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PCCP OF CIRCULAR GRAPH FAMILY WITH A FAN GRAPH

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Abstract: A function f : V(G) ∪ E(G) ∪ R(G) → C is said to be perfect coloring of the graph G, if f(x) ≠ f(y) for any two adjoint or incident elements x,y ∈ V(G) ∪ E(G) ∪ R(G). And the PC number $\chi^P(G)$ is the least number of colors needed to assign colors to a graph by using perfect coloring. In this paper, we prove the results for perfect chromatic number of corona product (PCCP) of circular (cycle) graph family and a fan graph, which leads to perfect chromatic number equivalent to Δ+1, where Δ is the largest degree of the resultant graph.

Keywords and Phrases: Graph coloring, corona product, perfect coloring.

2020 Mathematics Subject Classification: 05.

1. Introduction

Graph coloring is a preeminent element of graph theory. It is having huge implementations in abundant disciplines like aircraft scheduling, register allocation, sudoku, mobile networking etc. The four color theorem plays important role in graph coloring [2]. The result of four color theorem was proved using PRN of that graph by Bhapkar [7]. The graph coloring basically deals with vertex, region and edge coloring. The coloring of elements (vertex, region or edge) of a connected graph such that adjoining element should receive dissimilar colors is the graph coloring. And the least colors needed to color a graph is the chromatic number [4]. The total coloring is coloring the vertices and edges of the graph so that no two adjoint vertices, vertices and its incident edges should not receive the same color. The minimum number of colors required to color graph G using total coloring is total chromatic number ($\chi^{''}(G)$) of graph G. Behzad proposed the concept of total coloring. In this type of coloring adjoint vertices and incident edges receives distinct colors [3, 14]. Rosenfeld [12] proved that every cubic graph is having total chromatic number five. Rong [9] proved that the face-edge chromatic number is equivalent to edge chromatic hence degree of the graph for any 2-connected planar graph with $\Delta \geq 24$. S. Mohan et al. [10] proposed the tight bounds of vizing's conjecture on total coloring of corona product of two graphs. Bhange and Bhapkar [6] proposed the concept of perfect coloring and proved the results for some standard graphs. Bhange and Bhapkar [5] proved that PCCP of cycle graph with null, cycle and chain graph is Δ+1 of resultant graph. S Nada et al. [11] stated the cordiality of the corona between cycle graphs and path graphs. In this paper, the PCCP of cycle graph and it's family i.e. sunlet graph, tadpole graph etc. is determined with fan graph.

2. Preliminaries

All graphs assumed in this paper are directionless and planar graphs. In the paper, (g_j, g_k) denotes an edge with end vertices g_j and g_k in graph G. Also, $C\{g_i\}$ and $C\{g_j, g_k\}$ are the color of vertex g_i and edge (g_j, g_k) respectively.

Definition 2.1. *Consider a graph G = (V(G),E(G),R(G)) having set of vertices V(G), set of edges E(G) and set of regions R(G), then perfect coloring is the mapping h : V(G) ∪ E(G) ∪ R(G)→ S, where S is set of colors with following conditions:*

(i) $h(a) \neq h(b)$ *for any two adjoint vertices a and b of G,*

(ii) $h(x) \neq h(y)$ *for any two adjoint edges x and y of G,*

(iii) $h(R_i) \neq h(R_j)$ *for any two adjoint regions* R_i *and* R_j *of G,*

(iv) $h(e) \neq h(x) \neq h(y)$*, where e is an edge connecting points x and y and*

(v) $h(p_i) \neq h(l_j) \neq h(R)$*, where* p_i *and* l_j *are frontier vertices and edges respectively creating region R.*

PC number or Perfect chromatic number ($\chi^P(G)$) *is least number of colors needed to color any graph which obeys conditions of perfect coloring* [5], [6].

Definition 2.2. *Let M and N be two graphs. Consider a copy of graph M and* $|V(M)|$ *(number of vertices in graph M) copies of graph N and keeping* j^{th} *vertex of M adjacent to each vertex of* j^{th} *copy of graph N, this gives corona product of graph M and N (MoN). Frucht and Harary* [8] *defined this corona product.*

Definition 2.3. *The minimum number of colors required to color a graph formed by corona product of two graphs is termed as Perfect chromatic number of Corona Product (PCCP)* [8].

