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ON SOME PROPERTIES OF BINARY SCHUBERT CODE AND EXTENDED BINARY SCHUBERT CODE

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Abstract: Linear error correcting codes associated to higher dimensional algebraic varieties defined over finite fields have been topical interest. For example codes associated to Hermitian varieties, Grassman varieties, Schubert varieties and Flag varieties have been studied quite extensively. The codes associated to these types of varieties is the central interest. Codes associated with Schubert varieties in $G(2,4)$ over $\mathbb{F}_2$ have been studied in [16]. In this paper we have defined extended binary Schubert Code of the length 19, binary Schubert code of the length 18 and some properties corresponding to these codes.

Keywords and Phrases: Linear Codes, binary Schubert Code, Extended binary Schubert Code.

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1. Introduction

Let q be fixed prime power and l, m with $l \leq m$ are positive integers. Let $\mathbb{F}_q$ be the field with q elements and $\mathbb{F}_q^m$ be a m-dimensional linear space over $\mathbb{F}_q$. Let Gaussian binomial coefficient be given by $\begin{bmatrix} m \\ l \end{bmatrix}_q$ and $G(\ell, m)$ denotes the Grassmannian of all ℓ-planes of $\mathbb{F}_q^m$. Due to Plücker mapping the Grassmannian

$G(\ell,m)$ can be embedded into projective space $\mathbb{P}^{\binom{m}{l}-1}$. The image of $G(\ell,m)$ under this is a projective algebraic variety, due to [15] every subset of a projective space can associate a linear code. Hence linear code corresponding to Grassmannian $G(\ell,m)$ is denoted by$C(\ell,m)$. These Grassman Codes were introduced by C. T. Ryan in the series of papers [14, 13] and field used was over $\mathbb{F}_2$. Later D Yu Nogin [10] studied the linear code $C(l,m)$ associated to $G(\ell,m)$ over any finite field and proved that $C(l,m)$ is an q-ary $[n,k,d]$ linear code, where $n = \begin{bmatrix} m \\ l \end{bmatrix}_q$, $k = \binom{m}{l}$ and $d = q^{l(m-l)}$.

Let $\alpha = (\alpha_1, \alpha_2, . . . \alpha_l)$ be increasing sequence of positive integers satisfying the relations $1 \leq \alpha_1 \leq \alpha_2 \leq . . . . \leq \alpha_l \leq m$ and denote $\Omega_\alpha(\ell,m)$ be the Schubert varieties [16] in the Grassmannian $G(\ell,m)$. Note that Schubert varieties are sub-varieties of Grassmannian. As we know that the Grassmannian can be embeddable in Projective Space $\mathbb{P}^{\binom{m}{l}-1}$ via Plücker map. Therefore Plücker map also embeds Schubert Varieties in $\mathbb{P}^{\binom{m}{l}-1}$. The Linear code corresponding the Schubert variety $\Omega_\alpha(\ell,m)$ is called Schubert code and it is denoted by $C_\alpha(l,m)$. Ghorpade- Lachaud [2] initiated the study of Schubert Code and conjectured the minimum distance $d_\alpha = q^{\delta(\alpha)}$ where $\delta(\alpha) = \sum_{i=1}^{l}(\alpha_i - 1)$. Many attempts [1, 4, 5] have been done to settle this conjecture and it was settled in some special cases. The MDC (Minimum Distance Conjecture) first proved by Chen [1] and Guerra-Vincenti [5] for $l = 2$ Gorpade-Tsfasmann [4] proved that the Schubert-Code $C_\alpha(l,m)$ is q-ary $[n_\alpha, k_\alpha]$ code where $n_\alpha = \sum_{\beta\leq\alpha} q^{\delta(\beta)}$, the sum is over al l-touples $\beta = (\beta_1, \beta_2, . . . \beta_l)$ integers satisfying $1 \leq \beta_1 \leq \beta_2 \leq . . . \leq \beta_l \leq m$, $\beta_i \leq \alpha_i$ for $i = 1, 2, . . . l$ with $\delta(\beta) = \sum_{i=1}^{l}(\beta_i - i)$

and the dimension k_α=$\begin{vmatrix} \binom{\alpha_1}{1} & 1 & 0 & \dots & 0 \\ \binom{\alpha_2}{1} & \binom{\alpha_2-1}{1} & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ \binom{\alpha_l}{l} & \binom{\alpha_2-1}{l-1} & \binom{\alpha_3-2}{l-2} & \dots & \binom{\alpha_l-l+1}{1} \end{vmatrix}$

Xiang in [17] proved the MDC. An alternative Proof is given in [3]. The codes associated $G(2,5)$ over F_2 have been studied in [16]. In this paper we have defined extended binary Schubert code which will be denoted by $\bar{\Omega}_{19}$ and binary Schubert Code $\Omega_{(2,4)}$ along with some properties of these Codes.

