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SPECIAL SOLUTIONS TO THE FLOW OF INCOMPRESSIBLE FLUIDS COUPLING WITH MAGNETIC FIELD

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Abstract: To provide the extensive mathematical analysis, we have considered the ideal Magnetohydrodynamics (MHD) equations, which represent the flow of fluid in the atmosphere or ocean in the presence of a magnetic field. Alternatively, we can say that it represents the flow of electromagnetic fluids. We followed the procedure of Majda [4] that was implemented to find special solutions of the rotating stratified Boussinesq equations and found the exact solutions of an initial value problem as well as we provided a local analysis of incompressible electromagnetic fluids in the neighborhood of the origin. Further, we reduce these ideal MHD equations into a system of six-coupled ordinary differential equations, and we conclude that it is a completely integrable system. Hence, through the quadrature, we find its solutions. Thereby, we determine the critical point of a reduced system and which is a degenerate critical point. Finally, we obtained special solutions to the initial value problem. While providing examples of special solutions to ideal MHD equations, we come across the fact that, Mathematically, it is possible to find a flow of an ideal fluid in the presence of a magnetic field such that there is no pressure at every point of the fluid. But practically it is impossible because of zero pressure at a point, implying that there is no movement of fluid molecules. Whereas in the second example the pressure varies with space variable $\boldsymbol{x}$.

Keywords and Phrases: Ideal MHD equations, Special Solutions, Incompressible inviscid fluids, Stratified Boussinesq equations, Completely Integrable Systems.

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1. Introduction

The flow of fluid in the atmosphere and ocean is generally governed by the stratified Boussinesq equations. In his monograph, Andrew Majda [4] has provided the rigorous mathematical analysis of these stratified Boussinesq equations. In the literature survey, we saw that the Boussinesq approximation is referred to as the Oberbeck-Boussinesq approximation and for more details about these approximations, readers may see the article [5]. Furthermore, if we consider the flow of fluid in the atmosphere and the ocean in the presence of the magnetic field, it may be referred to as magnetohydrodynamics (MHD) equations. In concern with the MHD equations, we came across the research article published by Jiahong Wu [6]. In his paper, he has considered the following generalized MHD equations.

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -\nabla P + \boldsymbol{b}\cdot\nabla\boldsymbol{b} - \nu(-\Delta)^{\alpha}\boldsymbol{u}, \\ \partial_t \boldsymbol{b} + \boldsymbol{u}\cdot\nabla\boldsymbol{b} &= \boldsymbol{b}\cdot\nabla\boldsymbol{u} - \eta(-\Delta)^{\beta}\boldsymbol{b}, \end{aligned} \tag{1}$$

where $\nu \geq 0$, $\eta \geq 0$, $\alpha > 0$ and $\beta > 0$ are real numbers. In above equations (1), we have

$$\begin{aligned} \boldsymbol{x} = (x_1, x_2, x_3) &\ : \ \text{space variable}, \\ \boldsymbol{u} = (u^1, u^2, u^3) &\ : \ \text{fluid velocity}, \\ \boldsymbol{b} = (b^1, b^2, b^3) &\ : \ \text{magnetic field}, \\ P = P(x_1, x_2, x_3) &\ : \ \text{pressure}. \end{aligned}$$

In fact, Jiahong Wu [6] has provided the rigorous mathematical analysis and regularity of solutions of equations (1) by varying the parameters ν, η, α, and β. He has used singular operator method with kernel K. The convolution of kernel K with vorticity is the deformation tensor of the velocity vector and convolution of K with current density is deformation tensor of the magnetic field. On the other hand, we followed the procedure of Majda [4] to obtain the special solutions of the ideal MHD equations. That is, we provide a local analysis of MHD flows of fluid in the neighborhood of the origin. Furthermore, if we consider $\alpha = \beta = 1$ then (1) reduces to the usual MHD equations. In a literature survey, we found that there have been extensive studies of equations (1). Also, we saw that Duvaut and Lions [1] have obtained global weak solutions. Further, we observe that these global weak solutions are similar to those were obtained by Leray-Hopf [2, 3] for the following stratified Boussinesq equations.

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} &= -\nabla P, \\ \nabla\cdot\boldsymbol{u} &= 0. \end{aligned} \tag{2}$$

For more details the readers are advised to refer the articles [2, 3]. In his paper Zujin Zhang [7] has discussed the generalised MHD equations (1) for $\nu = \eta = 1$, $\alpha = \beta = 1$, and has proved that if one of the directional derivative of fluid velocity is exists in $L^p(0, T, L^q(\mathbb{R}^3))$, where $\frac{2}{p} + \frac{3}{q} = 1$, $3 < q \leq \infty$, then the solution of MHD equations exists and is smooth.

In this paper, we have obtained the special solutions of (1) with particular values $\nu = \eta = 0$. That is, for our investigation, we have considered the flow of incompressible fluid coupled with a magnetic field. Now, we can mathematically represent this flow of fluids by the following partial differential equations (PDEs):

$$\begin{aligned} \partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} &= -\nabla P + \boldsymbol{b} \cdot \nabla \boldsymbol{b}, \\ \partial_t \boldsymbol{b} + \boldsymbol{u} \cdot \nabla \boldsymbol{b} &= \boldsymbol{b} \cdot \nabla \boldsymbol{u}, \\ \nabla \cdot \boldsymbol{u} &= 0, \\ \nabla \cdot \boldsymbol{b} &= 0, \end{aligned} \tag{3}$$

with following initial conditions that are compatible with (3):

$$\boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \boldsymbol{b}(\boldsymbol{x}, 0) = \boldsymbol{b}_0(\boldsymbol{x}). \tag{4}$$

In fact, in the following section, we develop the solution to the initial value problem (3) together with initial conditions (4) in the neighbourhood of origin.