Definition 2.4. *Tadpole graph $T_{n,m}$ is formed by connecting path graph P_m ($m \geq 2$) with any vertex t_i ($i = 1 : n$) of the cycle graph C_n ($n \geq 3$) such that $t_i = p_1$, where p_1 is first vertex of the path graph. The tadpole graph is with degree 3. The tadpole graph $T_{n,m}$ is having m+n vertices and edges with two regions.*

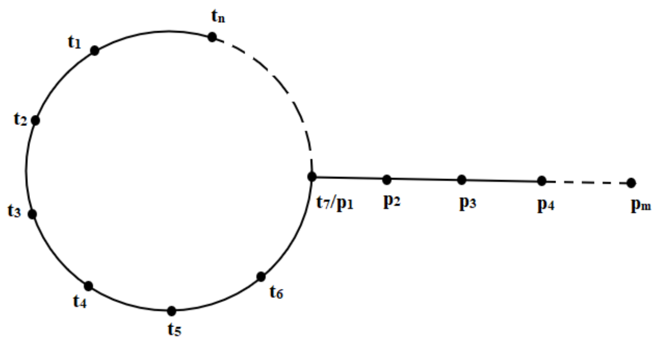

Figure 1: Tadpole graph $T_{n,m}$

Definition 2.5. *The sunlet graph is formed by the corona product of any cycle graph C_n with a null graph N_1 with one vertex. Sunlet graph S_n or corona graph is formed by connecting a pendent vertex to every vertex of the cycle graph C_n, with n vertices. The degree of corona graph is 3. The sunlet graph S_n is having 2n vertices and edges with two regions* [1], [13].

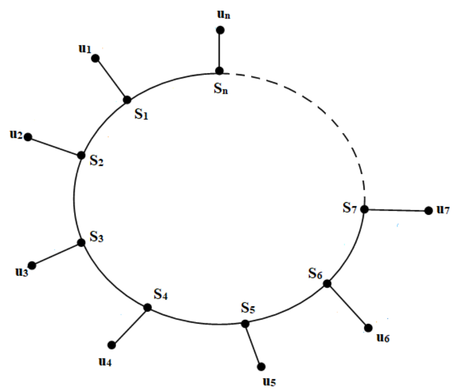

Figure 2: Sunlet graph S_n

Definition 2.6. *A fan graph F_n is formed by taking join of path graph P_n with n vertices ($n \geq 2$) with one vertex null graph O. The degree of the (n+1) vertex fan graph is n. The fan graph F_n is with n+1 vertices, 2n+1 edges and n+1 regions.*

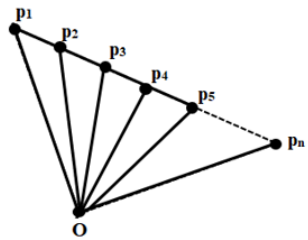

Figure 3: Fan graph F_n

3. PCCP with a Fan Graph

In this section, we prove the PCCP of Sunlet graph, tadpole graph and cycle graph with a fan graph F_m.

Theorem 3.1. *The PCCP of Sunlet graph S_n and fan graph F_m is $\Delta + 1$ or m+5, for all $n \geq 5$, $m \geq 4$.*

Proof. Consider the sunlet graph with 2n vertices with vertex set $\bigcup_{j=1}^{n} [\{s_j\} \cup \{u_j\}]$ and fan graph with m+1 vertices having vertex set $\bigcup_{k=1}^{m} [\{f_k\} \cup \{d\}]$. The corona product of sunlet graph S_n with fan graph F_m is as shown in Figure 4. Consider the corona product at any vertex s_j as it is highest degree vertex after corona product. For the sake of definiteness consider corona product at s_1. Hence the vertex set of resultant graph after corona product at s_1 is given as $\{\bigcup_{k=1}^{m} \{f_k\} \cup \{d\} \cup \{s_1\} \cup \{u_1\}\}$ Assign colors to vertices and edges of the resultant graph so that no two adjacent vertices, adjacent edges and vertices and it's incident edges should not receive the same color. The vertex s_1 is adjacent to vertex u_1 and all $f_k' s$. Also, the vertex d is adjacent to all $f_k' s$ and s_1. Hence assign colors to vertices and edges as below.