2. Preliminaries

2.1. Basic definitions

Let $\mathbb{F}_q$ be the finite field with q elements, $q = p^h$, p a prime and denote by $\mathbb{F}_q^n$ the n-dimensional vector space over $\mathbb{F}_q$. For any $x \in \mathbb{F}_q^n$, the *support* of x, $supp(x)$, is the set of nonzero coordinates in $x = (x_1, x_2, \cdots, x_n)$. The *support weight* (or *Hamming norm*) of x is defined by,

$$||x|| = |supp(x)|.$$

More generally, if D is a subspace of $\mathbb{F}_q^n$, the *support* of D, $Supp(D)$ is the set of positions where not all the vectors in D are zero and the *support weight* (or *Hamming norm*) of D is defined by,

$$||D|| = |supp(D)|.$$

A *linear* $[n,k]_q$*-code* is a k-dimensional subspace of $\mathbb{F}_q^n$. The parameters n and k are referred to as the *length* and the *dimension* of the corresponding code. The *minimum distance* $d = d(C)$ of C is defined by

$$d = d(C) = \min\{||x|| : x \in C, x \neq 0\}$$

More generally, given any positive integer r, the *rth higher weight* $d_r = d_r(C)$ is defined by

$$d_r = d_r(C) = \min\{||D|| : D \text{ is a subspace of } C \text{ with } dimD = r\}.$$

Note that $d_1(C) = d(C)$. It also follows that $d_i \leq d_j$ when $i < j$ and that $d_k = |supp(C)|$, where k is the dimension of code C. Thus we have $1 \leq d_1 < d_2 < \cdots < d_{k-1} < d_k$. The first weight d_1 is equal to the minimum distance and the last weight is equal to the length of the code.
An $[n,k]_q$-code is said to be *nondegenerate* if it is not contained in a coordinate hyperplane of $\mathbb{F}_q^n$. Two $[n,k]_q$-codes are said to be *equivalent* if one can be obtained from another by permuting coordinates and multiplying them by nonzero elements of $\mathbb{F}_q$. It is clear that this gives a natural equivalence relation on the set of $[n,k]_q$-codes.
The *(usual) spectrum (or weight distribution)* of a code $C \subseteq \mathbb{F}_q^n$ is the sequence $\{A_0, A_1, \cdots, A_n\}$ defined by

$$A_i = A_i(C) = |\{c \in C : ||c|| \neq 0\}|$$

More generally, the *rth higher weight spectrum (or rth support weight distribution)* of a code C is the sequence $\{A_0^r, A_1^r, \cdots, A_n^r\}$ defined by

$$A_i^r = |\{D \subseteq C : dimD = r, ||D|| = i\}|$$

This naturally allows us to define *rth support weight distribution function (or rth weight enumerator polynomial)* as

$$A^r(Z) = A_0^r + A_r^1 Z + \cdots + A_r^n Z^n$$

Hence for each $0 \leq r \leq k$, we have a weight enumerator polynomial. We can also define the *rth higher weight* as

$$d_r(C) = \min\{i : A_i^r \neq 0\}.$$

Note that $A^0(Z) = 1$. Also note that if $\bar{x} \in \mathbb{F}_q^n$, then

$$||x|| = ||\{\bar{x}\}|| = ||\{\lambda\bar{x} : \lambda \in \mathbb{F}_q\}||.$$

The r^{th} generalized spectrum of a $[n, k]_q$ projective system X is double sequence

$$(A_0^r, A_1^r, A_2^r, \cdots A_n^r)$$

of integers,where

$$A_i^r = A_i^r(X) := |\{\Pi \subseteq \mathbb{P}^{k-1} : |X \cap \Pi| = n - i, codimension \Pi = r\}|$$

for all i = 0, 1, 2, . . . , n and r = 1, 2, . . . k.

