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MULTIPLICATIVE VERSIONS OF BANHATTI INDICES

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Abstract: A topological index is a numeric quantity obtained from a graph structure that is invariant under graph isomorphism. Generally, vertex degree-based topological indices take into account the contributions of pairs of adjacent vertices. But, in Banhatti indices are contributions of pairs of incident elements. Particularly, the concept of Zagreb and Banhatti indices was established in chemical graph theory based on vertex degrees. Analogously, we initiate the study of multiplicative versions of the Banhatti indices of a graph. The main goal of this paper is to shed light on the relationship between the multiplicative Banhatti indices and other degree-based topological indices by using certain classical inequalities.

Keywords and Phrases: Graphical indices, Multiplicative Banhatti indices, multiplicative Zagreb indices and inequalities.

2020 Mathematics Subject Classification: 05C07, 05C09, 05C92.

1. Introduction

Throughout this paper, graph is simple, connected, undirected, and without loop. Let $G = (V, E)$ be graph with $|V(G)| = n$ vertices and $|E(G)| = m$ edges. The maximum and minimum degrees of the graphs represented by $\Delta = \Delta(G)$ and

$\delta = \delta(G)$ respectively. For a vertex $v \in V(G)$ and $d_G(v)$ denotes the degree of v and for an edge $e = uv \in E(G)$ and $d_G(e)$ denotes the degree of an edge e, which is defined as $d_G(e) = d_G(u) + d_G(v) - 2$. For graph theoretic notations and terminologies, we follow to [9].

A graph can be identified by a numeric number, a polynomial, or a matrix that represents the entire graph, and these representations are intended to be unique to the graph. The whole structure of the graph is characterised by a numeric quantity called the topological index. Numerous such indices have been considered in theoretical chemistry and have found some applications, especially in QSPR / QSAR / QSTR research, see [6, 16].

The first and second multiplicative versions of the Zagreb indices were defined in [3, 17, 18] and these indices are defined as,

$$\Pi_1(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)] \text{ and } \Pi_2(G) = \prod_{uv \in E(G)} [d_G(u) d_G(v)].$$

In [10], the first and second multiplicative hyper Zagreb indices are

$$H\Pi_1(G) = \prod_{uv \in E(G)} [d_G(u) + d_G(v)]^2 \text{ and } H\Pi_2(G) = \prod_{uv \in E(G)} [d_G(u) d_G(v)]^2.$$

Followed by the first and second forgotten indices [4], Bhanumathi [1] and Ghobadi [5] defined the multiplicative Forgotten index and is defined as

$$F\Pi_1(G) = \prod_{uv \in E(G)} [d_G(u)^2 + d_G(v)^2] \text{ and } F\Pi_2(G) = \prod_{uv \in E(G)} [d_G(u)^2 d_G(v)^2].$$

In 2004, Milicevic et al., [14] defined reformulated Zagreb indices in terms of edge-degree defined as follows,

$$EM_1(G) = \sum_{e \in E(G)} d_G(e)^2 \quad \text{and} \quad EM_2(G) = \sum_{e \sim f} d_G(e) d_G(f).$$

Bearing in mind, additive version of the reformulated Zagreb indices, the multiplicative reformulated Zagreb indices are defined as follows,

$$E\Pi_1(G) = \prod_{e \in E(G)} d_G(e)^2 \quad \text{and} \quad E\Pi_2(G) = \prod_{e \sim f} d_G(e) d_G(f).$$

The multiplicative Banhatti indices and multiplicative hyper Banhatti indices were defined in [11, 12, 13] and these indices are defined as follows,

$$B\Pi_1(G) = \prod_{ue} [d_G(u) + d_G(e)] \quad \text{and} \quad B\Pi_2(G) = \prod_{ue} [d_G(u) . d_G(e)].$$

$$HB\Pi_1(G) = \prod_{ue}[d_G(u) + d_G(e)]^2 \quad \text{and} \quad HB\Pi_2(G) = \prod_{ue}[d_G(u)d_G(e)]^2.$$

In 1972, the first Zagreb index was introduced by Gutman and Trinajstic [7]. It is an important molecular descriptor and has been closely correlated with many chemical properties. The first Zagreb index of G is defined as

$$M_1(G) = \sum_{u \in V(G)} d_G(u)^2 = \sum_{uv \in E(G)} \big[d_G(u) + d_G(v)\big].$$