$$\forall 1 \leq j \leq n/3 \begin{cases} C\{s_{3j-2}\} = C\{s_{3j-1}, s_{3j}\} = 1, \\ C\{s_{3j-1}\} = 3, \\ C\{s_{3j}\} = C\{s_{3j-2}, s_{3j-1}\} = 2. \end{cases} \tag{1}$$

$$\forall\ j = 1 : (n-3)/3, C\{s_{3j}, s_{3j+1}\} = C\{s_n, s_1\} = 3. \tag{2}$$

$$\forall 1 \leq j \leq n \begin{cases} C\{u_j\} = 3 \text{ if } C\{s_j\} = 1, \\ C\{u_j\} = 1 \text{ if } C\{s_j\} = 2, \\ C\{u_j\} = 2 \text{ if } C\{s_j\} = 3 \\ C\{s_j, u_j\} = 4. \end{cases}$$

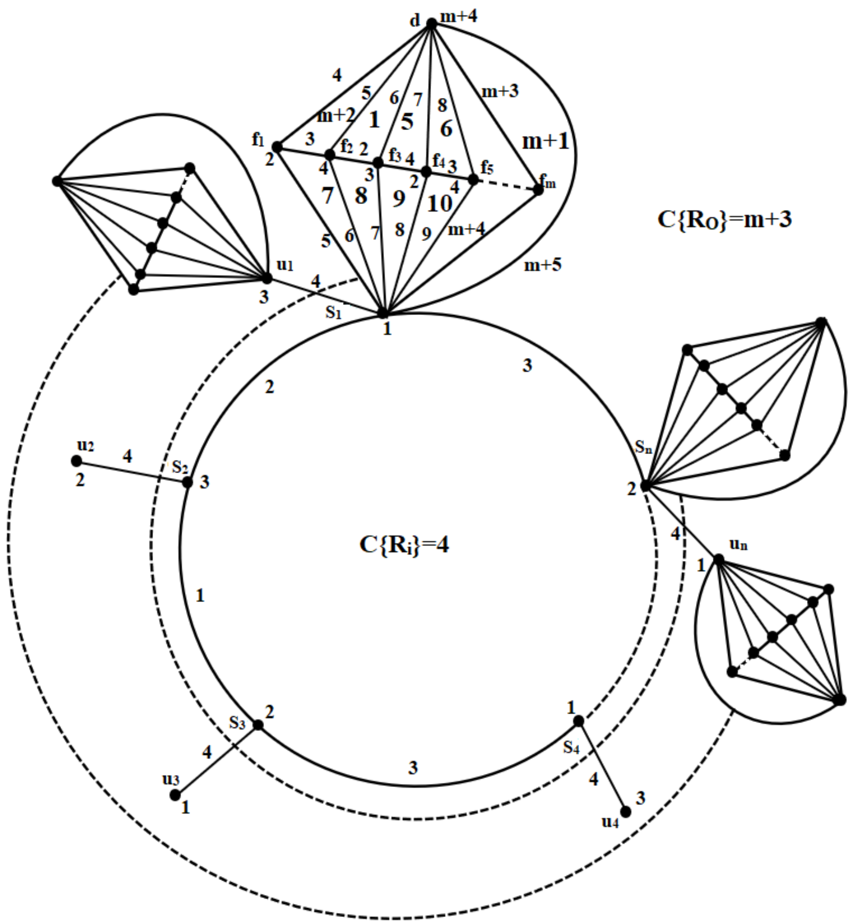

Figure 4: PCCP of sunlet graph S_n with fan graph F_m

$$\forall 1 \leq k \leq m/3 \begin{cases} C\{f_{3k-2}\} = C\{f_{3k-1}, f_{3k}\} = 2, \\ C\{f_{3k-1}\} = 4, \\ C\{f_{3k}\} = C\{f_{3k-2}, f_{3k-1}\} = 3. \end{cases} \quad (3)$$

