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A NEW APPROACH OF ITERATIVE METHODS FOR SOLVING NON-LINEAR EQUATIONS USING BOOLE'S QUADRATURE

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Abstract: In this paper, we produced two efficient iterative methods improvising two earlier methods for solving non-linear equations using a quadrature of higher precision. The convergence analysis of the methods are studied. Using these new methods, some non-linear equations have been solved numerically. The results are found to be more encouraging as compared to those by using some earlier established methods.

Keywords and Phrases: Newton-Raphson method, Iterative method, Boole's rule, Convergence analysis, Modified super Halley method.

2020 Mathematics Subject Classification: 65D32, 65H04, 65H05.

1. Introduction

Detecting zeros of a single variable non-linear equation $f(x) = 0$ is always fascinating problem in numerical analysis. It has massive implementations in applied sciences. Researchers use iterative methods in solving non-linear equations. Taylor's rule, quadrature rules act as foundations in forming iterative methods.

In this paper, our intension is to design an efficient method of solution for a simple root of a non-linear equation $f(x) = 0$, where $f : I \subset R \to R$ defined on an open interval I.

We introduce Boole's quadrature rule [4].

$$\int_a^b f(x)dx = \frac{2h}{45}(7f_0 + 32f_1 + 12f_2 + 32f_3 + 7f_4) + E \tag{1.1}$$

where $a = x_0 < x_1 < x_2 < ... < x_n = b,\ x_{i+1} - x_i = h,\ f_i = f(x_i)$, $E = -\frac{8h^7}{945} f^{vi}(\xi)$, $a < \xi < b$.
This is one of the basic quadrature rule having degree of precision 5. We have used Newton-Raphson method [3].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.2}$$

This method converges quadratically.
On the other hand, we implement modified super Halley [6].

$$x_{n+1} = x_n - \left[1 + \frac{1}{2} K_f(x_n) + \frac{1}{2} \frac{K_f(x_n)^2}{1 - \alpha K_f(x_n)}\right] \frac{f(x_n)}{f'(x_n)}. \tag{1.3}$$

Where α is a parameter

$$K_f(x_n) = \frac{f''[x_n - f(x_n)/(3f'(x_n))]f(x_n)}{(f'(x_n))^2}$$

$$a = 1$$

This method has fourth order of convergence.
We improved the above two methods and designed two algorithms using Boole's quadrature. We did convergence analysis of the improved methods. Then we computed the solutions of a bunch of test non-linear equations using the new algorithms. Our result is found to be more encouraging than the earlier papers.
The content of this paper is summarized as follows: Section-1 is introductory one, Sction-2 deals with newly developed iterative methods using Boole's rule, Section-3 deals with convergence analysis, Section-4 deals with numerical verification of some test non-linear equations and conclusion have been given in the last section.

Several researchers [1, 2, 5, 7, 9, 8] performs an outstanding work in this field of research.

2. Iterative Method

Consider nonlinear equation

$$f(x) = 0 \tag{2.1}$$

Now from (1.1) we have

$$\int_a^b f(x)dx = \frac{(b-a)}{90}\left[7f(a) + 32f\left(\frac{3a+b}{4}\right) + 12f\left(\frac{a+b}{2}\right) + 32f\left(\frac{a+3b}{4}\right) + 7f(b)\right] \tag{2.2}$$

Let α be a simple zero of sufficiently differentiable function $f:(a,b)\subset R\rightarrow R$ for an open interval (a,b) and x_0 is sufficiently close to α. We have

$$f(x)=f(x_n)+\int_{x_n}^{x}f'(t)dt \tag{2.3}$$

Now using (1.2), we obtain

$$\int_{x_n}^{x}f'(t)dt$$

$$=\frac{(x-x_n)}{90}\left[7f'(x_n)+32f'\left(\frac{3x_n+x}{4}\right)+12f'\left(\frac{x_n+x}{2}\right)+32f'\left(\frac{x_n+3x}{4}\right)+7f'(x)\right] \tag{2.4}$$

From (1.3) and (1.4), we obtain

$$x=x_n-\frac{90f(x_n)}{\left[7f'(x_n)+32f'\left(\frac{3x_n+x}{4}\right)+12f'\left(\frac{x_n+x}{2}\right)+32f'\left(\frac{x_n+3x}{4}\right)+7f'(x)\right]} \tag{2.5}$$

Using fixed point iteration method we conclude as

$$x_{n+1}=x_n-\frac{90f(x_n)}{\left[7f'(x_n)+32f'\left(\frac{3x_n+x_{n+1}}{4}\right)+12f'\left(\frac{x_n+x_{n+1}}{2}\right)+32f'\left(\frac{x_n+3x_{n+1}}{4}\right)+7f'(x_{n+1})\right]} \tag{2.6}$$

which is in implicit form.

Algorithm 2.1.

Replacing y_n in place of x_{n+1} on the right hand side of (2.6). For the given x_0, compute the approximate solution x_{n+1} by the following iterative scheme

$$x_{n+1}=x_n-\frac{90f(x_n)}{\left[7f'(x_n)+32f'\left(\frac{3x_n+y_n}{4}\right)+12f'\left(\frac{x_n+y_n}{2}\right)+32f'\left(\frac{x_n+3y_n}{4}\right)+7f'(y_n)\right]}, \tag{2.7}$$

where $y_n=x_n-\frac{f(x_n)}{f'(x_n)}$, $n=0,1,2,....$

Algorithm 2.2.