Bearing in mind the alternative form of above index can be consider a multiplicative version of the first Zagreb index $\Pi_1(G)$. But, $\Pi_1(G) = \prod_{uv \in E(G)}[d_G(u) + d_G(v)] \neq \prod_{u \in V(G)} d_G(u)^2$. For example, the Path P_3, their values are 4 and 9, respectively. Similarly, the multiplicative first Banhatti index $B\Pi_1(G) = \prod_{ue}[d_G(u) + d_G(e)] \neq \prod_{u \in V(G)} d_G(u)^2$. In view of the above facts, the Zagreb and Banhatti indices are closely related but not in general. The main goal of this paper is to shed light on the relationship between multiplicative Banhatti indices and other degree based topological indices. For more details, we refer to [2, 8].

2. Comparison of multiplicative Banhatti and Zagreb type indices

Theorem 2.1. *For any graph G, the first multiplicative Banhatti index is related to multiplicative Zagreb type indices as*

(i) $B\Pi_1(G) = 2H\Pi_1(G) + \Pi_2(G) - 6\Pi_1(G) + 4^m$

(ii) $B\Pi_1(G) = E\Pi_1(G) + H\Pi_1(G) - 2\Pi_1(G) + \Pi_2(G)$

(iii) $B\Pi_1(G) = 2F\Pi(G) + 5\Pi_2(G) - 6\Pi_1(G) + 4^m$

(iv) $B\Pi_1(G) = E\Pi_1(G) + F\Pi_1(G) - 2\Pi_1(G) + 3\Pi_2(G)$.

Proof. Let G be a (n, m)- graph with $n \geq 3$ and an edge degree $d_G(e) = d_G(uv) = d_G(u) + d_G(v) - 2$ for every $e = uv \in E(G)$.

(i) We have,

$$\begin{aligned} B\Pi_1(G) &= \prod_{ue}[d_G(u) + d_G(e)] \\ &= \prod_{uv \in E(G)} [d_G(u) + d_G(uv)] \times \prod_{uv \in E(G)} [d_G(v) + d_G(uv)] \\ &= \prod_{uv \in E(G)} \{2[d_G(u) + d_G(v)]^2 + d_G(u)d_G(v) - 6[d_G(u) + d_G(v)] + 4\} \end{aligned}$$

$$B\Pi_1(G) = 2H\Pi_1(G) + \Pi_2(G) - 6\Pi_1(G) + 4^m.$$

(ii) First, we consider the first multiplicative reformulated Zagreb index

$$\begin{aligned} E\Pi_1(G) &= \prod_{e\in E(G)} d_G(e)^2 = \prod_{uv\in E(G)} [d_G(u)+d_G(v)-2]^2 \\ &= \prod_{uv\in E(G)} [(d_G(u)+d_G(v))^2 - 4(d_G(u)+d_G(v))+4] \\ E\Pi_1(G) &= H\Pi_1(G) - 4\Pi_1(G) + 4^m. \end{aligned} \tag{2.1}$$

On simplification of the result (i) and an equation (2.1), we have

$$B\Pi_1(G) = E\Pi_1(G) + H\Pi_1(G) - 2\Pi_1(G) + \Pi_2(G).$$

(iii) We have

$$\begin{aligned} B\Pi_1(G) &= \prod_{ue} [d_G(u)+d_G(e)] \\ &= \prod_{uv\in E(G)} [d_G(u)+d_G(uv)] \times \prod_{uv\in E(G)} [d_G(v)+d_G(uv)] \\ &= \prod_{uv\in E(G)} \{2[d_G(u)^2+d_G(v)^2] + 5d_G(u)d_G(v) \\ &\quad - 6[d_G(u)+d_G(v)] + 4m\} \\ B\Pi_1(G) &= 2F\Pi(G) + 5\Pi_2(G) - 6\Pi_1(G) + 4^m. \end{aligned}$$

(iv) First, we consider the first multiplicative reformulated Zagreb index

$$\begin{aligned} E\Pi_1(G) &= \prod_{e\in E(G)} d_G(e)^2 = \prod_{uv\in E(G)} [d_G(u)+d_G(v)-2]^2 \\ &= \prod_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2 + 2d_G(u)d_G(v) \\ &\quad - 4(d_G(u)+d_G(v)) + 4] \\ E\Pi_1(G) &= F\Pi(G) + 2\Pi_2 - 4\Pi_1(G) + 4^m. \end{aligned} \tag{2.2}$$

On simplification of the result (iii) and an equation (2.2), we have

$$B\Pi_1(G) = E\Pi_1(G) + F\Pi_1(G) - 2\Pi_1(G) + 3\Pi_2(G).$$

Thus the result follows.

