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Hypergeometric relations among Jacobis theta functions

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Abstract: In this paper, an attempt has been made to evaluate and established certain hypergeometric relations containing Jacobi's theta functions.

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Mathematics subject Classification: 33A30, 33D15, 33D20

1. Introduction, Notations and Definitions

Jacobi in 1829 defined following four functions which are called Jacobi's theta – functions;

$$\theta_1(z,q) = 2\sum_{n=0}^{\infty}(-)^n q^{\left(n+\frac{1}{2}\right)^2}\sin(2n+1)z, \tag{1.1}$$

$$\theta_2(z,q) = 2\sum_{n=0}^{\infty} q^{\left(n+\frac{1}{2}\right)^2}\cos(2n+1)z, \tag{1.2}$$

$$\theta_3(z,q) = 1+2\sum_{n=1}^{\infty} q^{n^2}\cos 2nz, \tag{1.3}$$

and

$$\theta_4(z,q) = 1+2\sum_{n=1}^{\infty}(-)^n q^{n^2}\cos 2nz. \tag{1.4}$$

For the absolute convergence of these functions we need $|q| < 1$. Sometimes we use the additional notation $q = e^{\pi i\tau}$, where $\mathrm{Im}(\tau) > 0$.
It is easy to see that

$$\left.\begin{array}{l}\theta_1\left(z+\frac{\pi}{2},q\right) = \theta_2(z),\\ \theta_2\left(z+\frac{\pi}{2},q\right) = -\theta_1(z),\\ \theta_3\left(z+\frac{\pi}{2},q\right) = \theta_4(z)\\ \text{and}\\ \theta_4\left(z+\frac{\pi}{2},q\right) = \theta_3(z).\end{array}\right\} \tag{1.5}$$

From (1.5) we have

$$\left.\begin{array}{l}\theta_1(z+\pi,q)=-\theta_1(z),\theta_2(z+\pi,q)=-\theta_2(z)\\ \theta_3(z+\pi,q)=\theta_3(z) \quad \text{and} \quad \theta_4(z+\pi,q)=\theta_4(z).\end{array}\right\} \tag{1.6}$$

Also,

$$\left.\begin{array}{l}\theta_1\left(z+\frac{\pi\tau}{2},q\right)=-iq^{-\frac{1}{4}}e^{-zi}\theta_4(z,q),\\ \theta_2\left(z+\frac{\pi\tau}{2},q\right)=q^{-\frac{1}{4}}e^{-zi}\theta_3(z,q),\\ \theta_3\left(z+\frac{\pi\tau}{2},q\right)=q^{-\frac{1}{4}}e^{-zi}\theta_2(z,q)\\ \theta_4\left(z+\frac{\pi\tau}{2},q\right)=-iq^{-\frac{1}{4}}e^{-zi}\theta_1(z,q).\end{array}\right\} \tag{1.6(A)}$$

From (1.6(A)) we have

$$\left.\begin{array}{l}\theta_1(z+\pi\tau,q)=-q^{-1}e^{-2zi}\theta_1(z,q),\\ \theta_2(z+\pi\tau,q)=q^{-1}e^{-2zi}\theta_2(z,q),\\ \theta_3(z+\pi\tau,q)=q^{-1}e^{-2zi}\theta_3(z,q)\\ \theta_4(z+\pi\tau,q)=-q^{-1}e^{-2zi}\theta_4(z,q).\end{array}\right\}. \tag{1.6(B)}$$

Also,

$$\left.\begin{array}{l}\theta_1(z-\pi,q)=-\theta_1(z,q),\\ \theta_2(z-\pi,q)=-\theta_2(z,q),\\ \theta_3(z-\pi,q)=\theta_3(z,q),\\ \theta_4(z-\pi,q)=\theta_4(z,q)\end{array}\right\}. \tag{1.6(C)}$$

and

$$\left.\begin{array}{l}\theta_1(z-\pi\tau,q)=-q^{-1}e^{2zi}\theta_1(z,q),\\ \theta_2(z-\pi\tau,q)=q^{-1}e^{2zi}\theta_2(z,q),\\ \theta_3(z-\pi\tau,q)=q^{-1}e^{2zi}\theta_3(z,q),\\ \theta_4(z-\pi\tau,q)=-q^{-1}e^{2zi}\theta_4(z,q).\end{array}\right\}. \tag{1.6(D)}$$

Now, it is clear that $\theta_1(z,q)$ and $\theta_2(z,q)$ have common period 2π and $\theta_3(z,q)$,$\theta_4(z,q)$ have common period π.