2. Special Solutions of MHD Equations

In this section, we investigate special solutions to (3) in a large scale of motion. We are looking for the local behaviour of an incompressible electromagnetic fluid. Thus, we expand the smooth velocity and magnetic field in Taylor's series about some point $\boldsymbol{x}_0$:

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}, t) &= \boldsymbol{u}(\boldsymbol{x}_0, t) + \nabla \boldsymbol{u}|_{(\boldsymbol{x}_0, t)}(\boldsymbol{x} - \boldsymbol{x}_0) + O(|\boldsymbol{x} - \boldsymbol{x}_0|^2), \\ \boldsymbol{b}(\boldsymbol{x}, t) &= \boldsymbol{b}(\boldsymbol{x}_0, t) + \nabla \boldsymbol{b}|_{(\boldsymbol{x}_0, t)}(\boldsymbol{x} - \boldsymbol{x}_0) + O(|\boldsymbol{x} - \boldsymbol{x}_0|^2), \end{aligned} \tag{5}$$

where $\nabla \boldsymbol{u}$ is a 3×3 matrix whose $(i, j)^{\text{th}}$ entry is $\frac{\partial u^i}{\partial x_j}$, $i = 1, 2, 3$, $j = 1, 2, 3$. We can have a unique decomposition of $\nabla \boldsymbol{u}$ and $\nabla \boldsymbol{b}$ as the sum of symmetric and skew-symmetric matrices as follows:

$$\begin{aligned} \nabla \boldsymbol{u} &= \left(\frac{\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t}{2} \right) + \left(\frac{\nabla \boldsymbol{u} - (\nabla \boldsymbol{u})^t}{2} \right) \\ &= \mathcal{D}(\boldsymbol{x}_0, t) + \Omega(\boldsymbol{x}_0, t), \\ \nabla \boldsymbol{b} &= \left(\frac{\nabla \boldsymbol{b} + (\nabla \boldsymbol{b})^t}{2} \right) + \left(\frac{\nabla \boldsymbol{b} - (\nabla \boldsymbol{b})^t}{2} \right) \\ &= \mathcal{K}(\boldsymbol{x}_0, t) + \mathcal{L}(\boldsymbol{x}_0, t) \end{aligned} \tag{6}$$

where $\mathcal{D}$ and $\mathcal{K}$ are the symmetric parts of $\nabla \boldsymbol{u}$ and $\nabla \boldsymbol{b}$ respectively, and we call them as deformation matrices. The deformation matrix $\mathcal{D}$ has the property that its trace is equal to the divergence of the vector field $\boldsymbol{u}$. Similarly, the trace of matrix $\mathcal{K}$ is equal to the divergence of the magnetic field $\boldsymbol{b}$. On the other hand, Ω and $\mathcal{L}$ are the skew-symmetric parts of $\nabla \boldsymbol{u}$ and $\nabla \boldsymbol{b}$, respectively. Further, these matrices satisfy the following equation (7) for any arbitrary $\boldsymbol{h} = (h_1, h_2, h_3)^t \in \mathbb{R}^3$

$$\Omega \boldsymbol{h} = \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{h}, \quad \mathcal{L}\boldsymbol{h} = \frac{1}{2}\boldsymbol{j} \times \boldsymbol{h}, \tag{7}$$

where the current density vector $\boldsymbol{j} = \nabla \times \boldsymbol{b} = (j_1, j_2, j_3)^t$ and the vorticity vector $\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = (w_1, w_2, w_3)^t$.

Now, we can use vorticity vector $\boldsymbol{\omega}$ and current density vector $\boldsymbol{j}$ in (5) we get

$$\begin{aligned} \nabla \boldsymbol{u}|_{(\boldsymbol{x}_0,t)}\boldsymbol{h} &= \mathcal{D}(\boldsymbol{x}_0,t)\boldsymbol{h} + \tfrac{1}{2}\boldsymbol{\omega}(\boldsymbol{x}_0,t) \times \boldsymbol{h}, \\ \nabla \boldsymbol{b}|_{(\boldsymbol{x}_0,t)}\boldsymbol{h} &= \mathcal{K}(\boldsymbol{x}_0,t)\boldsymbol{h} + \tfrac{1}{2}\boldsymbol{j}(\boldsymbol{x}_0,t) \times \boldsymbol{h}. \end{aligned} \tag{8}$$

We assume that $\boldsymbol{u}(\boldsymbol{x}_0, t) = 0$ and $\boldsymbol{b}(x_0, t) = 0$. We take advantage of the local representation to determine certain special solutions to the ideal magnetohydrodynamics equations (3). Now taking the gradient of the equation of motion of fluid and magnetic field, we get

$$\begin{aligned} (u^i_{x_k})_t + \sum_j u^j (u^i_{x_k})_{x_j} + \sum_j \frac{\partial u^j}{\partial x_k}\frac{\partial u^i}{\partial x_j} &= -(p_{x_i})_{x_k} + \sum_j b^j (b^i_{x_k})_{x_j} \\ &+ \sum_j \frac{\partial b^j}{\partial x_k}\frac{\partial b^i}{\partial x_j}, \\ (b^i_{x_k})_t + \sum_j u^j (b^i_{x_k})_{x_j} + \sum_j \frac{\partial u^j}{\partial x_k}\frac{\partial b^i}{\partial x_j} &= \sum_j b^j (u^i_{x_k})_{x_j} + \sum_j \frac{\partial b^j}{\partial x_k}\frac{\partial u^i}{\partial x_j}. \end{aligned} \tag{9}$$

We introduce the notations $U = (u^i_{x_k})$, $B = (b^i_{x_k})$ and $\hat{P} = (p_{x_i})_{x_k}$ for the Hessian matrix of the gradient pressure ∇P. With the introduction of these notations (9) can be rewritten as follows:

$$\begin{aligned} \frac{\partial U}{\partial t} &+ (\boldsymbol{u} \cdot \nabla)U + U^2 = -\hat{P} + (\boldsymbol{b} \cdot \nabla)B + B^2 \\ \frac{\partial B}{\partial t} &+ (\boldsymbol{u} \cdot \nabla)B + BU = (\boldsymbol{b} \cdot \nabla)U + UB. \end{aligned} \tag{10}$$

Here, we notice that:

(a) The symmetric part of $U^2 = (\mathcal{D} + \Omega)^2$ is $\mathcal{D}^2 + \Omega^2$ and skew-symmetric part is $\mathcal{D}\Omega + \Omega\mathcal{D}$.

