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ANALYSIS AND MODELLING OF THE SEIR EPIDEMIC MODEL UNDER TREATMENT RATE USING HOMOGENEOUS TRANSMISSION FUNCTION

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Abstract: This paper is a study of the SEIR mathematical epidemic model using homogeneous transmission function. We worked over the rate of treatment on infected and susceptible individuals to boost the recovery among them. The endemic equilibrium and disease free equilibrium are calculated with certain conditions for their existence. Stability of these points are tested based on available treatment situation. Analytical results are illustrated using numerical values.

Keywords and Phrases: Epidemic model, Routh - Herwitz criterion, Lyapunov function, Dulac's criterion, Stability.

2020 Mathematics Subject Classification: 92D30, 93D05, 34D23.

1. Introduction

The history and present situation of human being is full of fear due to several diseases. A lot of infectious diseases have threatened the human beings. Human kind understands the importance of dynamical study of diseases to know and to stop them through various appropriate controlling actions.

Mathematical models are always helpful in dynamic study of any disease. They are

very useful to present different involved parameters, their functioning and sensitivity. By using these we are able to make different policies to control the diseases. After taking corrective actions and by rectifying different parameters, we receive the optimized solution of the real world problem.

2. Literature Survey

To know the behaviour of epidemics, it is necessary to find the steady state of model and its stability for which rate of incidence is must. It is the degree of variation at which susceptible individuals become infectious. Kermack and Mckendrick used the simple mass action λSI, λ is infection coefficient, S is susceptible individuals and I is infectious individuals. Capasso and Serio [3] introduced the saturated incidence rate to show a saturation level as I become larger, to calculate the intensity of infection when disease enters in susceptible individuals and to know the effect of inhibition due to behavioural change in susceptible when they increase in numbers or due to crowding effect of infectives. Pathak [8] considered the transmission rate $\varphi = KSI/1 + \alpha S + \beta I$ represent a saturation effect to know the value of maximum interaction due to spatial or social distribution of the population. A lot of new incidence rates are generalized by various authors with time as Kar and Batabyal [5] proposed non-monotonic incidence rate. Treatment is always an initial and necessary step to limit the spread of disease.

Wang [10] proposed a treatment function $T(I) = \begin{cases} rI, & \text{if} \quad 0 \leq I \leq I_0 \\ k, & \text{if} \quad I > I_0,\ k = rI_0 \end{cases}$.

Driven by the work of Al-Sheikh [1], in this paper we considered SEIR epidemic model under treatment rate proposed by Wang [12] to showcase the limitation of treatment and medical facilities. Homogeneous transmission function is to display the saturation effect. Here main concern is to present the stability of the model at local and global level.

This work is organized as follows: Coming section represents the proposed mathematical model and equilibrium points of system with conditions of their existence are calculated. In section 3, the mathematical model is presented as per the assumption. In section 4, the basic reproduction number is calculated. In section 5, equilibrium points are calculated. In section 6, stability of equilibrium is investigated by proving some theorems on stability. Section 7, analytical results are illustrated using numerical simulation. Section 8, concludes with the discussion of the results.

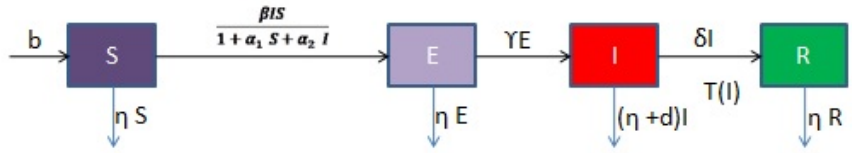
3. The Mathematical Model Formulation

To present the model we considered following assumptions:

1. The total population (N) is divided into four compartments, compartment

of susceptible individuals (S), compartment of exposed individuals (E), compartment of infected individuals (I), the recovered or removed individuals are in a compartment(R).

2. Recruitment rate is treated as constant.
3. The treatment function is defined by $T(I) = \begin{cases} rI, & \text{if } 0 \leq I \leq I_0 \\ k, & \text{if } I > I_0 \end{cases}$.
   Here treatment rate is increasing or decreasing with the increase or decrease of the number of infective till they are less than or equal to a certain value and it becomes stagnant afterwards. Practically it is possible when patients are more and the medical facilities are not sufficient.
4. Homogeneous transmission of disease $\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I}$ with α_1 = parameter which measures the sociological effects and α_2 = parameter which measures the effect of psychological or other mechanisms.
5. Disease related death rate is also considered.