As $z \to 0$ we find

$$\theta_1(q)=0,\theta_2(q)=2\sum_{n=0}^{\infty}q^{\left(n+\frac{1}{2}\right)^2}$$

$$=2q^{\frac{1}{4}}\sum_{n=0}^{\infty}q^{n^2+n}$$

So,

$$\theta_2(q)=q^{\frac{1}{4}}\left[\sum_{n=0}^{\infty}q^{n^2+n}+\sum_{n=0}^{\infty}q^{n^2+n}\right].$$

Putting n – 1 for n in the second summation we have,

$$\theta_2(q) = q^{\frac{1}{4}}\left[\sum_{n=0}^{\infty} q^{n^2+n} + \sum_{n=1}^{\infty} q^{n^2-n}\right].$$

Again, putting –n for n in the second summation we obtain,

$$\theta_2(q) = q^{\frac{1}{4}}\sum_{n=-\infty}^{\infty} q^{n^2+n}. \tag{1.7}$$

Similarly

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \theta_4(q) = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2}. \tag{1.8}$$

The most important identity involving these theta functions is

$$\theta_3^4(q) = \theta_2^4(q) + \theta_4^4(q), \tag{1.9}$$

which can also be written as,

$$q\left\{\sum_{n=-\infty}^{\infty} q^{n^2+n}\right\}^4 + \left\{\sum_{n=-\infty}^{\infty} (-)^n q^{n^2}\right\}^4 = \left\{\sum_{n=-\infty}^{\infty} q^{n^2}\right\}^4. \tag{1.10}$$

2. Some more identities involving theta functions;

(i)

$$\theta_3(q) + \theta_3(-q) = 2\theta_3\left(q^4\right) \tag{2.1}$$

Proof; Since,

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

$$\theta_3(-q) = \sum_{n=-\infty}^{\infty} (-q)^{n^2} = \sum_{n=-\infty}^{\infty} (-)^n (q)^{n^2}$$

So,

$$\theta_3(q) + \theta_3(-q) = \sum_{n=-\infty}^{\infty} \{1 + (-)^n\} q^{n^2}.$$

Now, taking n = 2k (even) we find

$$\theta_3(q) + \theta_3(-q) = 2\sum_{n=-\infty}^{\infty} q^{4k^2} = 2\sum_{n=-\infty}^{\infty} \left(q^4\right)^{k^2}$$

$$= 2\theta_3\left(q^4\right).$$

(ii)

$$\theta_3(q) - \theta_3(-q) = 2\theta_2\left(q^4\right) \tag{2.2}$$

Proof;

$$\theta_3(q) - \theta_3(-q) = \sum_{n=-\infty}^{\infty} \{1-(-)^n\} q^{n^2}$$

Taking n = 2k + 1 (odd) we get,

$$\theta_3(q) - \theta_3(-q) = 2q \sum_{n=-\infty}^{\infty} \left(q^4\right)^{k^2+k}$$

$$= 2\left(q^4\right)^{\frac{1}{4}} \sum_{k=-\infty}^{\infty} \left(q^4\right)^{k^2+k}$$

$$= 2\theta_2\left(q^4\right). \tag{2.3}$$

(iii)

$$\theta_3^2(q) - \theta_3^2(-q) = 4\theta_2\left(q^4\right)\theta_3\left(q^4\right) \tag{2.4}$$