Theorem 2.2. *For any connected graph G, the multiplicative second Banhatti index is related to the multiplicative Zagreb type indices as*

(i) $B\Pi_2(G) = \Pi_2(G)[H\Pi_1(G) - 4\Pi_1(G) + 4^m]$

(ii) $B\Pi_2(G) = \Pi_2(G)[F\Pi(G) + 2\Pi_2(G) - 4\Pi_1(G) + 4^m]$

(iii) $B\Pi_2(G) = \Pi_2(G).E\Pi_1(G)$.

Proof. Let G be a (n, m)- graph with $n \geq 3$ and an edge degree $d_G(e) = d_G(uv) = d_G(u) + d_G(v) - 2$.

(i) Consider

$$\begin{aligned}
B\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)] \\
&= \prod_{uv\in E(G)}[d_G(u)d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v)d_G(uv)] \\
&= \prod_{uv\in E(G)} d_G(u)d_G(v)[d_G(u) + d_G(v) - 2]^2 \\
&= \prod_{uv\in E(G)} d_G(u)d_G(v)[(d_G(u) + d_G(v))^2 \\
&\quad - 4(d_G(u) + d_G(v)) + 4] \\
B\Pi_2(G) &= \Pi_2(G)[H\Pi_1(G) - 4\Pi_1(G) + 4^m].
\end{aligned}$$

(ii) Consider

$$\begin{aligned}
B\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)] = \prod_{uv\in E(G)} d_G(u)d_G(v)[d_G(u) + d_G(v) - 2]^2 \\
&= \prod_{uv\in E(G)} d_G(u)d_G(v)[(d_G(u)^2 + d_G(v)^2 + 2d_G(u)d_G(v) - 4(d_G(u) \\
&\quad + d_G(v)) + 4] \\
B\Pi_2(G) &= \Pi_2(G)[F\Pi(G) + 2\Pi_2(G) - 4\Pi_1(G) + 4^m].
\end{aligned}$$

(iii) On simplification of the result (i) and an equation (2.1). Also, the result (iii) and an equation (2.2), we have

$$B\Pi_2(G) = \Pi_2(G).E\Pi_1(G).$$

Thus the result follows.

Theorem 2.3. *For any connected graph G, the multiplicative hyper Banhatti index is related to the multiplicative Zagreb type indices as*

$$(i)\ HB\Pi_1(G) = (H\Pi_1(G))^2 - 6\Pi_1(G)H\Pi_1(G) + 13H\Pi_1(G) - 12\Pi_1(G) + \Pi_2(G)H\Pi_1(G) + \frac{H\Pi_2(G)}{4} + 2\Pi_2(G) + 4^m$$

$$(ii)\ HB\Pi_1(G) = [F\Pi(G)]^2 + \frac{25}{4}H\Pi_2(G) + 5\Pi_2(G)F\Pi(G) - 6[\Pi_1(G)]^3 - 3\Pi_2(G)\Pi_2(G) + 13H\Pi_1(G) + 2\Pi_2(G) - 12\Pi_1(G) + 4^m$$

Proof. Let G be a (n, m)- graph with $n \geq 3$.