SEIR epidemic model

The SEIR model in epidemiology for the spread of an infection is formulated by the following simultaneous differential equations under treatment rate and homogeneous transmission function.

$$\begin{aligned} \frac{dS}{dt} &= b - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \eta S \\ \frac{dE}{dt} &= \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\eta)E \\ \frac{dI}{dt} &= \gamma E - (\delta+\eta+d)I - T(I) \\ \frac{dR}{dt} &= \delta I - \eta R + T(I) \end{aligned} \tag{1}$$

Variable and Parameters	Description
b	recruitment rate
β	infection rate
γ	rate of developing infectivity
η	natural death rate
δ	recovery rate
d	disease related death
α_1	Parameter which measures the sociological effects.
α_2	Parameter which measures the effect of psychological or other mechanisms.

Here, first three equations do not contain R, i.e. they are free from the variable R, so reduced system for analysis,

$$\begin{aligned}\frac{dS}{dt} &= b - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \eta S \\ \frac{dE}{dt} &= \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\eta)E \\ \frac{dI}{dt} &= \gamma E - (\delta+\eta+d)I - T(I)\end{aligned} \tag{2}$$

By summing up the first three equations of proposed model

$$\begin{aligned}\frac{d(S+E+I)}{dt} &= b - \eta S - \eta E - (\delta+\eta+d)I + T(I) \\ \frac{d(S+E+I)}{dt} &= b - \eta(S+E+I) - (\delta+\eta)I + T(I) \\ &\leq b - \eta(S+E+I)\end{aligned}$$

By variable separable,

$$\frac{d(S+E+I)}{b-\eta(S+E+I)} \leq dt$$

On integrating on both the side,

$$\Rightarrow \lim_{t\to\infty} Sup(S+E+I) \leq \frac{b}{\eta}$$

So the feasible region for the proposed system is

$$\boldsymbol{\varphi} = \{(S,E,I):\ S+E+I \leq \frac{b}{\eta},\ S>0,\ E\geq 0,\ I\geq 0\}$$

The region $\boldsymbol{\varphi}$ is positively invariant with respect to the system (2).

4. Basic Reproduction number (R_0)

Next generation method is used for finding the basic reproduction number of the model (2). System (2) always has the diseases free equilibrium $P_0(b/\eta, 0, 0)$. For this disease free equilibrium I is always less than or equal to $I_0(I \leq I_0)$ with $T(I) = rI$.

Let $\psi = (E, I, S)^T$, for system (2) can be written as

$$\frac{d\psi}{dt} = F(\psi) - V(\psi)$$

The Jacobian matrix of $F(\psi)$ and $V(\psi)$ at the diseases free equilibrium P_0 are, $DF(P_0) = \begin{bmatrix} F_1 & 0 \\ 0 & 0 \end{bmatrix}$, $DV(P_0) = \begin{bmatrix} V_1 & 0 \\ 0 & 0 \end{bmatrix}$ respectively

$$F_1 = \begin{bmatrix} 0 & \frac{\beta b}{\eta + \alpha_1 b} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad V_1 = \begin{bmatrix} \gamma + \eta & 0 & 0 \\ -\gamma & (\delta + \eta + d + r) & 0 \\ 0 & 0 & \eta \end{bmatrix}$$

$$V_1^{-1} = \begin{bmatrix} \frac{1}{(\gamma + \eta)} & 0 & 0 \\ \frac{\gamma}{(\gamma + \eta)(\delta + \eta + r + d)} & \frac{1}{(\delta + \eta + r + d)} & 0 \\ 0 & 0 & \frac{1}{\eta} \end{bmatrix}$$

$$F_1 V_1^{-1} = \begin{bmatrix} \frac{\beta b \gamma}{(\gamma + \eta)(\delta + \eta + r + d)(\eta + \alpha_1 b)} & \frac{\beta \Lambda}{(\eta + \alpha_1 b)(\delta + \eta + r + d)} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Here maximum Eigen value is $\dfrac{\beta b \gamma}{(\gamma + \eta)(\delta + \eta + r + d)(\eta + \alpha_1 b)}$ so the spectral radius R_0 of the matrix $F_1 V_1^{-1}$ is the basic reproduction number of the model. $R_0 = \rho(F_1 V_1^{-1})$, so, $R_0 = \dfrac{\beta b \gamma}{(\gamma + \eta)(\delta + \eta + r + d)(\eta + \alpha_1 b)}$

