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CONNECTED FORCING NUMBER OF CERTAIN GRAPHS

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Abstract: Given a simple graph $G = (V, E)$ with a set $S \subseteq V$, to be an initial set of coloured vertices called black vertices and all remaining vertices being uncoloured, called white vertices. At each integer valued time step, a coloured vertex in the set S with a single uncoloured neighbour will force that neighbour to get coloured and such a vertex is called a forcing vertex and the set S is called a forcing set, if by relatively applying the forcing process, all of V becomes coloured. The forcing number of a graph G is the cardinality of the smallest forcing set of G and it is denoted by $F(G)$. One of the variants of forcing, namely connected forcing, is a restriction of forcing in which initial set of coloured vertices induces a connected subgraph. The connected forcing number, $F_c(G)$ of a graph G, is the minimum cardinality among all connected forcing sets of G. In this paper, we determine $F_c(G)$ of degree splitting graphs and line graphs of certain graphs. Further we discuss on its bounds and the realizability theorem.

Keywords and Phrases: Forcing, Forcing number, Connected forcing, Connected forcing number, Degree splitting graphs, Line graphs.

2020 Mathematics Subject Classification: 05C15, 05C69.

1. Introduction

Unlike a static colouring process of a graph G, a dynamic colouring process, considers a subset $S \subseteq V(G)$ as an initial set of coloured vertices (black vertices) known as a zero forcing set or forcing set or propagation set or by several other different names by different authors, which can alter the colour of all the other vertices of G to black, by iteratively applying a subsequent colour change rule. The colour change rule is as follows: At each step, any black coloured vertex in the set S which has exactly one white coloured vertex as a neighbour, can change that white coloured vertex to a black coloured vertex. This vertex is called the *forcing vertex* and the set S is called the *forcing set* of G, if by applying this colour change rule iteratively, the vertex set $V(G)$ is coloured black. The minimum cardinality of all forcing sets of G is called the *forcing number* of G and it is denoted by $F(G)$. *Connected forcing* is one of the variants of forcing, in which the subgraph induced by a forcing set is connected. The minimum cardinality of a connected forcing set is called the *connected forcing number* of G and it is denoted by $F_c(G)$. The notion of forcing in graphs was first introduced in a workshop on Linear Algebra and Graph Theory in 2006 by R. A. Brualdi [1] and was used to bound the minimum rank of a graph. Connected forcing number was introduced by Boris Brimkov and Randy Davila in the year 2016 [2]. For further analytic study on forcing number and its variants, reader can refer to [4, 5, 6, 7, 9, 10, 11]. Forcing in graphs, finds extensive applications in scheduling, aircraft scheduling and mobile networks. In this paper connected forcing number of *degree splitting graphs* and *line graphs* of certain graphs are determined. Further we discuss on its bounds and realizability theorem.

2. Definitions and Preliminaries

For graph theoretic terminology and for definitions not mentioned here one can refer to [8]. The graphs considered in this paper are simple, finite and undirected graphs. Let the order and size of the graph $G = (V, E)$ be denoted by $n = |V(G)|$ and $m = |E(G)|$, respectively. Two vertices $v, w \in V(G)$ are said to be *adjacent*, or neighbours, if there exists an edge $vw \in E$. The number of edges incident to a vertex v is called the *degree* of the vertex v and it is denoted by $deg_G v$ or simply $degv$, in context to the graph. A vertex with degree zero is called an *isolated vertex* and a vertex with degree one is called a *pendant vertex* or a *leaf*. For any vertex $u \in V$, the open neighbourhood of u is the set $N(u) = \{v \in V : uv \in E\}$. The closed neighbourhood of a vertex in a graph is the vertex together with the set of adjacent vertices and it is denoted by $N[u]$. A *support vertex* is a vertex adjacent to atleast one leaf vertex. A *strong support* is a vertex which is adjacent to atleast two leaf vertices. A *cut edge* is an edge whose removal disconnects the graph. The *Corona product* of two graphs G and H is defined as the graph obtained by taking one copy of G and $|V(G)|$ copies of H and joining the i-th vertex of G to every vertex in the i-th copy of H. The *line graph* $L(G)$, of a graph G, is a simple graph whose vertices are the edges of G, with $ee^{'} \in E(L(G))$ when e and $e^{'}$ have common endpoint in G. In a graph G, the maximum degree denoted by $\Delta(G)$, is the degree of a vertex with maximum number of edges incident to it and the minimum degree denoted by $\delta(G)$, is the degree of a vertex with the minimum number of edges incident to it.

