan S., Notebooks, Volume I & II. Tata Institute of Fundamental Research, Bombay, 1957.

[11] Ramanujan S., The lost notebook and other unpublished papers, Narosa, New Delhi, 1988.

[12] Vasuki K. R., Some Studies in the Theory of Special Functions and Number Theory, PhD thesis, University of Mysore, 2000.