5. Equilibrium Points

After solving System (2) has the diseases free equilibrium $P_0(b/\eta, 0, 0)$ when $I \leq I_0$. When $0 < I \leq I_0$,

$$\begin{aligned} b - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \eta S &= 0 \\ \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\eta)E &= 0 \\ \gamma E - (\delta+\eta+d)I - rI &= 0 \end{aligned} \tag{3}$$

When $I > I_0$, then (2) becomes,

$$\begin{aligned} b - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \eta S &= 0 \\ \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\eta)E &= 0 \\ \gamma E - (\delta+\eta+d)I - k &= 0 \end{aligned} \tag{4}$$

And endemic equilibrium $P^e(S^e, E^*, I^e, R^e)$

$$S^e = \frac{(\gamma+\eta)(\gamma+\eta+r+d)(1+\alpha_2 I)}{\{\beta\gamma - \alpha_1(\gamma+\eta)(\delta+\eta+d+r)\}} = \frac{\dfrac{(1+\alpha_2 I)b}{\eta+\alpha_1 b}}{R_0 - \dfrac{\alpha_1 b}{\eta+\alpha_1 b}}$$

$$E^* = \frac{(\delta+\eta+r+d)I}{\gamma}$$

$$I^e = \frac{\gamma[b\beta\gamma - (\alpha_1 b+\eta)(\gamma+\eta)(\delta+\eta+r+d)]}{(\gamma+\eta)(\delta+\eta+r+d)\{\eta\gamma\alpha_2+\beta\gamma-\alpha(\gamma+\eta)(\delta+\eta+r+d)\}}$$

$$\text{or} \quad I^e = \frac{(\alpha_1 b+\eta)\gamma[R_0-1]}{\{\eta\gamma\alpha_2+\beta\gamma-\alpha_1(\gamma+\eta)(\delta+\eta+r+d)\}}$$

$$R^e = \frac{(\delta+r)I}{\eta}$$

$I^e \leq I_0$ if and only if $R_0 \leq 1 + \dfrac{\{\eta\gamma\alpha_2+\beta\gamma-\alpha(\gamma+\eta)(\delta+\eta+r+d)\}I_0}{(a_1 b+\eta)\gamma} \triangleq X_0$.
So, P^e is an endemic equilibrium of system (3) if and only if $1 < R_0 \leq X_0$. Which is a unique in nature but when we solve system (4) we receive quadratic equations for S^e it means two or more endemic equilibrium points are possible.

6. Stability

6.1. Local Stability

For local stability at disease free equilibrium,

Lemma 6.1. *If $R_0 < 1$, then the disease free equilibrium P_0 is asymptotically stable, P_0 is stable, if $R_0 = 1$ and P_0 is unstable if $R_0 > 1$. Consider,*

$$F_2 = b - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - \eta S$$
$$F_3 = \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I} - (\gamma+\eta)E \quad at \quad (S_0, E_0, I_0) = (b/\eta, 0, 0)$$
$$F_4 = \gamma E - (\delta+\eta+d+r)I$$

$$J_0(P_0) = \begin{bmatrix} -\eta & 0 & \dfrac{-\beta b}{\eta+b\alpha_1} \\ 0 & -(\gamma+\eta) & \dfrac{\beta b}{\eta+b\alpha_1} \\ 0 & \gamma & -(\delta+\eta+r+d) \end{bmatrix}$$