(i) Consider

$$\begin{aligned}
HB\Pi_1(G) &= \prod_{ue}[d_G(u) + d_G(e)]^2 \\
&= \prod_{uv\in E(G)} [d_G(u) + d_G(e)]^2 \times \prod_{uv\in E(G)} [d_G(v) + d_G(e)]^2 \\
&= \prod_{uv\in E(G)} [2d_G(u) + d_G(v) - 2]^2 \times \prod_{uv\in E(G)} [2d_G(v) + d_G(u) - 2]^2 \\
HB\Pi_1(G) &= (H\Pi_1(G))^2 - 6\Pi_1(G)H\Pi_1(G) + 13H\Pi_1(G) - 12\Pi_1(G) \\
&+ \Pi_2(G)H\Pi_1(G) + \frac{H\Pi_2(G)}{4} + 2\Pi_2(G) + 4^m
\end{aligned}$$

(ii) Consider

$$\begin{aligned}
HB\Pi_1(G) &= \prod_{ue}[d_G(u) + d_G(e)]^2 \\
&= \prod_{uv\in E(G)} [d_G(u) + d_G(e)]^2 \times \prod_{uv\in E(G)} [d_G(v) + d_G(e)]^2 \\
&= \prod_{uv\in E(G)} [2d_G(u) + d_G(v) - 2]^2 \times \prod_{uv\in E(G)} [2d_G(v) + d_G(u) - 2]^2 \\
&= \prod_{uv\in E(G)} \{4\left[d_G(u)^2 + d_G(v)^2\right]^2 + 20d_G(u)d_G(v) \\
&\times \left[d_G(u)^2 + d_G(v)^2\right] + 25d_G(u)^2 d_G(v)^2 \\
&- 24\left[d_G(u) + d_G(v)\right]^3 - 12d_G(u)d_G(v) \\
&\times \left[d_G(u) + d_G(v)\right] + 52\left[d_G(u) + d_G(v)\right]^2 \\
&- 48\left[d_G(u) + d_G(v)\right] + 8d_G(u)d_G(v) + 16\}
\end{aligned}$$

$$\begin{aligned} HB\Pi_1(G) &= [F\Pi(G)]^2 + \frac{25}{4} H\Pi_2(G) + 5\Pi_2(G)F\Pi(G) \\ &\quad - 6[\Pi_1(G)]^3 - 3\Pi_2(G)\Pi_2(G) \\ &\quad + 13H\Pi_1(G) + 2\Pi_2(G) - 12\Pi_1(G) + 4^m. \end{aligned}$$

Thus the result follows.

3. Bounds in terms of other multiplicative versions of degree-based indices

To prove the lower bound of $B\Pi_1(G)$ and $B\Pi_2(G)$ in terms of order and size, we make use of the following result.

Theorem 3.1. [15] *For any (n, m) - graph G with $n \geq 3$,*

$$\Pi_1(G) \leq \left(\frac{2m}{n}\right)^{2n} \textit{and}\, \Pi_2(G) \geq \left(\frac{2m}{n}\right)^{2m}.$$

Theorem 3.2. *For any (n, m) - graph G with $n \geq 3$,*

(i) $B\Pi_1(G) \geq 5\left(\frac{2m}{n}\right)^{2m} - 6\left(\frac{2m}{n}\right)^{2n} + 4^m.$

(ii) $B\Pi_2(G) \geq 4\left(\frac{2m}{n}\right)^{2m}\left[\left(\frac{2m}{n}\right)^{2m} - \left(\frac{2m}{n}\right)^{2n} + 1\right].$

Proof. Let G be a (n, m) - graph with $n \geq 3$. Then the first multiplicative Banhatti index

$$\begin{aligned} B\Pi_1(G) &= \prod_{ue}[d_G(u) + d_G(e)] \\ &= \prod_{uv\in E(G)} [d_G(u) + d_G(uv)] \times \prod_{uv\in E(G)} [d_G(v) + d_G(uv)] \\ B\Pi_1(G) &= \prod_{uv\in E(G)} \{2[d_G(u) + d_G(v)]^2 + d_G(u)d_G(v) \\ &\quad - 6[d_G(u) + d_G(v)] + 4\}. \end{aligned}$$

Since, $(d_G(u) + d_G(v))^2 \geq d_G(u)^2 + d_G(v)^2 \geq 2d_G(u)d_G(v)$. Hence

$$B\Pi_1(G) \geq 5\Pi_2(G) - 6\Pi_1(G) + 4^m. \tag{3.1}$$

Now, the second multiplicative Banhatti index

$$\begin{aligned} B\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)] \\ &= \prod_{uv\in E(G)}[d_G(u)d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v)d_G(uv)) \\ &= \prod_{uv\in E(G)} d_G(u)d_G(v)[d_G(u)+d_G(v)-2]^2 \\ &= \prod_{uv\in E(G)} d_G(u)d_G(v)[d_G(u)^2+d_G(v)^2 \\ &+ 2d_G(u)d_G(v)-4(d_G(u)+d_G(v))+4] \end{aligned}$$