(b) Similarly, the symmetric part of $B^2 = (\mathcal{K} + \mathcal{L})^2$ is $\mathcal{K}^2 + \mathcal{L}^2$ and skew symmetric part is $\mathcal{K}\mathcal{L} + \mathcal{L}\mathcal{K}$.

(c) As we have $(\boldsymbol{u} \cdot \nabla)U = (\boldsymbol{u} \cdot \nabla)(\mathcal{D} + \Omega)$, so that $(\boldsymbol{u} \cdot \nabla)\mathcal{D}$ is a symmetric matrix and $(\boldsymbol{u} \cdot \nabla)\Omega$ is skew symmetric.

(d) With similar calculations, we see that the symmetric and skew-symmetric parts of $(\boldsymbol{b}\cdot\nabla)B$ are $(\boldsymbol{b}\cdot\nabla)\mathcal{K}$ and $(\boldsymbol{b}\cdot\nabla)\mathcal{L}$ respectively. Whereas, $(\boldsymbol{b}\cdot\nabla)\mathcal{D}$ and $(\boldsymbol{b}\cdot\nabla)\Omega$ are respectively the symmetric and skew symmetric parts of $(\boldsymbol{b} \cdot \nabla)U$

(e) In a similar fashion, we notice that the symmetric and skew symmetric parts of $UB - BU$ are $(\mathcal{D}\mathcal{L} - \mathcal{L}\mathcal{D} + \Omega\mathcal{K} - \mathcal{K}\Omega)$ and $(\mathcal{D}\mathcal{K} - \mathcal{K}\mathcal{D} + \Omega\mathcal{L} - \mathcal{L}\Omega)$ respectively.

Let us assume that the matrix

$$\mathcal{D} = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{12} & k_{22} & k_{23} \\ k_{13} & k_{23} & k_{33} \end{bmatrix}. \tag{11}$$

Since $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{b} = 0$ so that

$$d_{11} + d_{22} + d_{33} = 0 = k_{11} + k_{22} + k_{33}. \tag{12}$$

The matrices Ω and $\mathcal{L}$ can be expressed in terms of the components of $\boldsymbol{\omega}$ and $\boldsymbol{j}$ as follows.

$$\Omega = \frac{1}{2}\begin{bmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{bmatrix}, \quad \mathcal{L} = \frac{1}{2}\begin{bmatrix} 0 & -j_3 & j_2 \\ j_3 & 0 & -j_1 \\ -j_2 & j_1 & 0 \end{bmatrix}. \tag{13}$$

We proceed towards expressing the evolution of vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$, so we need to present the following lemmas.

Lemma 2.1. *A vorticity vector $\boldsymbol{\omega}$ satisfy the relation.*

$$\boldsymbol{\omega} \cdot \nabla\boldsymbol{u} = \boldsymbol{\omega} \cdot (\nabla\boldsymbol{u})^t \tag{14}$$

Proof. For any $\boldsymbol{h} \in \mathbb{R}^3$, we have an identification by (7) that

$$0 = \frac{1}{2}\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \boldsymbol{h}) = \frac{1}{2}\boldsymbol{\omega} \cdot ((\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^t)\boldsymbol{h}) = \frac{1}{2}\boldsymbol{\omega} \cdot (\nabla\boldsymbol{u} - (\nabla\boldsymbol{u})^t)\boldsymbol{h}$$

since $\boldsymbol{h}$ is arbitrary, the result follows.

Lemma 2.2. *A current density vector satisfy*

$$\boldsymbol{j} \cdot \nabla \boldsymbol{b} = \boldsymbol{j} \cdot (\nabla \boldsymbol{b})^t. \tag{15}$$

The proof of this lemma is just follows by (7).

Proposition 2.1. *The evolution of vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$ are governed by the following system of equations:*

$$\begin{aligned} \frac{D\boldsymbol{\omega}}{Dt} &- \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = (\boldsymbol{b} \cdot \nabla)\boldsymbol{j} - \boldsymbol{j} \cdot \nabla \boldsymbol{b}, \\ \frac{D\boldsymbol{j}}{Dt} &= (\boldsymbol{b} \cdot \nabla)\boldsymbol{\omega} + 2\boldsymbol{T} + \frac{1}{2}\boldsymbol{\omega} \times \boldsymbol{j}, \end{aligned} \tag{16}$$

where the vector $\boldsymbol{T} = (T_{23}, T_{13}, T_{12})^t$ has the property that; for any $\boldsymbol{h} \in \mathbb{R}^3$

$$\boldsymbol{T} \times \boldsymbol{h} = (\mathcal{D}\mathcal{K} - \mathcal{K}\mathcal{D})\boldsymbol{h}. \tag{17}$$