Proof;
Multiplying (2.1) and (2.2) we get (2.4).
(iv) $\theta_3(q)\,\theta_4(q) = \theta_4^2\left(q^2\right)$
Applying Jacobi's triple product identity, viz.,

$$\sum_{n=-\infty}^{\infty} (-)^n q^{kn^2}\left(z^l\right)^n = \left(q^{2k};q^{2k}\right)_\infty \left(q^k z^l;q^{2k}\right)_\infty \left(q^k/z^l;q^{2k}\right)_\infty,$$

where

$$\left(a;q^k\right)_\infty = \prod_{r=0}^{\infty}\left(1-aq^{kr}\right),$$

we have

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \left(q^2;q^2\right)_\infty \left(-q;q^2\right)_\infty^2 \tag{2.5}$$

and

$$\theta_4(q) = \sum_{n=-\infty}^{\infty} (-)^n q^{n^2} = \left(q^2;q^2\right)_\infty \left(q;q^2\right)_\infty^2. \tag{2.6}$$

Multiplying (2.5) and (2.6) we have

$$\begin{aligned}\theta_3(q)\,\theta_4(q) &= \left(q^2;q^2\right)_\infty^2 \left\{\left(-q;q^2\right)_\infty \left(q;q^2\right)_\infty\right\}^2 \\ &= \left(q^2;q^2\right)_\infty^2 \left(q^2;q^4\right)_\infty^2 \\ &= \left(q^2;q^4\right)_\infty^2 \left(q^4;q^4\right)_\infty^2 \left(q^2;q^4\right)_\infty^2 \\ &= \left(q^4;q^4\right)_\infty^2 \left(q^2;q^4\right)_\infty^4 \\ &= \theta_4^2\left(q^2\right). \qquad (2.7)\end{aligned}$$

(v)

$$\theta_3^2(q) + \theta_4^2(q) = \theta_3^2\left(q^2\right). \qquad (2.8)$$

$$\theta_3^2(q) = \sum_{m=-\infty}^{\infty} q^{m^2} \sum_{n=-\infty}^{\infty} q^{n^2} = \sum_{m,n=-\infty}^{\infty} q^{m^2+n^2}$$

So,

$$\theta_3^2(q) + \theta_4^2(q) = \sum_{m,n=-\infty}^{\infty} \left\{1 + (-)^{m+n}\right\} q^{m^2+n^2}. \qquad (2.9)$$

Taking m + n = 2k (even),
we have m − n = 2j (even),
which give m = k + j and n = k − j.
Putting these values in (2.9) we get,

$$\begin{aligned}\theta_3^2(q) + \theta_4^2(q) &= 2\sum_{k,j=-\infty}^{\infty} \left(q^2\right)^{k^2+j^2} \\ &= 2\theta_3^2\left(q^2\right). \qquad (2.10)\end{aligned}$$

(vi) $\theta_3^4(q) - \theta_4^4(q) = 8\theta_2\left(q^4\right)\theta_3\left(q^4\right)\theta_3^2\left(q^2\right).$
we know that

$$\theta_3(-q) = \theta_4(q)$$

So, from (2.4) we find

$$\theta_3^2(q) - \theta_4^2(q) = 4\theta_2\left(q^4\right)\theta_3\left(q^4\right). \qquad (2.11)$$

Multiplying (2.10) and (2.11) we get,

$$\theta_3^4(q) - \theta_4^4(q) = 8\theta_2\left(q^4\right)\theta_3\left(q^4\right)\theta_3^2\left(q^2\right). \qquad (2.12)$$

3. Hypergeometric representation of theta function

Let us consider the following transformation;

$$ {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\left(\frac{1-x}{1+x}\right)^2\right]=(1+x)\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;x^2\right]. \tag{3.1}$$

[Bruce ; (3.2) p. 98]

Taking

$$\frac{1-x}{1+x}=\frac{\theta_4^2\left(q\right)}{\theta_3^2\left(q\right)},$$

we get

$$x=\frac{\theta_3^2\left(q\right)-\theta_4^2\left(q\right)}{\theta_3^2\left(q\right)+\theta_4^2\left(q\right)}$$

and

$$x^2=1-\frac{4\theta_3^2\left(q\right)\theta_4^2\left(q\right)}{\left\{\theta_3^2\left(q\right)+\theta_4^2\left(q\right)\right\}^2}. \tag{3.2}$$