For a graph G with vertex set $V(G) = S_1 \cup S_2 \cup S_3 \cup ... \cup S_t \cup T$, where each S_i is a set of vertices with atleast two vertices having same degree and $T = V - \bigcup_{i=1}^{t} S_i$, the degree splitting graph of G, denoted by $DS(G)$ is obtained by adding the vertices $w_1, w_2, ..., w_t$ and joining w_i to each vertex of S_i, $1 \leq i \leq t$, respectively. A Path on n vertices is denoted by P_n and a Cycle on n vertices by C_n. A complete graph is a simple graph whose vertices are pairwise adjacent. K_n denotes a complete graph on n vertices. A graph G is *bipartite* if its vertex set V can be partitioned into partite sets V_1 and V_2 such that if uv is an edge of G then $u \in V_1$ and $v \in V_2, V = V_1 \cup V_2$. A graph G is said to be a *complete bipartite graph* if every vertex in V_1 is adjacent to every vertex of V_2 and it is denoted by $K_{m,n}$. *A* $t-$*partite* graph is one whose vertices set can be partitioned into t partite sets so that no edge has both end in any one partite set. A *complete* $t-$*partite* graph is one in which every vertex is joined to every other vertex which is not in the same set and it is denoted by $K_{t_1,t_2,t_3,...,t_p}$. For any set $S \subseteq V$, the induced subgraph $G[S]$ is the maximal subgraph of G with vertex set S. A *connected graph* having no cycles is called a *tree.* A *rooted tree* is a tree in which one of the vertices is distinguished from others. The distinguished vertex is called the root of the tree. The length of the path from the root v to a vertex x is the *depth* of x in T. Let $P(x)$ be a unique $x - v$ path. The parent of x is its neighbour on $P(x)$; The *children* of x are its other neighbours. A complete binary tree is a rooted tree in which all leaves have the same depth and all internal vertices have degree three, except the root vertex which is of degree two. If T is a complete *binary tree* with root vertex v, the set of all vertices with depth k are called vertices at *level* k. A *spider graph* is a graph $P_{n,m}$ obtained by identifying the end points of m paths, each one has length n. The spider graph is a tree with one vertex of degree atleast three called the center of the spider and all other with degree atmost two. The spider $P_{n,m}$ is called a *Regular spider* if it consists of one central vertex u connected with m number of paths P_n of same length. A *Star* S_k is complete bipartite graph $K_{1,k}$ is a tree with one internal node and k leaves. An n-star graph S_n is graph with n-vertices. *Banana tree* graph $B_{n,k}$ is a graph obtained by connecting one leaf of each of n copies of a k-star graph with a single root vertex that is distinct for all the stars. The *wheel graph* w_n is a cycle graph C_{n-1} with an additional central vertex adjacent to all the vertices of the cycle C_{n-1}.

In this paper, we compute the $F_c(G)$ of degree splitting graphs and line graphs of certain graphs. Further we study on its bounds and the realizability theorem. In Section 3, connected forcing number of degree splitting graphs of certain standard graphs like paths, cycles, complete graphs, complete $t-$partite graphs, regular spider and wheel graphs are determined. Bounds and realizability theorem are also discussed. In section 5, Line graphs $L(T)$ of certain classes of trees T, like complete binary tree, regular spider and banana trees, $F_c(L(T))$ are evaluated. The Following are the results referred for subsequent study in this paper.

Proposition 2.1. [2] *For a connected graph G different from a path, $F_c(G) \geq p$, where p is the number of leaves in G and the bound is sharp.*

Theorem 2.2. [2] *Let $T = (V, E)$ be a tree. Then,*

$$F_c(T) = \begin{cases} 1 & if \quad \Delta(T) < 3 \\ |R_1| + |R_2| + \mathcal{L} & if \quad \Delta(T) \geq 3 \end{cases}$$

where $R_1(G) = \{v \in V : G - V$ has atleast 3 connected components$\}$
$R_2(G) = \{v \in V : G - V$ has 2 connected components ,neither of which is a path$\}$
$R_3(G) = \{v \in V : v$ is adjacent to atleast one leaf$\}$
$\mathcal{L}$ is the number of leaves coloured.

Theorem 2.3. [5] *Let $G = (V, E)$ be a connected graph of order $n \geq 2$. Then $F_c(G) = n - 1$ if and only if $G = K_n$, $n \geq 2$, or $G = K_{1,n-1}$, $n \geq 4$.*

Observation 2.4. [3] *For every connected graph G, it holds that $F(G) \leq F_c(G)$.*

Degree splitting graphs find extensive applications in communication networks, power network monitoring and information in social networks.