Since $d_G(u)^2+d_G(v)^2 \geq 2d_G(u)d_G(v)$. Hence

$$\begin{aligned} B\Pi_2(G) \geq \prod_{uv\in E(G)} & d_G(u)d_G(v)[2d_G(u)d_G(v)+2(d_G(u)d_G(v) \\ & -4(d_G(u)+d_G(v)+4] \end{aligned}$$

$$B\Pi_2(G) \geq 4\Pi_2(G)[\Pi_2(G)-\Pi_1(G)+1]. \tag{3.2}$$

Substitute Theorem 3.1 in equations 3.1 and 3.2, we have the desired results of (i) and (ii).

Theorem 3.3. *For any graph connected G,*

(i) $(3\delta-2)^{2m} \leq B\Pi_1(G) \leq (3\Delta-2)^{2m}$.

(ii) $[2\delta(\delta-1)]^{2m} \leq B\Pi_2(G) \leq [2\Delta(\Delta-1)]^{2m}$.

Further, equalities of (i) and (ii) in both lower and upper bounds hold if and only if G is regular.

Proof. Let G be any connected graph.

(i) Consider

$$\begin{aligned} B\Pi_1(G) &= \prod_{ue}[d_G(u)+d_G(e)] \\ &= \prod_{uv\in E(G)}[d_G(u)+d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v)+d_G(uv)] \\ &= \prod_{uv\in E(G)}[2d_G(u)+d_G(v)-2] \times [d_G(u)+2d_G(v)-2]. \end{aligned}$$

We know that $2\delta \leq d_G(u) + d_G(v) \leq 2\Delta$. This implies that

$$(3\delta - 2) \leq (2d_G(u) + d_G(v) - 2) \leq (3\Delta - 2) \tag{3.3}$$

$$(3\delta - 2) \leq (d_G(u) + 2d_G(v) - 2) \leq (3\Delta - 2) \tag{3.4}$$

Multiplying an equations (3.3) and (3.4), we have,
$(3\delta - 2)^2 \leq (2d_G(u) + d_G(v) - 2)(d_G(u) + 2d_G(v) - 2) \leq (3\Delta - 2)^2$

$$(3\delta - 2)^{2m} \leq B\Pi_1(G) \leq (3\Delta - 2)^{2m}.$$

(ii) Consider

$$\begin{aligned} B\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)] \\ &= \prod_{uv\in E(G)}[d_G(u)d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v)d_G(uv)] \\ &= \prod_{uv\in E(G)} d_G(u)d_G(v)[d_G(u) + d_G(v) - 2]^2. \end{aligned}$$

We know that $2\delta \leq d_G(u) + d_G(v) \leq 2\Delta$. This implies that

$$4(\delta - 1)^2 \leq (d_G(u) + d_G(v) - 2)^2 \leq 4(\Delta - 1)^2 \tag{3.5}$$

$$\delta^2 \leq d_G(u)d_G(v) \leq \Delta^2 \tag{3.6}$$

Multiplying equations (3.5) and (3.6), we have

$$4\delta^2(\delta - 1)^2 \leq d_G(u)d_G(v)(d_G(u) + d_G(v) - 2)^2 \leq 4\Delta^2(\Delta - 1)^2$$

$$[4\delta^2(\delta - 1)^2]^m \leq B\Pi_2(G) \leq [4\Delta^2(\Delta - 1)^2]^m$$

$$[2\delta(\delta - 1)]^{2m} \leq B\Pi_2(G) \leq [2\Delta(\Delta - 1)]^{2m}.$$

Further, equalities of (i) and (ii) will hold if and only if $d_G(u) + d_G(v) = 2\delta(G) = 2\Delta(G)$ and $d_G(u) \times d_G(v) = \delta(G)^2 = \Delta(G)^2$ respectively for each $uv \in E(G)$, which implies that G is a regular graph.

Theorem 3.4. *For any connected graph G,*

(i) $1 \leq B\Pi_1(G) \leq (3n-5)^{2m}$.