Proof. Let us proceed with equating the skew-symmetric parts of equations (10), we get

$$\begin{aligned} \frac{D\Omega}{Dt} &+ \mathcal{D}\Omega + \Omega\mathcal{D} = (\boldsymbol{b} \cdot \nabla)\mathcal{L} + \mathcal{K}\mathcal{L} + \mathcal{L}\mathcal{K}, \\ \frac{D\mathcal{L}}{Dt} &= (\boldsymbol{b} \cdot \nabla)\Omega + (\mathcal{D}\mathcal{K} - \mathcal{K}\mathcal{D}) + (\Omega\mathcal{L} - \mathcal{L}\Omega). \end{aligned} \tag{18}$$

Hence the action of any $\boldsymbol{h} \in \mathbb{R}^3$ by post multiplication to (18), we have

$$\begin{aligned} \frac{1}{2}\frac{D\boldsymbol{\omega}}{Dt} \times \boldsymbol{h} &+ (\mathcal{D}\Omega + \Omega\mathcal{D})\boldsymbol{h} = \frac{1}{2}(\boldsymbol{b} \cdot \nabla)\boldsymbol{j} \times \boldsymbol{h} + (\mathcal{K}\mathcal{L} + \mathcal{L}\mathcal{K})\boldsymbol{h}, \\ \frac{1}{2}\frac{D\boldsymbol{j}}{Dt} \times \boldsymbol{h} &= \frac{1}{2}(\boldsymbol{b} \cdot \nabla)\boldsymbol{\omega} \times \boldsymbol{h} + (\mathcal{D}\mathcal{K} - \mathcal{K}\mathcal{D})\boldsymbol{h} + (\Omega\mathcal{L} - \mathcal{L}\Omega)\boldsymbol{h}. \end{aligned} \tag{19}$$

We use (12) to compute the matrix $\mathcal{D}\Omega + \Omega\mathcal{D}$ as follows:

$$(\mathcal{D}\Omega + \Omega\mathcal{D}) = \\ \frac{1}{2}\begin{bmatrix} 0 & d_{13}w_1 + d_{23}w_2 + d_{33}w_3 & -d_{12}w_1 - d_{22}w_2 - d_{23}w_3 \\ -d_{13}w_1 - d_{23}w_2 - d_{33}w_3 & 0 & d_{11}w_1 + d_{12}w_2 + d_{13}w_3 \\ d_{12}w_1 + d_{22}w_2 + d_{23}w_3 & -d_{11}w_1 - d_{12}w_2 - d_{13}w_3 & 0 \end{bmatrix}. \tag{20}$$

Let us define a vector $\boldsymbol{c}$ as

$$\boldsymbol{c} = \begin{bmatrix} c_{23} \\ c_{13} \\ c_{12} \end{bmatrix} = \begin{bmatrix} -d_{11}w_1 - d_{12}w_2 - d_{13}w_3 \\ -d_{12}w_1 - d_{22}w_2 - d_{23}w_3 \\ -d_{13}w_1 - d_{23}w_2 - d_{33}w_3 \end{bmatrix}. \tag{21}$$

With this definition of vector $\boldsymbol{c}$, we can rewrite the matrix $(\mathcal{D}\Omega+\Omega\mathcal{D})$ as follows.

$$(\mathcal{D}\Omega+\Omega\mathcal{D}) = \frac{1}{2}\begin{bmatrix} 0 & -c_{12} & c_{13} \\ c_{12} & 0 & -c_{23} \\ -c_{13} & c_{23} & 0 \end{bmatrix}. \tag{22}$$

Hence for any $\boldsymbol{h}\in\mathbb{R}^3$, we have

$$(\mathcal{D}\Omega+\Omega\mathcal{D})\boldsymbol{h} = \frac{1}{2}\boldsymbol{c}\times\boldsymbol{h}. \tag{23}$$

Further, we notice that

$$\begin{aligned} \boldsymbol{c} &= \begin{bmatrix} -d_{11}w_1 - d_{12}w_2 - d_{13}w_3 \\ -d_{12}w_1 - d_{22}w_2 - d_{23}w_3 \\ -d_{13}w_1 - d_{23}w_2 - d_{33}w_3 \end{bmatrix} = -\begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{12} & d_{22} & d_{23} \\ d_{13} & d_{23} & d_{33} \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} \\ &= -\mathcal{D}\boldsymbol{\omega} = -\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}. \end{aligned} \tag{24}$$

Thus, we have

$$(\mathcal{D}\Omega+\Omega\mathcal{D})\boldsymbol{h} = -\frac{1}{2}\left(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\right)\times\boldsymbol{h}. \tag{25}$$

With the same line of above arguments, we can compute the matrices $(\mathcal{K}\mathcal{L}+\mathcal{L}\mathcal{K})$, $(\Omega\mathcal{L}-\mathcal{L}\Omega)$ and action of any $\boldsymbol{h}\in\mathbb{R}^3$ by post multiplication on these matrices are as follows.

$$\begin{aligned} (\mathcal{K}\mathcal{L}+\mathcal{L}\mathcal{K})\boldsymbol{h} &= -\frac{1}{2}\left(\boldsymbol{j}\cdot\nabla\boldsymbol{b}\right)\times\boldsymbol{h} \\ (\Omega\mathcal{L}-\mathcal{L}\Omega)\boldsymbol{h} &= \frac{1}{4}(\boldsymbol{\omega}\times\boldsymbol{j})\times\boldsymbol{h}. \end{aligned} \tag{26}$$

Furthermore, we compute that

$$(\mathcal{D}\mathcal{K}-\mathcal{K}\mathcal{D}) = \begin{bmatrix} 0 & -T_{12} & T_{13} \\ T_{12} & 0 & -T_{23} \\ -T_{13} & T_{23} & 0 \end{bmatrix}. \tag{27}$$

Hence for $\boldsymbol{h}\in\mathbb{R}^3$, the matrix $(\mathcal{D}\mathcal{K}-\mathcal{K}\mathcal{D})$ satisfies the equation (17).