Lemma 2.1. *If C is a code with the dimension k over $\mathbb{F}_2$ then we have to for $Z = 1$*

$$A^r(1) = \begin{bmatrix} k \\ r \end{bmatrix}_2$$

where $\begin{bmatrix} k \\ r \end{bmatrix}_2 = \frac{(2^k-1)(2^k-2)\cdots(2^k-2^{r-1})}{(2^r-1)(2^r-2)\cdots(2^r-2^{r-1})}$, *which is the number of subspaces of the dimension r in a k dimensional space.*

2.2. Dual Codes

The standard *inner product* on $\mathbb{F}_q^n$ is defined by

$$< x, y >:= \sum_{i=1}^{n} x_i y_i$$

Definition 2.1. *The Dual of a code $C \subseteq \mathbb{F}_q^n$ is the code*

$$C^{\perp} := \left\{x \in \mathbb{F}_q^n :< x, c >= 0 \forall\, c \in \mathbb{F}_q^n\right\}.$$

2.3. Self Orthogonal and Self Dual Code

Definition 2.2. *Let C be a linear code and $C^{\perp}$be dual over $\mathbb{F}_q$ then* *(i) C is said to be Self Orthogonal if $C \subseteq C^{\perp}$*

(ii) C is said to be Self Dual if $C = C^{\perp}$

2.4. Grassman and Schubert Varieties

2.4.1. Grassmanian

Definition 2.3. *The Grassmannian of all l dimensional subspaces of $\mathbb{F}_q^m$ is denoted by $G(\ell, m)$*

$$G(\ell, m) := \{W \subseteq \mathbb{F}_q^m : W \textit{ is Subspace of } \mathbb{F}_q^m \textit{ and } dim(W) = l\}$$

2.4.2. Plücker Embedding

The aim of this section is to prove that Grassmannian $G(\ell, m)$ is a projective variety. Let us fix an integer ℓ such that $1 \le \ell \le m$ and $I(\ell, m) := \{\alpha = (\alpha_1, \ldots, \alpha_\ell) : \alpha_i \in \mathbb{Z}_+, 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_\ell \le m\}$.
e. g. $I(2,5) = \{(1,2),(1,3),(1,4),(1,5),(2,3),(2,4),(2,5),(3,4),(3,5),(4,5)\}$. It is easy to see that $|I(\ell, m)| = \binom{m}{\ell}$.
Let W be an ℓ-dimensional subspace of a vector space V. Hence $W \in G_{\ell,m}$. As an element $W \in G(\ell, m)$can be represented by an $\ell \times m$ matrix A of rank ℓ [7], we get a matrix A of *rank* ℓ. For any $\alpha \in I(\ell, m)$, consider maximal minors of the matrix A, so let $p_\alpha(A)$ be the minor of the matrix whose columns are labeled by α and is given by

$$p_\alpha(A) = \begin{vmatrix} a_{1\alpha_1} & a_{1\alpha_2} & \cdots & a_{1\alpha_\ell} \\ a_{2\alpha_1} & a_{2\alpha_2} & \cdots & a_{2\alpha_\ell} \\ \vdots & \vdots & \ddots & \vdots \\ a_{\ell\alpha_1} & a_{\ell\alpha_2} & \cdots & a_{\ell\alpha_\ell} \end{vmatrix}$$

One can form $\binom{m}{\ell}$ such minors. Hence we get a map $\phi : I(\ell, m) \to \mathbb{K}$, given by $\alpha \mapsto p_\alpha(A)$. It may happen that B and A are two matrices representing the same subspace W, then we know that they are related by the equation , $B = CA$, for some $C \in GL(\ell, \mathbb{K})$. But then, $p_\alpha(B) = det(C).\ p_\alpha(A)$, $\forall\ \alpha \in I(\ell, m)$. Hence, up to some non-zero scalar in $\mathbb{K}$, we will get a uniquely determined $\binom{m}{\ell}$ tuple as $(\ldots, p_\alpha(A), \ldots)$. Note that, the tuple $(\ldots, p_\alpha(A), \ldots)$ is independent of choice of basis, but it depends only on the subspace with which we have started. As a result of which, we get a function

$$\pi : G_{\ell,m} \to \mathbb{P}^{\binom{m}{\ell}-1} \quad \text{which is defined as}$$
$$\pi(A) = (\ldots, p_\alpha(A), \ldots),$$

where A is an $\ell \times m$ matrix representing subspace W.