(ii) $1 \leq B\Pi_2(G) \leq 4^m(n-1)^{4m}$.

Proof. Let G be any connected graph.

(i) Consider

$$\begin{aligned} B\Pi_1(G) &= \prod_{ue}[d_G(u)+d_G(e)] \\ &= \prod_{uv\in E(G)}[d_G(u)+d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v)+d_G(uv)] \\ &= \prod_{uv\in E(G)}[2d_G(u)+d_G(v)-2] \times [d_G(u)+2d_G(v)-2]. \end{aligned}$$

We know that $2 \leq d_G(u)+d_G(v) \leq 2(n-1)$. This implies that

$$1 \leq (2d_G(u)+d_G(v)-2) \leq (3n-5) \tag{3.7}$$

$$1 \leq (d_G(u)+2d_G(v)-2) \leq (3n-5) \tag{3.8}$$

Multiplying an equations (3.7) and (3.8), we have,

$1 \leq (2d_G(u)+d_G(v)-2)(d_G(u)+2d_G(v)-2) \leq (3n-5)^2$

$$1 \leq B\Pi_1(G) \leq (3n-5)^{2m}.$$

(ii) Consider

$$\begin{aligned} B\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)] \\ &= \prod_{uv\in E(G)}[d_G(u)d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v)d_G(uv)] \\ &= \prod_{uv\in E(G)} d_G(u)d_G(v)[d_G(u)+d_G(v)-2]^2. \end{aligned}$$

We know that $2 \leq d_G(u)+d_G(v) \leq 2(n-1)$. This implies that

$$1 \leq (d_G(u)+d_G(v)-2)^2 \leq 4(n-2)^2 \tag{3.9}$$

$$1 \leq d_G(u)d_G(v) \leq (n-1)^2 \tag{3.10}$$

Multiplying equations (3.9) and (3.10), we have

$$1 \leq d_G(u)d_G(v)(d_G(u) + d_G(v) - 2)^2 \leq 4(n-1)^4$$
$$1 \leq B\Pi_2(G) \leq 4^m(n-1)^{4m}.$$

Theorem 3.5. *For any connected graph G,*

(i) $(3\delta - 2)^{4m} \leq HB\Pi_1(G) \leq (3\Delta - 2)^{4m}$.

(ii) $[16\delta^4(\delta-1)^4]^m \leq HB\Pi_2(G) \leq [16\Delta^4(\Delta-1)^4]^m$.

Further, equalities of (i) and (ii) in both lower and upper bounds are attained if and only if G is regular.

Proof. Let G be any connected graph.

(i) Consider

$$\begin{aligned} HB\Pi_1(G) &= \prod_{ue}[d_G(u) + d_G(e)]^2 \\ &= \prod_{uv \in E(G)} [d_G(u) + d_G(e)]^2 \times \prod_{uv \in E(G)} [d_G(v) + d_G(e)]^2 \\ &= \prod_{uv \in E(G)} [2d_G(u) + d_G(v) - 2]^2 \times \prod_{uv \in E(G)} [2d_G(v) + d_G(u) - 2]^2. \end{aligned}$$

We know that $3\delta \leq 2d_G(u) + d_G(v) \leq 3\Delta$ and $3\delta \leq d_G(u) + 2d_G(v) \leq 3\Delta$. This implies that

$$(3\delta - 2)^2 \leq (2d_G(u) + d_G(v) - 2)^2 \leq (3\Delta - 2)^2 \tag{3.11}$$

$$(3\delta - 2)^2 \leq (d_G(u) + 2d_G(v) - 2)^2 \leq (3\Delta - 2)^2 \tag{3.12}$$

Multiplying equations (3.11) and (3.12), we have

$$(3\delta - 2)^4 \leq (2d_G(u) + d_G(v) - 2)^2(d_G(u) + 2d_G(v) - 2)^2 \leq (3\Delta - 2)^4$$
$$(3\delta - 2)^{4m} \leq HB\Pi_1(G) \leq (3\Delta - 2)^{4m}.$$