Now, we use (17), (25) and (26) to (19), we get

$$\begin{aligned} \frac{1}{2}\frac{D\boldsymbol{\omega}}{Dt}\times\boldsymbol{h} &- \frac{1}{2}\left(\boldsymbol{\omega}\cdot\nabla\boldsymbol{u}\right)\times\boldsymbol{h} = \frac{1}{2}(\boldsymbol{b}\cdot\nabla)\boldsymbol{j}\times\boldsymbol{h} - \frac{1}{2}\left(\boldsymbol{j}\cdot\nabla\boldsymbol{b}\right)\times\boldsymbol{h}, \\ \frac{1}{2}\frac{D\boldsymbol{j}}{Dt}\times\boldsymbol{h} &= \frac{1}{2}(\boldsymbol{b}\cdot\nabla)\boldsymbol{\omega}\times\boldsymbol{h} + \boldsymbol{T}\times\boldsymbol{h} + \frac{1}{4}(\boldsymbol{\omega}\times\boldsymbol{j})\times\boldsymbol{h}. \end{aligned} \tag{28}$$

Since $\boldsymbol{h}$ is arbitrary, we have the result that the evolution of vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$ are governed by the equations (16). Thus, we complete the proof.

Theorem 2.1. *If the gradients of velocity and magnetic field are commutes, then the ideal MHD equations (3) admits the special solution of the form*

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \mathcal{D}(t)\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{\omega}(t)\times\boldsymbol{x}, \\ \boldsymbol{b}(\boldsymbol{x},t) &= \mathcal{K}(t)\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{j}(t)\times\boldsymbol{x}, \\ P &= \frac{1}{2}\hat{P}(t)\boldsymbol{x}\cdot\boldsymbol{x}, \end{aligned} \tag{29}$$

where $\mathcal{D}(t)$ and $\mathcal{K}(t)$ are symmetric matrices with zero trace and are defined by (6); furthermore, the vorticity vector $\boldsymbol{\omega}(t)$ and the current density vector $\boldsymbol{j}(t)$ satisfies the system of ODEs;

$$\begin{aligned} \frac{d\boldsymbol{\omega}(t)}{dt} &= \mathcal{D}(t)\boldsymbol{\omega}(t) - \mathcal{K}(t)\boldsymbol{j}(t), \\ \frac{d\boldsymbol{j}(t)}{dt} &= 0 \end{aligned} \tag{30}$$

and the matrix $\hat{P}(t)$ is given by

$$-\hat{P}(t) = \frac{d\mathcal{D}}{dt} + \mathcal{D}^2 + \Omega^2 - \mathcal{K}^2 - \mathcal{L}^2, \tag{31}$$

where the matrices Ω and $\mathcal{L}$ are defined in (6) through the linear maps given by (7).

Proof. We proceed to show that the velocity, magnetic field and pressure functions defined by equations (29) satisfies the equations (3) provided that these functions satisfies the equations (30). Indeed, (29) is the solution of (3) such that $\nabla\boldsymbol{u}$ and $\nabla\boldsymbol{b}$ are commutes; provided that $\boldsymbol{\omega}$, $\boldsymbol{j}$ satisfies (30) and scalar function $\hat{P}(t)$ determined by the equation (31).

The conditions $\text{div}\boldsymbol{u} = 0$ and $\text{div}\boldsymbol{b} = 0$ follows from the fact that the matrices $\mathcal{D}$ and $\mathcal{K}$ have zero trace. To verify the momentum equations of fluid velocity and magnetic field, note that $\boldsymbol{u}$ and $\boldsymbol{b}$ are linear in $\boldsymbol{x}$ say $\boldsymbol{u} = U\boldsymbol{x}$ and $\boldsymbol{b} = B\boldsymbol{x}$, where $U = \nabla\boldsymbol{u} = \mathcal{D} + \Omega$ and $B = \nabla\boldsymbol{b} = \mathcal{K} + \mathcal{L}$ are functions of time alone. Therefore, the advection terms become

$$\begin{aligned} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} &= (U\boldsymbol{x}\cdot\nabla)U\boldsymbol{x} = U^2\boldsymbol{x}, \\ (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} &= (U\boldsymbol{x}\cdot\nabla)B\boldsymbol{x} = UB\boldsymbol{x}. \end{aligned} \tag{32}$$

Hence, we have the convective terms

$$\begin{aligned} \frac{D\boldsymbol{u}}{Dt} &= \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = \frac{d\mathcal{D}}{dt}\boldsymbol{x} + \frac{d\Omega}{dt}\boldsymbol{x} + U^2\boldsymbol{x}, \\ \frac{D\boldsymbol{b}}{Dt} &= \frac{\partial \boldsymbol{b}}{\partial t} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{b} = \frac{d\mathcal{K}}{dt}\boldsymbol{x} + \frac{d\mathcal{L}}{dt}\boldsymbol{x} + UB\boldsymbol{x}. \end{aligned} \tag{33}$$

The symmetric parts of (10) can be recast as follows

$$\begin{aligned} \frac{D\mathcal{D}}{Dt} &+ \mathcal{D}^2 + \Omega^2 = -\hat{P} + (\boldsymbol{b}\cdot\nabla)\mathcal{K} + \mathcal{K}^2 + \mathcal{L}^2, \\ \frac{D\mathcal{K}}{Dt} &= (\boldsymbol{b}\cdot\nabla)\mathcal{D} + \mathcal{D}\mathcal{L} - \mathcal{L}\mathcal{D} + \Omega\mathcal{K} - \mathcal{K}\Omega. \end{aligned} \tag{34}$$