Definition 2.4. (Plücker Map) [11, 8, 6] *The function π defined above is called as Plücker Map and the coordinates of $\pi(A) = (\ldots, p_\alpha(A), \ldots)$ are called as Plücker coordinates.*

2.4.3. Schubert Varieties

Ghorpade and Lachaud in [2] proposed the generalization of Grassmann code as Schubert code. The Schubert code are indexed by the elements of the set

$$I(\ell, m) := \{\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_\ell) \in \mathbb{Z} : 1 \leq \alpha_1 < \cdots < \alpha_\ell \leq m\}.$$

Given any $\alpha \in I(\ell, m)$, the corresponding *Schubert code* is denoted by $C_\alpha(\ell, m)$, and it is the code obtained from the projective system defined by the Schubert variety $\Omega_\alpha(\ell, m)$ in $G(\ell, m)$ with a nondegenerate embedding induced by the Plücker embedding. We define Ω_α as

$$\Omega_\alpha = \{W \in G(\ell, m) : \dim(W \cap A_{\alpha_i}) \geq i \text{ for } i = 1, 2, \cdots, \ell\},$$

where A_j denotes the span of the first j vectors in a fixed basis of V, for $1 \leq j \leq m$.

Definition 2.5. *$\Omega_\alpha(\ell, m)$ as*

$$\Omega_\alpha(\ell, m) = \{W \in G(\ell, m) : dim(W \cap A_{\alpha_i}) \geq i \text{ for } i = 1, 2, \cdots, \ell\},$$

where A_j denotes the span of the first j vectors in a fixed basis of V, for $1 \leq j \leq m$. Ghorpade and Tsfasman in [4], *determined the length n_α and the dimension k_α of $C_\alpha(\ell, m)$. It was conjectured by Ghorpade in* [2], *that*

$$d(C_\alpha(\ell, m)) = q^{\delta_\alpha}$$

where

$$\delta_\alpha := \sum_{i=1}^{\ell} (\alpha_i - i) = \alpha_1 + \alpha_2 + \cdots + \alpha_\ell - \frac{\ell(\ell+1)}{2}$$

.2.4.4. Extended Linear Code [9] [Chapter 5, Section 5.1]

Definition 2.6. *Let C be linear code over $\mathbb{F}_q$,the extended code of C, denoted by $\bar{C}$ is given by,*

$$\bar{C} = \{(c_1, c_2, \ldots c_n, -\sum_{i=1}^{n} c_i) : (c_1, c_2, \ldots c_n,) \in C\}$$

here extra coordinate added is known as parity-check coordinate and if $q = 2$ *then it may say that* $-\sum_{i=1}^{n} c_i = \sum_{i=1}^{n} c_i$

Definition 2.7. Generator Matrix of Linear code [9] [Chapter 4, Section 4.5] *If* C *is a linear code over* $\mathbb{F}_q$ *then generator matrix is a such matrix whose row vectors forms basis for* C *over* $\mathbb{F}_q$.

Definition 2.8. Parity check Matrix of Linear code [9] [Chapter 4, Section 4.5] *Let* C *be a code and* $C^{\perp}$ *be its dual. A generator matrix of* $C^{\perp}$ *is known as a parity check matrix of* C.

Remark 2.2.

(1) Since basis for any linear code is not unique,so its generator matrix for need not be unique.

(2) A generator matrix having form $[I_k|X]$ *,is called standard form generator matrix .*

(3) A parity check matrix having form $[Y|I_{n-k}]$*, is called standard form standard form parity check matrix.*

While certain linear codes may not have a generator matrix in standard form, after a suitable permutation of the coordinates of the codewords and possibly multiplying certain coordinates with some nonzero scalars, one can always arrive at a new code which has a generator matrix in standard form.

Definition 2.3. *Equivalence of linear codes* [9]
Two $[n, k]_q$*-codes are said be equivalent if one can be obtained from the other by a combination of operations of the following types:*

(1) permutation of the n *digits of the codewords;*

(2) multiplication of the symbols appearing in a fixed position by a nonzero scalar.

Theorem 2.4. [9] [Chapter 4, section 4] *If* $G = [I_k|X]$ *is standard form generator matrix of* $[n, k]_q$*linear code then a parity check matrix for* C *is* $H = [-X^T|I_{n-k}]$.

Theorem 2.5. [9] [Chapter 4, section 4] *Let* C *be a linear code and let* H *be a parity check matrix for* C*. Then* C *has the minimum distance* d *if and only if any* $d-1$ *columns of* H *are linearly independent and parity check matrix* H *has* d *columns which are linearly dependent.*

The above results and remarks will be used to prove properties of codes defined in next section. The linear code associated to Schubert Variety associated to

$\alpha = (2,4)$ have been studied in [16] with linear code generated we have defined extended binary Schubert Code of the length 19 and binary Schubert Code of the length 18 in the next section.