The characteristics equation of system (2) at P_0 is as follows,

$$|J_0(P_0) - \lambda I| = -(\eta+\lambda)\left\{(\delta+\eta+r+d+\lambda)(\gamma+\eta+\lambda) - \frac{\beta b\gamma}{\eta+b\alpha_1}\right\} = 0$$
$$\Rightarrow (n+\lambda)\{\lambda^2 + \lambda(2\eta+r+\gamma+d+\delta) + (\gamma+\eta)(\delta+\eta+r+d)[1-R_0]\} = 0$$
$$\Rightarrow (n+\lambda)\{\lambda^2 + \lambda p + q\} = 0$$

where, $a = (2\eta+r+\gamma+\delta+d)$, $b = (\gamma+\eta)(\delta+\eta+r+d)[1-R_0]$, $b > 0$ when $R_0 < 1$ all the Latent values of the Jacobian matrix gives negative real values therefore at P_0 the system is locally asymptotically stable when $R_0 < 1$. Which verifies the theorem.

Now for local stability at endemic equilibrium point.

Lemma 6.2. *If $R_0 > 1$, then the endemic equilibrium P^e is locally asymptotically stable.*

$$J_e(P^e) = \begin{bmatrix} -\eta - \left\{\dfrac{\beta I^e(1+\alpha_2 I^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2}\right\} & 0 & \dfrac{-\beta S^e(1+\alpha_1 S^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2} \\ \dfrac{\beta I^e(1+\alpha_2 I^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2} & -(\gamma+\eta) & \dfrac{\beta S^e(1+\alpha_1 S^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2} \\ 0 & \gamma & -(\delta+\eta+r+d) \end{bmatrix}$$

Let $Z_1 = -\left[\eta + \left\{\frac{\beta I^e(1+\alpha_2 I^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2}\right\}\right]$, $Z_2 = \frac{\beta S^e(1+\alpha_1 S^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2}$, $Z_3 = \frac{\beta I^e(1+\alpha_2 I^e)}{(1+\alpha_1 S^e+\alpha_2 I^e)^2}$

$$J_e(P^e) = \begin{bmatrix} Z_1 & 0 & -Z_2 \\ Z_3 & -(\gamma+\eta) & Z_2 \\ 0 & \gamma & -(\delta+\eta+r+d) \end{bmatrix}$$

Consider

$$|P^e - \lambda I| = \begin{bmatrix} Z_1-\lambda & 0 & -Z_2 \\ Z_3 & -(\gamma+\eta)-\lambda & Z_2 \\ 0 & \gamma & -(\delta+\eta+r+d)-\lambda \end{bmatrix}$$

$$\Rightarrow (Z_1-\lambda)\{(\gamma+\eta+\lambda)(\delta+\eta+r+d+\lambda) - Z_2\gamma\} - Z_2(Z_3\gamma) = 0$$
$$a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$$
$$a_3 = 1, a_2 = (\delta+2\eta+r+d+\gamma-Z_1),$$
$$a_1 = [(\delta+\eta+r+d)(\mu+\gamma-Z_1) - Z_1(\eta+\gamma) + Z_2\gamma],$$
$$a_0 = \{Z_2\gamma(Z_3+Z_1) - Z_1(\delta+\eta+r+d)(\eta+\gamma)\}$$

So by using Routh Criterion, $a_n > 0$ *and* $a_1a_3 - a_0a_2 > 0$. $a_1a_3 > a_0a_2$ *for* $z_1 < 0$ *so all the Eigen values are negative hence endemic equilibrium is stable locally. Therefore, by Routh-Herwitz criteria, we conclude that the Latent values of* P^e *are all negative real values when* $R_0 > 1$. *Which proves the theorem i.e. If* $R_0 > 1$, P^e *is locally asymptotically stable.*

6.2. Global Stability

To validate the global stability at disease free equilibrium, we create Lyapunov function

$$L = \gamma E + (\gamma+\eta)I$$
$$L' = \gamma E' + (\gamma+\eta)I'$$
$$L' = (\gamma+\eta)(\delta+\eta+d+r)\left\{\frac{\beta S\gamma}{(1+\alpha_1 S+\alpha_2 I)(\gamma+\eta)(\delta+\eta+d+r)} - 1\right\} I = 0$$

Which indicates, at $I = 0 \Rightarrow L' = 0$ and if $I \neq 0$, won't be able to put restriction on R_0 as here denominator is $1+\alpha_1 S+\alpha_2 I$ [6]. Therefore by using Dulac function [9] with Poincare Bendixson concept for the same.