3. Degree Splitting Graphs

Theorem 3.1. *For a path P_n, $n \geq 2$, $F_c(DS(P_n)) = \begin{cases} 2, & if \quad 2 \leq n \leq 4 \\ 3 & otherwise. \end{cases}$*

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$, $G = DS(P_n)$ and S be a connected forcing set of G.

Case (a): $2 \leq n \leq 4$.
Since $\delta(G) = 2$, we have $|S| \geq 2$. The set $\{v_1, v_2\}$ is a minimum connected forcing set of G. Hence $F_c(G) = 2$.

Case (b): $n \geq 5$.
In this case, to claim that $F_c(G) = 3$, we prove $|S| \geq 3$. Suppose on contradiction, $|S| = 2$. It is clear that any two vertices of G cannot force G to be completely coloured. Hence $|S| \geq 3$. Therefore the set $\{v_1, v_2, w\}$, where $deg_G w = n - 2$, is a

minimum connected forcing set of G. Hence $F_c(G) = 3$.

As the proofs are analogous to the proof of the Theorem 3.1, the following theorems for cycles and complete graphs are stated without proofs .

Theorem 3.2. For a cycle C_n , $n \geq 3$, $F_c(DS(C_n)) = 3$.

Theorem 3.3. *For a complete graph K_n, $F_c(DS(K_n)) = n$, where $n \geq 3$.*
For a complete $t-$partite graph $G = K_{t_1,t_2,t_3,\ldots,t_p}$, $0 < t_1 = t_2 = t_3 = \ldots = t_p = 1$, $DS(K_{t_1}, K_{t_2}, \ldots, K_{t_p} \cong DS(K_p)$, therefore in this case $F_c(DS(G)) = p$

Theorem 3.4. *For a complete $t-$partite graph $K_{t_1,t_2,t_3,\ldots,t_p}$, $0 < t_1 \leq t_2 \leq t_3 \ldots \leq t_p$ $F_c(DS(K_{t_1,t_2,t_3,\ldots,t_p})) = t_1 + t_2 + t_3 + \ldots + t_p - 1$.*
Proof. Let $G = DS(K_{t_1,t_2,t_3,\ldots,t_p})$ and S be a connected forcing set of G. Consider $V(G) = \bigcup\limits_{i=1}^{p} V_i \cup u_1, u_2, \ldots, u_p$, where $V_1, V_2, \ldots, V_p$ are the partite sets with cardinality, $|V_i| = t_i$, $1 \leq i \leq p$ and $u_1, u_2, \ldots, u_p$ are the newly introduced vertices in the construction of G. Also by definition of degree splitting graph, it follows that if $|V_i| = |V_j|$ for $i \neq j$ then $u_i = u_j$, and if $|V_i| = 1$ for some i, $1 \leq i \leq p$, then u_i does not exists. Represent the vertices of the partite sets V_i, $1 \leq i \leq p$ by v_{ji}, $1 \leq i \leq t_i$. Now claim that $|S| \geq t_1 + t_2 + t_3 + \ldots + t_p - 1$. Suppose on contradiction, assume $|S| = t_1 + t_2 + t_3 + \ldots + t_p - 2$.
We have two cases : $|V_i| = |V_j|$ for atleast any two i, j with $i \neq j$, $1 \leq i \leq p$, $1 \leq j \leq p$ and $|V_i| \neq |V_j|$ for every $i \neq j$, $1 \leq i \leq p$, $1 \leq j \leq p$. In both the cases choose a vertex u_i, $1 \leq i \leq p$ for which $deg u_i$ is minimum. Let u_i be a black vertex. Since $\delta(G) > 1$ and for each of the vertices v_{ji}, $1 \leq j \leq t_i$, $t_i > 1$ is adjacent to every other vertices in V_k, $k \neq i$, $1 \leq k \leq p$ none of the vertices in $V_i \cup \{u_i\}$ can force any of the other white vertices in $\bigcup\limits_{\substack{k \neq i \\ 1 \leq k \leq p}} V_k$. Hence $S' = V_i - \{v_{ki}\}$ for some k, $1 \leq k \leq t_i$ are black vertices and this vertex v_{ki} can be forced by u_i.
In the case when $t_i = t_j = 1, u_i = u_j$, for atleast some $i \neq j$ and $1 \leq i \leq p, 1 \leq j \leq p$ and if $deg u_i = \delta(G)$ then $S' = \bigcup\limits_{k \neq i} V_k - \{v_{1i}\}$. Therefore $S' \cup \{u_i\} \subset S$. But again none of the vertices in V_i can force any of the vertices in the other partite sets and the vertices in $\bigcup\limits_{\substack{k=1 \\ k \neq i}}^{p} u_k$. Hence $\sum\limits_{\substack{k=1 \\ k \neq i}}^{p} t_k + |S'| + 1 = \sum\limits_{k=1}^{p} t_k$ vertices belong to S. Since by assumption $|S| = \sum\limits_{k=1}^{p} t_k - 2$, without loss of generality, either let the partite set say V_l , $l \neq i$, consists of $t_l - 2$ black vertices or V_l, V_m, $l \neq m \neq i$ consists of $t_l - 1$ and $t_m - 1$ black vertices respectively. In either of the cases vertices in S can force