As the gradient of velocity and magnetic field commutes, so that $UB - BU = 0$ and hence we get $\mathcal{D}\mathcal{L} - \mathcal{L}\mathcal{D} + \Omega\mathcal{K} - \mathcal{K}\Omega = 0$. Because of this, we can rewrite (34) as follows:

$$\begin{aligned} \frac{D\mathcal{D}}{Dt} &+ \mathcal{D}^2 + \Omega^2 = -\hat{P} + (\boldsymbol{b}\cdot\nabla)\mathcal{K} + \mathcal{K}^2 + \mathcal{L}^2, \\ \frac{D\mathcal{K}}{Dt} &= (\boldsymbol{b}\cdot\nabla)\mathcal{D}. \end{aligned} \tag{35}$$

Also, we conclude that the term $2T + \frac{1}{2}\boldsymbol{\omega}\times\boldsymbol{j}$ in the second equation of (16) vanishes. Therefore (16) can be written as

$$\begin{aligned} \frac{D\boldsymbol{\omega}}{Dt} &- \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{j} - \boldsymbol{j}\cdot\nabla\boldsymbol{b}, \\ \frac{D\boldsymbol{j}}{Dt} &= (\boldsymbol{b}\cdot\nabla)\boldsymbol{\omega} \end{aligned} \tag{36}$$

Since $\mathcal{D}$, Ω, $\mathcal{K}$ and $\mathcal{L}$ are functions of time alone, so that all the terms $(\boldsymbol{u}\cdot\nabla)\Omega$, $(\boldsymbol{u}\cdot\nabla)\mathcal{D}$, $(\boldsymbol{u}\cdot\nabla)\mathcal{K}$, $(\boldsymbol{u}\cdot\nabla)\mathcal{L}$, $(\boldsymbol{b}\cdot\nabla)\Omega$, $(\boldsymbol{b}\cdot\nabla)\mathcal{D}$, $(\boldsymbol{b}\cdot\nabla)\mathcal{K}$, and $(\boldsymbol{b}\cdot\nabla)\mathcal{L}$ vanishes. Thus we have

$$\begin{aligned} \frac{d\boldsymbol{\omega}}{dt} &= \boldsymbol{\omega}\cdot\nabla\boldsymbol{u} - \boldsymbol{j}\cdot\nabla\boldsymbol{b}, \\ \frac{d\boldsymbol{j}}{dt} &= \boldsymbol{0}. \end{aligned} \tag{37}$$

Using simple calculations, we see that $\boldsymbol{\omega}\cdot\nabla\boldsymbol{u} = \mathcal{D}(t)\boldsymbol{\omega}$ and $\boldsymbol{j}\cdot\nabla\boldsymbol{b} = \mathcal{K}(t)\boldsymbol{j}$. Hence, the vectors $\boldsymbol{\omega}$ and $\boldsymbol{j}$ satisfies the equations (30). Also, if we look at the skew-symmetric part of (18), then we see that it is equivalent to the following equations.

$$\begin{aligned} \frac{d\Omega}{dt} &+ \mathcal{D}\Omega + \Omega\mathcal{D} = \mathcal{K}\mathcal{L} + \mathcal{L}\mathcal{K}, \\ \frac{d\mathcal{L}}{dt} &= 0. \end{aligned} \tag{38}$$

Furthermore, we see that (35) are results into the following.

$$\begin{aligned} \frac{d\mathcal{D}}{dt} &+ \mathcal{D}^2 + \Omega^2 = -\hat{P} + \mathcal{K}^2 + \mathcal{L}^2, \\ \frac{d\mathcal{K}}{dt} &= 0. \end{aligned} \tag{39}$$

Using (38) and (39) to (33), we get

$$\begin{aligned} \frac{D\boldsymbol{u}}{Dt} &= -\mathcal{D}^2\boldsymbol{x} - \Omega^2\boldsymbol{x} - \hat{P}\boldsymbol{x} + \mathcal{K}^2\boldsymbol{x} + \mathcal{L}^2\boldsymbol{x} \\ &- \mathcal{D}\Omega\boldsymbol{x} - \Omega\mathcal{D}\boldsymbol{x} + \mathcal{K}\mathcal{L}\boldsymbol{x} + \mathcal{L}\mathcal{K}\boldsymbol{x} + U^2\boldsymbol{x}. \end{aligned} \tag{40}$$

By simple calculations, we have $U^2\boldsymbol{x} = (\mathcal{D}^2 + \Omega^2 + \mathcal{D}\Omega + \Omega\mathcal{D})\boldsymbol{x}$ and $B^2\boldsymbol{x} = (\mathcal{K}^2 + \mathcal{L}^2 + \mathcal{K}\mathcal{L} + \mathcal{L}\mathcal{K})\boldsymbol{x}$ and substituting these terms in (40), it will get result into following equation.

$$\frac{D\boldsymbol{u}}{Dt} = -\hat{P}\boldsymbol{x} + B^2\boldsymbol{x}. \tag{41}$$

Now, we calculate the terms $(\boldsymbol{b}\cdot\nabla)\boldsymbol{b}$ and gradient of pressure function P. By simple calculations, we see that $(\boldsymbol{b}\cdot\nabla)\boldsymbol{b} = B^2\boldsymbol{x}$ and $\nabla P = \nabla\left(\frac{1}{2}\hat{P}(t)\boldsymbol{x}\cdot\boldsymbol{x}\right) = \hat{P}(t)\boldsymbol{x}$. Thus, by substituting the $B^2\boldsymbol{x}$ and $\hat{P}(t)\boldsymbol{x}$ into (41) that gives us the momentum equation of fluid velocity in (3). To verify the momentum equation of magnetic field, we substitute (38) and (39) into the second equation of (33), we get

$$\frac{D\boldsymbol{b}}{Dt} = \frac{d\mathcal{K}}{dt}\boldsymbol{x} + \frac{d\mathcal{L}}{dt}\boldsymbol{x} + UB\boldsymbol{x} = UB\boldsymbol{x} = BU\boldsymbol{x} = (\boldsymbol{b}\cdot\nabla)\boldsymbol{u}. \tag{42}$$

Thus, we see that fluid velocity, magnetic field, and pressure are given by (29) satisfy the equations (3). Therefore $\boldsymbol{u}$, $\boldsymbol{b}$ and P given by (29) is the special solution of (3). Hence, it completes the proof.