$B(E,I) = 1/E.I$ where $E > 0, I > 0$

$$(B.F) = \frac{1}{E.I}\left\{\frac{\partial(B.F_3)}{\partial E} + \frac{\partial(B.F_4)}{\partial I}\right\}$$

$$\nabla.(B.F) = \frac{-\beta SI}{E^2 I(1+\alpha_1 S + \alpha_2 I)} - \frac{\gamma}{I^2} < 0$$

Indicates there is no closed orbit; therefore, P_0 is globally asymptotically stable. Applying the same to validate the global stability at P^e of system (3)
$B_1(S,E,I,R) = 1/S.E.I.R$ where $S > 0, E > 0, I > 0, R > 0$

$$\nabla(B_1.F) = \frac{1}{S.E.I.R}\left\{\frac{\partial(B.F_2)}{\partial S} + \frac{\partial(B.F_3)}{\partial E} + \frac{\partial(B.F_4)}{\partial I} + \frac{\partial(B.F_5)}{\partial R}\right\}$$

$$= -\left[\frac{b}{S^2 EIR} + \frac{\beta\alpha_1}{ER(1+\alpha_1 S + \alpha_2 I)^2}\right] - \left[\frac{\beta}{E^2 R(1+\alpha_1 S + \alpha_2 I)}\right]$$

$$-\left[\frac{\gamma}{SI^2 R}\right] - \frac{\delta + r}{SER^2} < 0$$

Represents there is no closed orbit in the first quadrant. Therefore, the endemic equilibrium is globally asymptotically stable.

7. Numerical Simulation

To check the dynamical behaviour of model, we solve the simultaneous ordinary equations by using the parameters and their values based on previous papers.

Case I. Suppose the parameters are $\delta = 1.2$, $\eta = 2$, $\alpha_1 = 3$, $d = 1.1$, $r = 0.8$, $b = 10$, $\beta = 12$, $\gamma = 1.5$, $\alpha_2 = 5$ and the initial number for (S,E,I,R) are $(2,1,1,1)$. So $P_0(5,0,0,0)$ and the basic reproduction number $R_0 = 0.3151 < 1$ which validate the Lemma 6.1.

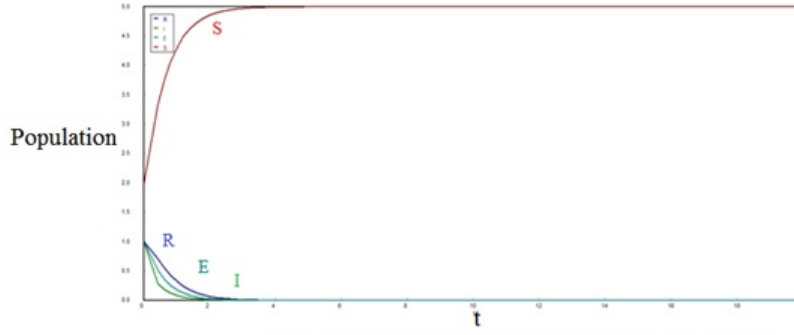


Fig. 1 The figure represents that the disease dies out

Case II. Suppose the parameters are $\delta = 0.7$, $\eta = 0.21$, $\alpha_1 = 2$, $d = 1.5$, $r = 0.1$, $b = 10$, $\beta = 9$, $\gamma = 1.5$, $\alpha_2 = 3$ and the initial number for (S, E, I, R) are $(2, 1, 1, 1)$. So $P^e(3.4925, 1.6733, 5.073, 13.8095)$ and the basic reproduction number $R_0 = 3.4585 < 1$. Therefore endemic equilibrium Pe is a global attractor in the interior of first octant. Figure represents $S(t)$, $E(t)$, $I(t)$ and $R(t)$ tends to their steady state value which represents disease is endemic in nature.