(ii) Consider

$$\begin{aligned} HB\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)]^2 \\ &= \prod_{uv \in E(G)} [d_G(u)d_G(e)]^2 \times \prod_{uv \in E(G)} [d_G(v)d_G(e)]^2 \\ &= \prod_{uv \in E(G)} d_G(u)^2 d_G(v)^2 [d_G(u) + d_G(v) - 2]^4. \end{aligned}$$

We know that $2\delta \leq d_G(u) + d_G(v) \leq 2\Delta$. This implies that $(2\delta - 2)^4 \leq (d_G(u) + d_G(v) - 2)^4 \leq (2\Delta - 2)^4$, we have

$$16(\delta - 1)^4 \leq (d_G(u) + d_G(v) - 2)^4 \leq 16(\Delta - 1)^4 \tag{3.13}$$

$$\delta^4 \leq (d_G(u)d_G(v))^2 \leq \Delta^4 \tag{3.14}$$

Multiplying equations (3.13) and (3.14), we have

$$16\delta^4(\delta - 1)^4 \leq (d_G(u)d_G(v))^2(d_G(u) + d_G(v) - 2)^4 \leq 16\Delta^4(\Delta - 1)^4$$
$$[16\delta^4(\delta - 1)^4]^m \leq HB\Pi_2(G) \leq [16\Delta^4(\Delta - 1)^4]^m.$$

Since $d_G(u) = d_G(v)$. Hence the equalities of (i) and (ii) in both lower and upper bounds are attained if and only if G is regular.

Theorem 3.6. *For any connected graph G,*

(i) $1 \leq HB\Pi_1(G) \leq (3n - 5)^{4m}$.

(ii) $0 \leq HB\Pi_2(G) \leq 16(n - 1)^{4m}$.

Proof. Let G be any connected graph.

(i) Consider

$$\begin{aligned} HB\Pi_1(G) &= \prod_{ue}[d_G(u) + d_G(e)]^2 \\ &= \prod_{uv\in E(G)} [d_G(u) + d_G(e)]^2 \times \prod_{uv\in E(G)} [d_G(v) + d_G(e)]^2 \\ &= \prod_{uv\in E(G)} [2d_G(u) + d_G(v) - 2]^2 \times \prod_{uv\in E(G)} [2d_G(v) + d_G(u) - 2]^2. \end{aligned}$$

We know that $3 \leq 2d_G(u) + d_G(v) \leq 3(n - 1)$ and $3 \leq d_G(u) + 2d_G(v) \leq 3(n - 1)$. This implies that

$$1 \leq (2d_G(u) + d_G(v) - 2)^2 \leq (3n - 5)^2 \tag{3.15}$$

$$1 \leq (d_G(u) + 2d_G(v) - 2)^2 \leq (3n - 5)^2 \tag{3.16}$$

Multiplying equations (3.15) and (3.16), we have

$$1 \leq (2d_G(u) + d_G(v) - 2)^2(d_G(u) + 2d_G(v) - 2)^2 \leq (3n - 5)^4$$
$$1 \leq HB\Pi_1(G) \leq (3n - 5)^{4m}.$$

(ii) Consider

$$\begin{aligned}HB\Pi_2(G) &= \prod_{ue}[d_G(u)d_G(e)]^2 \\ &= \prod_{uv\in E(G)}[d_G(u)d_G(e)]^2 \times \prod_{uv\in E(G)}[d_G(v)d_G(e)]^2 \\ &= \prod_{uv\in E(G)} d_G(u)^2 d_G(v)^2[d_G(u)+d_G(v)-2]^4.\end{aligned}$$

We know that $2 \leq d_G(u) + d_G(v) \leq 2(n-1)$. This implies that $0 \leq (d_G(u) + d_G(v) - 2)^4 \leq (2(n-1))^4$, we have

$$0 \leq (d_G(u) + d_G(v) - 2)^4 \leq 16(n-1)^4 \tag{3.17}$$

$$1 \leq (d_G(u)d_G(v))^2 \leq (n-1)^2 \tag{3.18}$$

Multiplying equations (3.17) and (3.18), we have

$$0 \leq (d_G(u)d_G(v))^2(d_G(u) + d_G(v) - 2)^4 \leq 16(n-1)^4$$

$$0 \leq HB\Pi_2(G) \leq 16(n-1)^{4m}.$$

Theorem 3.7. *For any connected graph G with η pendent vertices and minimal non- pendent vertices degree $\delta_1(G) = \delta_1$,*