In the following section 3, we focus on the qualitative nature of system (3) through the ODE reductions (30) come across in Theorem 2.1.

3. Qualitative Analysis

We established in Section 2 that if we look at the solutions of (3) in the form of (29) and the gradient of fluid velocity commutes with the gradient of the magnetic field, then the vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$ must satisfy (30). So in this view, we present here the qualitative nature of the solutions of (3) which are in the form of (29).

Now we suppose that the gradients of velocity and magnetic field commutes and solutions of (3) exists in the form of (29). In fact, if the deformation matrices $\mathcal{D}$ and $\mathcal{K}$ are continuous functions of time t then the existence of such solutions are guaranteed.

3.1. Integrability of an Ideal MHD Equations

Let us consider ideal MHD equations (3) in which the gradients of fluid velocity and magnetic field are commutes. Furthermore, the deformation matrices $\mathcal{D}$ and $\mathcal{K}$ are continuous functions of time-variable $t \in \mathbb{R}$. Now we are looking at the following initial value problem.

$$\begin{aligned} \frac{d\boldsymbol{\omega}(t)}{dt} &= \mathcal{D}(t)\boldsymbol{\omega}(t) - \mathcal{K}(t)\boldsymbol{j}(t), \quad \boldsymbol{\omega}(0) = \boldsymbol{\omega}_0, \\ \frac{d\boldsymbol{j}(t)}{dt} &= \mathbf{0}, \quad \boldsymbol{j}(0) = \boldsymbol{j}_0. \end{aligned} \tag{43}$$

It is very clear from the second equation of (43) that $\boldsymbol{j}(t) = \boldsymbol{j}_0$ is a constant vector. Thereby substituting in the first equation of (43) it results in the non homogeneous linear equation. Furthermore, $\mathcal{D}(t)$, $\mathcal{K}(t)$ are matrices with all entries are real valued continuous functions. So that there exists a unique solution to the system (43) and through the quadrature, we can solve the system proving the complete integrability of the system (43).

Suppose that the pressure P is given by $P = \frac{1}{2}\hat{P}(t)\boldsymbol{x} \cdot \boldsymbol{x}$, where $\hat{P}(t)$ is determined by the equation (31). Let us consider $\mathcal{R} = \{(\boldsymbol{x}, t) \in \mathbb{R}^3 \times \mathbb{R} : \nabla\boldsymbol{u}(\boldsymbol{x}, t)\nabla\boldsymbol{b}(\boldsymbol{x}, t) = \nabla\boldsymbol{b}(\boldsymbol{x}, t)\nabla\boldsymbol{u}(\boldsymbol{x}, t)\}$ be the domain in $(\boldsymbol{x}, t)$-space of the system (43). Whereas, $\mathcal{D}$ and $\mathcal{K}$ are matrices and all elements of these matrices are real valued continuous functions of variable $t \in \mathbb{R}$. Now we are looking at the existence of solution of initial value problem (43) in the domain $\mathcal{R}$. The system (43) is a linear system and all elements of coefficient matrices are continuous real valued functions. Hence, the initial value problem (43) determine the curl of fluid velocity and magnetic field uniquely. Consequently, we determine velocity and magnetic field uniquely. Furthermore, pressure function is determined through (31). From second equation of (43), we determine that $\boldsymbol{j}(t) = \boldsymbol{j}_0$. Consequently, we can solve the linear system (43) and following is the solution:

$$\begin{aligned} \boldsymbol{\omega}(t) &= -e^{\int_0^t \mathcal{D}(s)ds}\left[\int_0^t \left\{e^{-\int_0^s \mathcal{D}(a)da}\mathcal{K}(s)\boldsymbol{j}_0\right\} ds\right] + e^{\int_0^t \mathcal{D}(s)ds}\boldsymbol{\omega}_0, \\ \boldsymbol{j}(t) &= \boldsymbol{j}_0 \end{aligned} \tag{44}$$

This implying that the system (43) is completely integrable.

3.2. Critical point Analysis

In this section, first we find the critical points of the system (30) and we will investigate their nature. It is clear that $(\boldsymbol{\omega}, \boldsymbol{j}) = (\mathbf{0}, \mathbf{0})$ is the critical point.

Now at the critical point $(\boldsymbol{\omega}, \boldsymbol{j}) = (\mathbf{0}, \mathbf{0})$ the system (30) can be written in matrix form as follows:

$$\begin{bmatrix} \dfrac{d\boldsymbol{\omega}}{dt} \\ \dfrac{d\boldsymbol{j}}{dt} \end{bmatrix} = \begin{bmatrix} \mathcal{D}(t) & -\mathcal{K}(t) \\ O & O \end{bmatrix} \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{j} \end{bmatrix} = A(t) \begin{bmatrix} \boldsymbol{\omega} \\ \boldsymbol{j} \end{bmatrix} \quad \text{(say)}, \tag{45}$$

where O's are 3×3 null matrices. In the above equation, we have a matrix $A = A(t)$ is an upper triangular block matrix and its diagonal blocks are $\mathcal{D}$, a deformation matrix and O, a null matrix. Since the eigenvalues of matrix A are the sum of eigenvalues of matrices $\mathcal{D}$ and O. Furthermore, $\mathcal{D}$ is a real symmetric matrix and its entries are continuous real valued functions so that all the eigenvalues of matrix A are real numbers for every t. Since the matrix A has a diagonal block matrix O of order three, therefore we have $\lambda = 0$ is at least threefold eigenvalue of a matrix A. This shows that the critical point $(\mathbf{0}, \mathbf{0})$ is degenerate. Thus, it is non-trivial to determine the nature of critical point $(\mathbf{0}, \mathbf{0})$. Before concluding the nature of critical point we need to investigate some important concept of linear algebra of the matrix system in which entries of matrices are continuous functions of time variable t. But we intuitively come to know that the solutions provided by (44) are asymptotically unstable and it will be proved in our next article.