the other white vertices. Hence a contradiction to S being a connected forcing set of G. Therefore $|S| \geq t_1, t_2, ..., t_p - 1$. Hence $S = \bigcup_{\substack{k=1 \\ k\neq i,j}}^{p} V_k \cup S' \cup \{u_i\} \cup S''$ where $S'' = V_j - \{v_{kj}\}$, for k, $1 \leq k \leq t_j$ is a connected forcing set of G with minimum cardinality $\sum_{k=1}^{p} t_k - 1$.

Theorem 3.5. *For a regular spider graph $P_{n,m}$, $m > 2$, $F_c(DS(P_{n,m})) = m + 1$.*

Proof. Let $G = (DS(P_{n,m}))$ and D be a minimum connected forcing set of G. Assume u to be the central vertex of $P_{n,m}$ and $N(u) = \{u_1, u_2, ..., u_m\}$. Let $V(G) = V(P_{n,m}) \cup \{v, w\}$, where $degw = m, degv = m(n-1)$ and $N(w) = \{w_1, w_2, ..., w_m\}$. To prove $|D| = m + 1$. Suppose on contradiction $|D| < m + 1$, say $|D| = m$.
Now we claim the following:
Claim 1: $v \in D$.
Suppose $v \notin D$. Since $|D| = m$, the m black vertices, in their forcing process leaves vertices in G with two white vertices as neighbours. Thus leaving the graph G with uncoloured vertices. But this is a contradiction to D being a connected forcing set of G. Hence $v \in D$.
Claim 2: $u \in D$

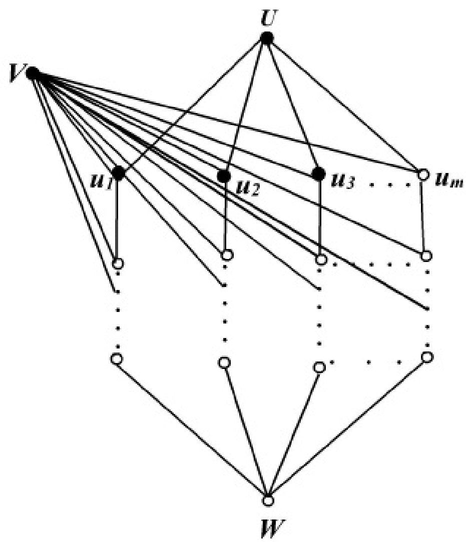

Figure **3.1**: **Shaded vertices denote a minimum connected forcing set of $\mathbf{DS(P_{n,m})}$**

Suppose $u \notin D$. Since D is a connected set, the $m-$vertices in D are connected by the vertex v. As the vertex $u \notin D$, the m vertices (black) in D, in their forcing process leaves atleast a vertex u_i, $1 \leq i \leq m$ with two white vertices as neighbours, a contradiction to D being a connected forcing set of G. Therefore $u \in D$. Hence the Claim 2. Therefore $\{v, u\} \subset D$, hence $D = \{v, u\} \cup \left(\bigcup_{i=1}^{m-1} u_i \right)$ is a minimum connected forcing set of G with $|D| = m + 1$. Therefore $F_c(DS(P_{n,m})) = m + 1$ (Refer to Figure 3.1).

Theorem 3.6. *For a wheel graph W_n on n vertices, $F_c(DS(W_n)) = 4$.*
Proof. Let G=$DS(W_n)$ and S be a connected forcing set of G Consider $V(G) = \{v_1, v_2, ..., v_n, w\}$, where $degv_n = n-1$, $degv_i = 4$, $1 \leq i \leq n-1$ and $degw = n-1$. Since $\delta(G) = 4$, $F_c(W_n) = 3$ and $F(G) \leq F_c(G)$, we have $|S| \geq 4$. Hence the set $\{w, v_n, v_1, v_2\}$ is a minimum connected forcing set of G. Therefore $F_c(G) = 4$.