$\forall$ k=1:(m-3)/3 , $C\{f_{3k}, f_{3k+1}\} = 4.$

$$\forall 1 \leq k \leq m \begin{cases} C\{s_j, f_k\} = k+4, \\ C\{d, f_k\} = k+3. \end{cases} \tag{4}$$

Also,

$$C\{d, s_j\} = m+5, C\{d\} = m+4.$$

The same result can be proved for $C\{s_j\} = 2$ and $C\{s_j\} = 3$.
Hence the total chromatic number is $\chi^{''}(S_n o F_m)$ = m+5 = Δ+1.
To find the PC (perfect chromatic) number of the graph, Assign colors to the regions as below.
Consider R_i as internal region of the sunlet graph. Assign color to this region using color apart from colors assigned to vertices and edges forming this region. Hence

$$C\{R_i\} = 4$$

Now, divide internal region of graph created by corona product with fan graph into two parts. Denote lower region as R_{sk} and upper region as R_{dk}. Here, corresponding $R'_{sk}s$ and $R'_{dk}s$ (k = 1:m) are adjacent to each other. Hence assign colors so that no two regions sharing common edge should receive the same color. Assign colors to these regions so that it should not receive the same color as its adjacent elements (vertices, edges, regions).

$$\forall 1 \leq k \leq m-2 \begin{cases} C\{R_{sk}\} = k+6, \\ C\{R_{s(m-1)}\} = C\{s_j, f_1\}. \end{cases}$$

Also,

$$C\{R_{d(1)}\} = m+2,$$
$$C\{R_{d(2)}\} = 1,$$

and

$$C\{R_{dk}\} = k+4; \forall 3 \leq k \leq m-1.$$

Finally,

$$C\{R_{ds}\} = m+1$$

and

$$C\{R_O\} = m+3.$$

where, R_{ds} is the region between d and s_j and R_O is open region. Which gives,

$$\chi^P(S_n o F_m) = m+5 = \Delta+1.$$

Hence the proof.

Theorem 3.2. *The PCCP of tadpole graph $T_{n,r}$ and fan graph F_m is $\Delta + 1$ or m+5, for all $n \geq 5$, $r \geq 2$, $m \geq 4$.*
Proof. The tadpole graph is formed by connecting path graph P_r with cycle graph C_n. Consider c_p is the vertex of cycle graph from which the path graph begins, so that $c_p = p_1$. Hence c_p is the highest degree vertex among all $c_i's$, $i = 1 : n$. So consider the corona product of the tadpole graph with fan graph at the vertex c_p only. The path graph is having total chromatic number 3. Hence the result can be proved using analogy of Theorem 3.1.

Theorem 3.3. *The PCCP of cycle graph C_n and fan graph F_m is $\Delta + 1$ or m+4, for all $n \geq 5$, $m \geq 4$.*
Proof.