3. Extended binary Schubert Code and Binary Schubert Code

Definition 3.1. Extended Binary Schubert Code *Let* $G{=}[I_5|X]$ *where* X *is* $5{\times}14$ *be a matrix given by*
$$X=\begin{bmatrix} 0&0&0&0&0&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&0&0&0&0&0\\ 0&1&0&1&1&1&1&0&0&1&0&0&0&1\\ 1&0&1&0&1&0&1&1&0&0&1&0&1&0\\ 1&1&0&0&1&0&1&0&0&1&1&1&0&0 \end{bmatrix}$$
The binary linear code whose generator matrix is G *is called extended binary Schubert code and it is denoted by* $\bar{\Omega}_{(2,4)}$

One can note that the last columns in the matrix G is in a such way that sum of all 18 coordinates in each row is last element of each row so one can define another code by defining generator matrix by deleting last coordinates of each rows in G and we can define binary Schubert Code in following way.

Definition 3.2. Binary Schubert Code *Let* $G_1{=}[I_5|Y]$, *where* Y *is* $5{\times}13$ *be a matrix given by*
$$Y=\begin{bmatrix} 0&0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&0&0&0&0\\ 0&1&0&1&1&1&1&0&0&1&0&0&0\\ 1&0&1&0&1&0&1&1&0&0&1&0&1\\ 1&1&0&0&1&0&1&0&0&1&1&1&0 \end{bmatrix}$$
The binary linear code whose generator matrix G_1 *is called binary Schubert code and it is denoted by* $\Omega_{(2,4)}$

We now discus the properties of these two linear codes in the next section.

4. Properties of Extended Binary Schubert code and Binary Schubert Code

Proposition 4.1. *(Properties of Extended Binary Schubert Code)*
Let $\bar{\Omega}_{(2,4)}$ *be the extended binary code generated by matrix G which is defined in a above section,then following properties holds for the corresponding linear code* $\bar{\Omega}_{(2,4)}$

(1) The length of $\bar{\Omega}_{(2,4)}$ *is 19.*

(2) The dimension of $\bar{\Omega}_{(2,4)}$ *is 5.*

(3) A parity check matrix for $\bar{\Omega}_{(2,4)}$ *is given by* $[X^T|I_{14}]$ *matrix*

(4) $\bar{\Omega}_{(2,4)}$ *is self orthogonal linear code.*

(5) The weight of every codeword in $\bar{\Omega}_{(2,4)}$ is multiple of 2

(6) A linear code $\bar{\Omega}_{(2,4)}$ is three error correcting code

Proof.

(1) By a generator matrix, one can see easily the length of the code to be 19. Alternatively one can also see that these are 19 $\mathbb{F}_2$-rational points as in a projective system the number of $\mathbb{F}_2$ -rational points corresponding $\Omega_{(2,4)}$ is given by

$$\begin{aligned} n &= \sum_{\beta\leq\alpha} q^{\delta(\beta)} \\ &= 2^{1+2-3} + 2^{1+3-3} + 2^{1+4-3} + 2^{2+3-3} + 2^{2+4-3} \\ &= 1+2+4+4+8 \\ &= 19 \end{aligned}$$

(2) The dimension of $\bar{\Omega}_{(2,4)}$ is 5 since Generator matrix of consist of I_5 the identity matrix of order 5 Since this generator matrix is in standard form so rank of this will be the dimension of the extended binary Schubert code $\bar{\Omega}_{(2,4)}$. Alternatively since $\alpha = (2,4)$

$$\begin{aligned} \dim(C) &= \#\{\beta : \beta \leq (2,4)\} \\ &= \#\{(1,2),(1,3),(1,4),(2,3),(2,4)\} \\ &= 5 \end{aligned}$$

(3) $G = [I_5|X]$ is generator matrix of $\bar{\Omega}_{(2,4)}$ where X is 5×14 matrix given by

$$\text{X}=\begin{bmatrix} 0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\ 0&0&1&1&1&1&1&1&1&0&0&0&0&0 \\ 0&1&0&1&1&1&1&0&0&1&0&0&0&1 \\ 1&0&1&0&1&0&1&1&0&0&1&0&1&0 \\ 1&1&0&0&1&0&1&0&0&1&1&1&0&0 \end{bmatrix}$$

here G is standard form generator matrix of $\bar{\Omega}_{(2,4)}$ using algorithm in [9] to convert this matrix into the form $H_1 = [X^T|I_{14}]$

(4) Note that the rows of G are mutually orthogonal, i.e if R_i and R_j (for $i \neq j$) then $R_i.\ \ R_j = 0$;this implies that $G \subseteq G^{\perp}$ thus $\bar{\Omega}_{(2,4)}$ is self orthogonal code.