Theorem 3.7. *For a graph G, $F_c(DS(G)) \leq F_c(G) + i$, $i \geq 1$.*
Proof. Let D be a minimum connected forcing set of G. Then we have $|D| = F_c(G)$. Construct $DS(G)$, such that $w_1, w_2, ..., w_i$, $i \geq 1$ to be the newly introduced vertices corresponding to the subsets S_i, $i \geq 1$ and $V(G) = \left(\bigcup_{i \geq 1} S_i \right) \cup T$, where $T = V(G) - \left(\bigcup_{i \geq 1} S_i \right)$. By definition of $DS(G)$, $V(DS(G) = V(G) \cup \left(\bigcup_{i \neq 1} w_i \right)$, In either of the cases when $T = \phi$ or $T \neq \phi$, we have $D \cap \bigcup_{i \geq 1} S_i \neq \phi$. Hence the set $D \cup \left(\bigcup_{i \geq 1} w_i \right)$ is a connected forcing set of $DS(G)$. Therefore $F_c(DS(G)) \leq |D| + i$ or $F_c(DS(G)) \leq F_c(G) + i$.

4. Realizability

Given a set of positive integers a, b with $2 < a \leq b$, it is realizable to find a graph G for which $F(G) = a$ and $F_c(G) = b$.
For a graph G and a vertex $v \in V(G)$, *attaching* a complete graph, say H to v, we mean identifying the vertices v and w, where $w \in V(H)$.

Theorem 4.1. (Realizability Theorem) *Given two positive integers a and b, with $2 < a \leq b$, there exists a graph G, such that $F(G) = a$ and $F_c(G) = b$.*
Proof. Case (a): $1 < a = b$
Consider a complete graph on $(a + 1)$ vertices. Then the resulting graph $G \cong K_{(a+1)}$ has $F(G) = a = F_c(G)$.
Case (b): $2 < a < b$
Consider a path P on $(b - a) + 1$ vertices. Attach $(a - 2)K_3'$s to one end vertex

of the path P and a K_3 to the other end vertex of P. Then the resulting graph G has $F(G) = a$ and $F_c(G) = b$ (Refer to Figure 4.1).

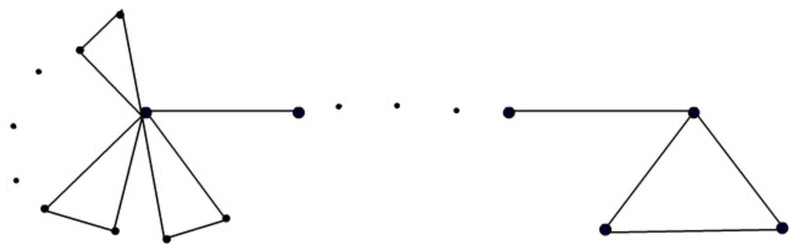

Figure **4.1**: **A graph illustrating Case(b)**

Edge colouring problem in a graph G can be viewed as a vertex colouring problem in line graph $L(G)$ of G. Time table scheduling of exams and scheduling of lectures in institutes are applications of this property of line graphs. Line graph is a powerful visual tool in marketing finance and other areas. Hence we attempt to study on the connected forcingness in line graphs of certain graphs.

5. Line Graphs

Theorem 5.1. *For a complete binary tree T of level l, $F_c(L(T)) = \sum_{k=1}^{l-1} 2^k + 2^{l-1}$.*