Putting these values in (3.1) we obtain

$$ {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4\left(q\right)}{\theta_3^4\left(q\right)}\right]=\frac{2\theta_3^2\left(q\right)}{\left\{\theta_3^2\left(q\right)+\theta_4^2\left(q\right)\right\}}\times$$

$$\times{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{4\theta_3^2\left(q\right)\theta_4^2\left(q\right)}{\left\{\theta_3^2\left(q\right)+\theta_4^2\left(q\right)\right\}^2}\right]. \tag{3.3}$$

Now using (2.7) and (2.8) we have,

$$ {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4\left(q\right)}{\theta_3^4\left(q\right)}\right]=\frac{\theta_3^2\left(q\right)}{\theta_3^2\left(q^2\right)}{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4\left(q^2\right)}{\theta_3^4\left(q^2\right)}\right]. \tag{3.4}$$

Iterating this process m times and writing $n=2^m$ we find,

$$ {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4\left(q\right)}{\theta_3^4\left(q\right)}\right]=\frac{\theta_3^2\left(q\right)}{\theta_3^2\left(q^n\right)}{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4\left(q^n\right)}{\theta_3^4\left(q^n\right)}\right]. \tag{3.5}$$

Since $|q|<1$, so, as $n\to\infty$
we find

$$\theta_3\left(q^n\right)=\theta_3\left(0\right)=1$$

$$\theta_4\left(q^n\right)=\theta_4\left(0\right)=1$$

Thus (3.5) yields,

$$\theta_3^2(q) = {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4(q)}{\theta_3^4(q)}\right]. \tag{3.6}$$

Applying the identity (1.9) we get,

$$\theta_3^2(q) = {}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_2^4(q)}{\theta_3^4(q)}\right]. \tag{3.7}$$

Again, consider another transformation,

$${}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\left(\frac{1-x}{1+x}\right)^2\right] = \frac{1+x}{2}\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{4x}{(1+x)^2}\right]. \tag{3.8}$$

Taking $\dfrac{1-x}{1+x} = \dfrac{\theta_4^2(q)}{\theta_3^2(q)}$ in (3.8) we get,

$${}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q)}{\theta_3^4(q)}\right] = \frac{\theta_3^2(q)}{\{\theta_3^2(q)+\theta_4^2(q)\}}\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q^2)}{\theta_3^4(q^2)}\right]. \tag{3.9}$$

Applying (2.8), (3.9) yields;

$${}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q)}{\theta_3^4(q)}\right] = \frac{\theta_3^2(q)}{2\theta_3^2(q^2)}\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q^2)}{\theta_3^4(q^2)}\right]. \tag{3.10}$$

Iterating it m times and putting $n = 2^m$ we get,

$${}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q)}{\theta_3^4(q)}\right] = \frac{\theta_3^2(q)}{n\theta_3^2(q^n)}\,{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q^n)}{\theta_3^4(q^n)}\right]. \tag{3.11}$$

Now dividing (3.5) by (3.11), multiplying both sides by $-\pi$ and then taking exponential we obtain,

$$\exp\left[-\pi\frac{{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4(q)}{\theta_3^4(q)}\right]}{{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q)}{\theta_3^4(q)}\right]}\right] = \left\{\exp\left[-\pi\frac{{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;1-\frac{\theta_4^4(q^n)}{\theta_3^4(q^n)}\right]}{{}_2F_1\left[\frac{1}{2},\frac{1}{2};1;\frac{\theta_4^4(q^n)}{\theta_3^4(q^n)}\right]}\right]\right\}^n \tag{3.12}$$

We can write it as;

$$F\left(\frac{\theta_4^4(q)}{\theta_3^4(q)}\right) = F^n\left(\frac{\theta_4^4(q^n)}{\theta_3^4(q^n)}\right). \tag{3.13}$$

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