(5) Due to [12], let $\alpha = (\alpha_1, m)$ be strictly increasing sequence and $C_\alpha(2, m)$ be the corresponding Schubert code and if α_1 is even number then weight of every codeword in $C_\alpha(2, m)$ is divisible by $q^{\alpha_1 - 1}$, here $q = 2, \alpha_1 = 2$ implies that weight of every codeword is divisible by 2 alternatively one can see the weight of every row vector of generator matrix of $\bar{\Omega}_{(2,4)}$ is divisible by 2 and these row vectors being basis hence weight of every codeword is divisible by 2.

(6) Since $H_1 = [X^T | I_{14}]$ is parity check matrix for $\bar{\Omega}_{(2,4)}$ where

$$\text{X=}\begin{bmatrix} 0&0&0&0&0&1&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&0&0&0&0&0\\ 0&1&0&1&1&1&1&0&0&1&0&0&0&1\\ 1&0&1&0&1&0&1&1&0&0&1&0&1&0\\ 1&1&0&0&1&0&1&0&0&1&1&1&0&0 \end{bmatrix}$$

hence distance of $\bar{\Omega}_{(2,4)}$ is 8 and hence $\bar{\Omega}_{(2,4)}$ is an 3 error correcting code

Proposition 4.2. *(Properties of Binary Schubert Code) Let $\Omega_{(2,4)}$ be the binary code generated by matrix G_1 which is defined in a above section then following properties holds for the corresponding linear code*

(1) The length of $\Omega_{(2,4)}$ is 18.

(2) The dimension of $\Omega_{(2,4)}$ is 5.

(3) The parity check matrix for $\Omega_{(2,4)}$ is given by $[Y^T | I_{14}]$ matrix

(4) The extension of binary Schubert code is extended binary Schubert Code.

(5) The binary Schubert Code $\Omega_{(2,4)}$ is exactly 3 error correcting code.

Proof.

(1) The length is clear from generator matrix of $\Omega_{(2,4)}$ i.e. number of columns in the generator matrix is equal to the length of $\Omega_{(2,4)}$ which is 18

(2) since G_1=$[I_5|Y]$ is generator matrix of $\Omega_{(2,4)}$ where Y is 5×13 matrix given by

$$\text{Y=}\begin{bmatrix} 0&0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&1&1&1&1&1&1&1&0&0&0&0\\ 0&1&0&1&1&1&1&0&0&1&0&0&0\\ 1&0&1&0&1&0&1&1&0&0&1&0&1\\ 1&1&0&0&1&0&1&0&0&1&1&1&0 \end{bmatrix}$$

has rank 5 hence the dimension of $\Omega_{(2,4)}$ is equal to rank(G_1) i.e. 5.

(3) G_1=$[I_5|Y]$ is generator matrix of $\Omega_{(2,4)}$ where Y is 5×13 matrix given by

$$\mathrm{Y}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

G_1 is in standard form hence parity check matrix of $\Omega_{(2,4)}$ is $[Y^T|I_{14}]$

(4) In the generator matrix G_1=$[I_5|Y]$ of $\Omega_{(2,4)}$ where Y is given in the definition the sum of each row vectors gives rise the last column of generator matrix G hence extension of binary Schubert code is extended binary Schubert code .

(5) Since $\bar{\Omega}_{(2,4)}$ has distance 8 hence distance of $\Omega_{(2,4)}$ must be 8-1 i.e. 7 hence $\Omega_{(2,4)}$ is an exactly three error correcting code.

5. Conclusions

Thus due to projective space geometry we could able to define extended binary Schubert Code and binary Schubert code. The extended binary Schubert code denoted by $\bar{\Omega}_{(2,4)}$ is binary $[19, 5, 8]$ self orthogonal linear code where as $\Omega_{(2,4)}$ is binary $[18, 5, 7]$ linear code. Both the codes are 3-error correcting code.

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