In the following section, we describe the examples of two and three dimensional flow of fluid in the presence of a magnetic field for which we can determine the solutions of (3) in the form (29).

4. Examples

First, we consider the two dimensional flow of a fluid in the presence of a magnetic field, then after that we provide example of three dimensional flow.

Example 4.1. Consider a two dimensional time independent flow for which the constant vorticity and current density vectors are unit vectors in vertical direction. That is, $\boldsymbol{\omega} = \hat{\boldsymbol{e_3}} = (0, 0, 1)$ and $\boldsymbol{j} = \hat{\boldsymbol{e_3}} = (0, 0, 1)$ respectively. Let us assume that the deformation matrices $\mathcal{D}$ and $\mathcal{K}$ be given by;

$$\mathcal{D} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & -\lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, we obtained the special solutions of two dimensional ideal MHD fluid as follows:

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= \mathcal{D}(t)\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{\omega}(t)\times\boldsymbol{x} = \left(\lambda x_1 - \frac{1}{2}x_2,\, -\lambda x_2 + \frac{1}{2}x_1,\, 0\right), \\ \boldsymbol{b}(\boldsymbol{x},t) &= \mathcal{K}(t)\boldsymbol{x} + \tfrac{1}{2}\boldsymbol{j}(t)\times\boldsymbol{x} = \left(\lambda x_1 - \frac{1}{2}x_2, -\lambda x_2 + \frac{1}{2}x_1, 0\right), \\ P(\boldsymbol{x},t) &= \frac{1}{2}\hat{P}(t)\boldsymbol{x}\cdot\boldsymbol{x} = 0. \end{aligned} \tag{46}$$

Mathematically, it is possible to find a flow of an ideal fluid in the presence of magnetic field such that there is no pressure at every point of the fluid. But practically, it is impossible, because zero pressure at a point, implying that the temperature of molecule at that point is zero. Consequently, there is no molecular movement possible.

The following example is an example of three dimensional flow with realistic deformation of fluid particles.

Example 4.2. Consider a three dimensional flow of fluid in the presence of a magnetic field in which deformation matrices of fluid velocity and magnetic field are respectively as follows:

$$\mathcal{D} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -\lambda+1 \end{bmatrix}, \quad \mathcal{K} = \begin{bmatrix} \frac{1}{4}(-3-5\lambda) & 0 & 0 \\ 0 & \frac{1}{4}(5+3\lambda) & 0 \\ 0 & 0 & \frac{1}{2}(-1+\lambda) \end{bmatrix}.$$

The vorticity and current density vectors are respectively $\boldsymbol{\omega} = (0,0,-2)$ and $\boldsymbol{j} = (0,0,4)$. Now by using Theorem-2.1 we obtained a special solution of an ideal MHD equations (3) as follows.

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x},t) &= (\lambda x_1 + x_2, -x_1 - x_2, -(-1+\lambda)x_3)\,, \\ \boldsymbol{b}(\boldsymbol{x},t) &= \left(-\tfrac{1}{4}(3+5\lambda)x_1 - 2x_2,\, 2x_1 + \tfrac{1}{4}(5+3\lambda)x_2,\, \tfrac{1}{2}(-1+\lambda)x_3\right), \\ P(\boldsymbol{x},t) &= \tfrac{3}{32}(-1+\lambda)\left[(13+3\lambda)x_1^2 + (13+3\lambda)x_2^2 - 4(-1+\lambda)x_3^2\right]. \end{aligned}$$

5. Conclusion

We have considered an ideal MHD equations (3) for which we obtained its special solutions in the form of (5). Through the Proposition 2.1 we have shown that the vorticity $\boldsymbol{\omega}$ and current density $\boldsymbol{j}$ satisfies the PDEs (16). Furthermore, if gradients of fluid velocity and magnetic field commutes then the system (30) is the ODE reduction. Further, we proved that the system (30) is completely integrable. Also, we conclude that the system (30) has a degenerate critical point $(\boldsymbol{\omega}, \boldsymbol{j}) = (\mathbf{0}, \mathbf{0})$. Finally, we have provided the examples of special solutions in the section of the examples.

References

[1] Duvaut G. and Lions J. L., Inéquations en thermoélasticicité et magnétohydrodynamique, Archive for Rational Mechanics and Analysis, Vol. 46 (1972), 241-279.

[2] Hopf E., Üer die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Mathematische Nachrichten, Vol. 4 (1951), 213-231.

[3] Leray J., Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Mathematica, Vol. 63, No. 1 (1934), 193-248.

[4] Majda A., Introduction to PDEs and Waves for the Atmosphere and Ocean, American Mathematical Society, Providence, Rhode Island, 2003.

[5] Rajagopal K. R. and Ruzickz M. and Srinivasa A. R., On the Oberbeck-Boussinesq Approximation, Mathematical Models and Methods in Applied Sciences, Vol. 6 (1996), 1157-1167.

[6] Wu J., Generalized MHD equations, Journal of Differential Equations, Vol. 195 (2003), 284-312.

[7] Zhang Z., MHD Equations with Regularity in One Direction, International Journal of Partial Differential Equations, Vol. 2014, Article ID: 213083.