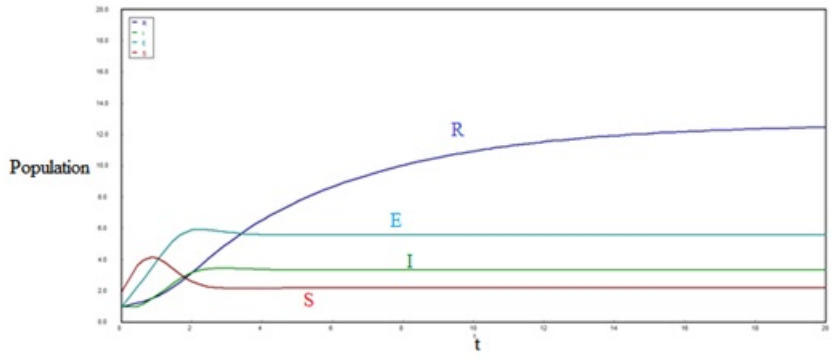


Fig. 2 The figure represents that the disease dies out endemic

Case III. If we change the α_1, α_2 values by keeping other parameters fixed, to measure the effect of sociology, psychology or their inhibition and mechanism it has been observed E^* decreases as α_1, α_2 are increases. The steady state value of E^* reduces with the increment of α_1, α_2 figure 3 and 4 verifies our results.

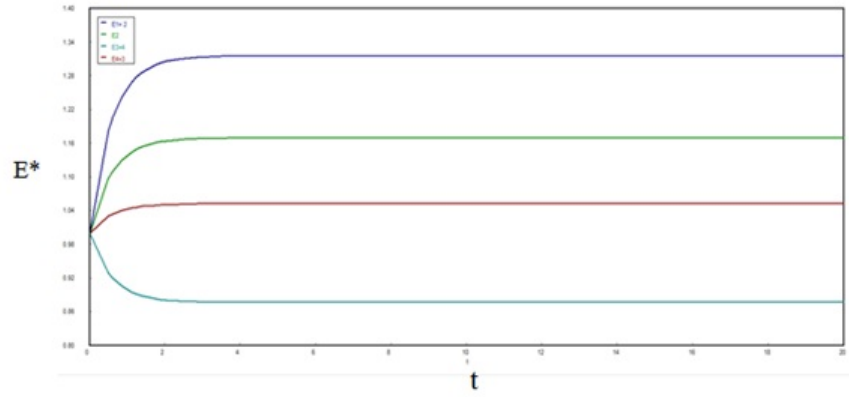


Fig. 3 The effect of soiological rate dependence of E*

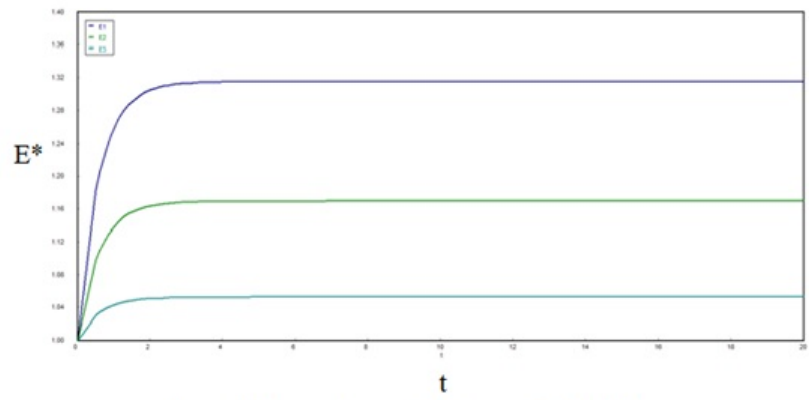


Fig. 4 the dependence of E* on psychological rate

8. Conclusions

We have studied the stability of the SEIR epidemic model under treatment rate using homogeneous transmission function. We observed if the basic reproduction number is less than unit at disease free equilibrium then it is locally and globally asymptotically stable which means the disease will vanish after some time and if basic reproduction is greater than one at endemic equilibrium shows it is locally and globally asymptotically stable means disease is endemic in nature. These dieses free and endemic equilibriums are stable under some specified parametric situations. Further, possible multiple endemic equilibrium points could be explored. The treatment function helped in understanding and controlling the disease. It has been observed that R_0 is not depending on α_2 but the numerical simulation shows that at the situation of endemic for disease the steady state for exposed goes down with the increase in the value of α_2. Which indicates, the spread of disease will decrease with the increase in social, psychological measures for protection.

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