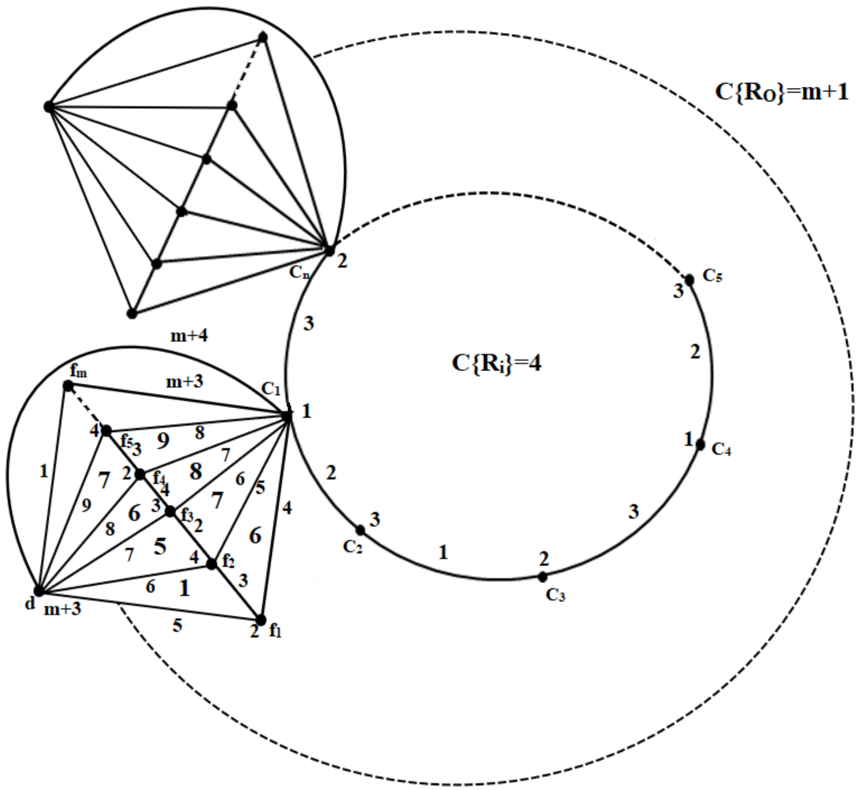

Figure 5: PCCP of C_n with fan graph F_m

Consider a cycle graph C_n with n vertices and fan graph F_m with m+1 vertices.

The vertex set of cycle graph is given as $\bigcup_{j=1}^{n}\{c_j\}$. The vertex set of fan graph is $\bigcup_{k=1}^{m}\{f_k\} \cup \{d\}$. Consider the corona product at any vertex c_j of cycle graph as all vertices are with unique degree. For the sake of definiteness, consider the corona product at vertex c_1 as shown in Figure 5. Hence the vertex set after corona product at vertex c_1 is $\bigcup_{k=1}^{m}\{f_k\} \cup \{d\} \cup \{c_1\}$. Assign colors to the vertices and edges so that no two adjacent vertices, adjacent edges and vertices and it's incident edges should receive non identical colors. Assign colors to the cycle graph using equation (1) and (2). Color the attached fan graph using equation (3).

$$C\{d, c_j\} = m + 4$$

$$C\{c_j, f_k\} = k + 3; \forall k = 1 : m$$

$$C\{d, f_k\} = k + 4; \forall k = 1 : m - 1$$

and

$$C\{d, f_m\} = 1, C\{d\} = m + 3$$

In the similar way, the result can be proved for $C\{c_j\} = 2$ and $C\{c_j\} = 3$.
Which gives

$$\chi''(C_n o F_m) = m + 4 = \Delta + 1.$$

To get the PC (perfect chromatic) number of the graph, assign colors to the regions as below. Assign color to the internal region R_i of cycle graph using color apart from the colors of vertices and edges forming the region. Hence

$$C\{R_i\} = 4$$

Now, divide the internal region formed by corona product with fan graph into two parts.
Denote the lower region as R_{ck} and the upper region as R_{dk}. Assign separate colors to adjacent regions of $R'_{ck}s$ and $R'_{dk}s$ (k=1 :m).
Hence

$$\forall 1 \leq k \leq m - 1,\ C\{R_{ck}\} = k + 5,$$

Also,

$$C\{R_{d(1)}\} = 1.$$

$$\forall 2 \leq k \leq m - 1,\ C\{R_{dk}\} = k + 3$$

Finally,

$$C\{R_{dc}\} = 5,\ C\{R_O\} = m + 1.$$

where, R_{dc} is region between d and c_j and R_O is open region. Which proves Perfect chromatic number is

$$\chi^P(C_n o F_m) = m + 4 = \Delta + 1.$$

Hence the proof.

4. Conclusions

The PCCP of cycle graph and it's family with a fan graph having m vertices has been explored. It is shown that total chromatic number and perfect chromatic number are equivalent to Δ+1, where Δ is highest degree of the graph in corona product. The perfect coloring is used in different types of scheduling, timetabling, register allocation, sudoku, GSM mobile network allocation etc.

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