(i) $(3\delta_1 - 2)^{2(m-\eta)}[\delta_1(2\delta_1 - 1)]^\eta \leq B\Pi_1(G) \leq (3\Delta - 2)^{2(m-\eta)}[\Delta(2\Delta - 1)]^\eta$

(ii) $[4\delta_1^2(\delta_1 - 1)^2]^{(m-\eta)}[\delta_1(\delta_1 - 1)^2]^\eta \leq B\Pi_2(G) \leq [4\Delta^2(\Delta - 1)^2]^{(m-\eta)}[\Delta(\Delta - 1)^2]^\eta$

Proof.

(i) Let G be a graph with η pendent vertices and minimal non- pendent vertices degree $\delta_1(G) = \delta_1$.

$$\begin{aligned}B\Pi_1(G) &= \prod_{ue} d_G(u) + d_G(e) \\ &= \prod_{uv\in E(G)}[d_G(u) + d_G(uv)] \times \prod_{uv\in E(G)}[d_G(v) + d_G(uv)] \\ &= \prod_{uv\in E(G)}[2d_G(u) + d_G(v) - 2][d_G(u) + 2d_G(v) - 2] \\ &= \prod_{uv\in E(G);d_G(u),d_G(v)\neq 1}[2d_G(u) + d_G(v) - 2][d_G(u) + 2d_G(v) - 2]\end{aligned}$$

$$\times \prod_{uv\in E(G);d_G(u)=1} [2d_G(u)+d_G(v)-2][d_G(u)+2d_G(v)-2]$$
$$B\Pi_1(G) \leq (3\Delta-2)^{2(m-\eta)}[\Delta(2\Delta-1)]^{\eta}$$

Thus the upper bound follows.
Similarly,

$$B\Pi_1(G) \geq (3\delta_1-2)^{2(m-\eta)}[\delta_1(2\delta_1-1)]^{\eta}$$

Hence the lower bound follows.

(ii) For any (n, m) graph with η pendent vertices and minimal non- pendent vertices degree $\delta_1(G)$, then

$$\begin{aligned}
B\Pi_2(G) &= \prod_{ue} d_G(u)d_G(e) \\
&= \prod_{uv\in E(G)} [d_G(u)d_G(uv)] \times \prod_{uv\in E(G)} [d_G(v)d_G(uv)] \\
&= \prod_{uv\in E(G);d_G(u),d_G(v)\neq 1} d_G(u)d_G(v)[d_G(u)+d_G(v)-2]^2 \\
&\times \prod_{uv\in E(G);d_G(u)=1} d_G(u)d_G(v)[d_G(u)+d_G(v)-2]^2 \\
B\Pi_2(G) &\leq [4\Delta^2(\Delta-1)^2]^{(m-\eta)}[\Delta(\Delta-1)^2]^{\eta}
\end{aligned}$$

Thus the upper bound follows.
Similarly,

$$\begin{aligned}
B\Pi_2(G) &\geq \prod_{uv\in E(G);d_G(u),d_G(v)\neq 1} d_G(u)d_G(v)[d_G(u)+d_G(v)-2]^2 \\
&+ \prod_{uv\in E(G);d_G(u)=1} d_G(u)d_G(v)[d_G(u)+d_G(v)-2]^2 \\
B\Pi_2(G) &\geq [4\delta_1^2(\delta_1-1)^2]^{(m-\eta)}[\delta_1(\delta_1-1)^2]^{\eta}
\end{aligned}$$

Hence the lower bound follows.

4. Conclusion

Being new multiplicative versions of topological index of a graph G in terms of incident vertex-edge degrees, the multiplicative Banhatti index is an invariant, whose properties are relatively unknown. For the comparative advantages, applications and mathematical point of view, many questions are suggested by this research, among them are the following.

1. Find the extremal values and extremal graphs of the multiplicative Banhatti indices.

2. Find some bounds (in terms of other degree based topological indices) and characterizations of multiplicative Banhatti indices.

3. Find the values of the multiplicative Banhatti indices of all classes of chemical graphs and compare with other degree based topological indices, when $\Delta(G) \leq 4$. Also, explore some results towards QSPR / QSAR / QSTR Model.

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