Proof. Let $G = L(T)$ and $E(T) = \{e_{ij} : 1 \leq i \leq l, 1 \leq j \leq 2^l, 1 > 0\}$. Then $V(G) = \{e_{ij} : 1 \leq i \leq l, 1 \leq j \leq 2^l, l > 0\}$. Consider a set S to be a connected forcing set of G. Clearly the edge $e_{11}e_{12}$ is a cut edge of G and since $\delta(G) = 3$, where $deg_G e_{11} = deg_G e_{12} = 3$, at least one of the vertices e_{2j}, $j \in \{1, 2\}$ which are adjacent to e_{11} belong to S or otherwise e_{11} is not a forcing vertex. Similarly, at least one of the vertices e_{2j}, $j \in \{3, 4\}$ which are adjacent to e_{12} belong to S or otherwise e_{12} cannot be a forcing vertex. Therefore, the vertices e_{11}, e_{12} and two of the vertices in the set $\{e_{2j} : 1 \leq j \leq 4\}$ belong to S. Without loss of generality, let $\{e_{21}, e_{23} \subset S\}$. Hence the vertices e_{11} and e_{12} force their adjacent vertices say e_{22} and e_{24} respectively. Since $deg_G e_{2j} = 4, 1 \leq j \leq 2^2$, none of the vertices $\{e_{2j} : 1 \leq i \leq 2^2\}$ can force the vertices in $\{e_{3j} : 1 \leq i \leq 2^3\}$. Continuing the argument in a similar manner, for each vertex corresponding to each edge up to the level $l - 1$, we obtain $|S| \geq \sum_{k=1}^{l-1} 2^k + 2^{l-1}$. Therefore, the set $\{e_{ij} : 1 \leq i \leq 1, 1 \leq j \leq 2^{l-1}\} \cup \{e_{lk} : k \equiv 1(mod)2, 1 \leq k \leq 2^l\}$ is a minimum connected forcing set of G. Hence $F_c(L(T)) = \sum_{k=1}^{l-1} 2^k + 2^{l-1}$.

Theorem 5.2. *For a regular spider $T = P_{n,m}$, $m > 2$, $F_c(L(T)) = m$.*
Proof. Let $G = L(T)$ and S be a connected forcing set of G. Denote by $E(T)$ the edge set of T as follows: $E(T) = \{e_i : 1 \leq i \leq m\} \cup \{e_{ij} : 1 \leq i \leq m, 1 \leq j \leq (n-1)\} \cup \{u\}$. Let e_i, $1 \leq i \leq m$ be the edges of T adjacent to the central vertex u of T. Then the induced subgraph $\langle e_1, e_2, e_3, ..., e_m \rangle \cong K_m$, in G. Also since G consists of m leaves, by Proposition 2.1, $|S| \geq m$. Therefore, the set of vertices $\{e_i, e_2, ..., e_m\}$ is a minimum connected forcing set of G, of cardinality m. Hence $F_c(G) = m$.
Since $L(P_{2,m}) \cong K_m \cdot K_1$, as an immediate consequence of Theorem 4.2, we have the following corollary.

Corollary 5.3. *For the Corona graph $K_m \cdot K_1$, $m \geq 3$, $F_c(K_m \cdot K_1) = m$.*

Theorem 5.4. *For a banana tree $T = B_{n,k}$, $F_c(L(T)) = n + n(k-1)$, $k \geq 2$.*
Proof. Let $G = L(T)$ and $E(T) = \{e_i : 1 \leq i \leq n\} \cup \{e_{ij} : 1 \leq i \leq n, 1 \leq j \leq k\}$, where $e_i^{'}$s $1 \leq i \leq n$ are the edges incident to the central vertex u with $degu = n$ and $e_{ij}, 1 \leq i \leq n, 1 \leq j \leq k$ are the edges of the star $K_{1,k}$ with edges $e_{i1}, 1 \leq i \leq n$ incident with the edges $e_i, 1 \leq i \leq n$, respectively. Since $e_1, e_2, e_3, ..., e_n$ are the cut-vertices of the graph G with $deg_G e_i = n-1, \{e_1, e_2, e_3, ..., e_n \subseteq S\}$, where S is a connected forcing set of G with $deg_G e_{i1} > 1, 1 \leq i \leq n$. Also since the edges $e_i, e_{i1}, 1 \leq i \leq n$, are the cut edges of G with $deg_G e_{i1} > 1, 1 \leq i \leq n$ and the subgraph induced by the vertices $\{e_{1j} : 1 \leq j \leq k\}, \{e_{2j} : 1 \leq j \leq k\}, \{e_{3j} : 1 \leq j \leq k\}, ..., \{e_{nj} : 1 \leq j \leq k\}$ are complete graphs K_n respectively, by Theorem 2.3, $F_c(K_k) = (k-1)$. Hence $\{e_{ij} : 1 \leq i \leq n, 1 \leq j \leq k-1\} \subseteq S$ and we have $|S| \geq n + n(k-1)$. Therefore $\{e_1, e_2, e_3, ..., e_n\} \cup \{e_{ij} : 1 \leq i \leq n, 1 \leq j \leq k-1\}$ is a minimum connected forcing set of G. Hence $F_c(L(T)) = n + n(k-1)$.

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