Replacing y_n^* in place of x_{n+1} in right hand side of (2.6). For a given x_0, we can compute the approximate solution of x_{n+1} of the equation by

$$x_{n+1}=x_n-\frac{90f(x_n)}{\left[7f'(x_n)+32f'\left(\frac{3x_n+y_n^*}{4}\right)+12f'\left(\frac{x_n+y_n^*}{2}\right)+32f'\left(\frac{x_n+3y_n^*}{4}\right)+7f'(y_n^*)\right]}, \tag{2.8}$$

where $y_n^* = x_{n+1} = x_n - \left[1 + \frac{1}{2}K_f(x_n) + \frac{1}{2}\frac{K_f(x_n)^2}{1-\alpha K_f(x_n)}\right]\frac{f(x_n)}{f'(x_n)}$,
$K_f(x_n) = \frac{f''[x_n - f(x_n)/(3f'(x_n))]f(x_n)}{(f'(x_n))^2}$, $a = 1$, $n = 0, 1, 2, ...$

3. Convergence Analysis

Theorem 3.1. *Let $\alpha \in (a, b)$ be a simple zero of sufficiently differentiable function $f : (a, b) \to R$ for an open interval (a, b). If an initial approximation x_0 is adequately close to α, then the iterative method defined in (2.7) has order of convergence 3 and satisfies the error equation*

$$e_{n+1} = \frac{13}{15}c_2^2 e_n^3 + O(e_n^4)$$

where $c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}$, $k = 1, 2, 3, ...,$ $e_n = x_n - \alpha$ and $a, b \in R$.

Proof. Suppose α be simple zero of $f(x)$. Since $f(x)$ is sufficiently differentiable, using Taylor's expansion of $f(x_n)$, $f'(x_n)$ and $f''(x_n)$ about $x_n = \alpha$, we obtain

$$f(x_n) = f'(\alpha)\left[e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O(e_n^6)\right] \quad (3.1.1)$$

$$f'(x_n) = f'(\alpha)\left[1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 5c_5e_n^4 + O(e_n^5)\right] \quad (3.1.2)$$

$$f''(x_n) = f'(\alpha)\left[2c_2 + 6c_3e_n + 12c_4e_n^2 + 20c_5e_n^3 + O(e_n^4)\right] \quad (3.1.3)$$

From algorithm-2.1

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 + 2(c_3 - c_2^2)e_n^3 + O(e_n^4)$$

$$k_n = \frac{3x_n + y_n}{4} = \alpha + \frac{3}{4}e_n + \frac{c_2}{4}e_n^2 + \frac{(c_3 - c_2^2)}{2}e_n^3 + O(e_n^4)$$

$$(k_n - \alpha)^2 = \frac{9}{16}e_n^2 + \frac{3c_2}{8}e_n^3 + O(e_n^4)$$

$$(k_n - \alpha)^3 = \frac{27}{64}e_n^3 + O(e_n^4)$$

Using Taylor series, we get

$$f'(k_n) = f'(\alpha)\left[1 + \frac{3c_2}{2}e_n + \left(\frac{c_2^2}{2} + \frac{27c_3}{16}\right)e_n^2 + O(e_n^3)\right] \quad (3.1.4)$$

$$l_n = \frac{x_n + y_n}{2} = \alpha + \frac{1}{2}e_n + \frac{c_2}{2}e_n^2 + O(e_n^3)$$

$$(l_n - \alpha)^2 = \frac{1}{4}e_n^2 + O(e_n^3)$$

Applying Taylor series expansion of $f(l_n)$ about $l_n = \alpha$

$$f'(l_n) = f'(\alpha)\left[1 + c_2 e_n + \left(c_2^2 + \frac{3c_3}{4}\right) e_n^2 + O(e_n^3)\right] \tag{3.1.5}$$

$$q_n = \frac{x_n + 3y_n}{4} = \alpha + \frac{1}{4}e_n + \frac{3c_2}{4}e_n^2 + O(e_n^3)$$

$$(q_n - \alpha)^2 = \frac{1}{16}e_n^2 + O(e_n^3)$$

$$\therefore \quad f'(q_n) = f'(\alpha)\left[1 + \frac{c_2}{2}e_n + \left(\frac{3c_2^2}{2} + \frac{3c_3}{16}\right) e_n^2 + O(e_n^3)\right] \tag{3.1.6}$$

Again,

$$\begin{aligned} y_n - \alpha &= c_2 e_n^2 + (e_n^3) \\ f'(y_n) &= f'(\alpha)[1 + 2c_2^2 e_n^2 + O(e_n^3)] \qquad (3.1.7) \\ X_n &= 7f'(x_n) + 32f'(k_n) + 12f'(l_n) + 32f'(q_n) + 7f(y_n) \\ &= f'(\alpha)[90 + 90c_2 e_n + (78c_2^2 + 90c_3)e_n^2 + O(e_n^3)] \end{aligned}$$

Now,

$$M_n = \frac{90f(x_n)}{X_n} = \frac{[e_n + c_2 e_n^2 + c_3 e_n^3 + O(e_n^4)]}{[1 + c_2 e_n + \left(c_3 + \frac{13}{15}c_2^2\right) e_n^2 + O(e_n^3)]} = e_n - \frac{13}{15}c_2^2 e_n^3 + O(e_n^4)$$

Hence, from (2.7)

$$\begin{aligned} x_{n+1} &= x_n - M_n = (e_n - \alpha) - \left[e_n - \frac{13}{15}c_2^2 e_n^3 + O(e_n^4)\right] \\ &= \alpha + \frac{13}{15}c_2^2 e_n^3 + O(e_n^4) \\ \therefore \quad e_{n+1} &= \frac{13}{15}c_2^2 e_n^3 + O(e_n^4). \end{aligned}$$

Which implies that (2.7) converges cubically.

Theorem 3.2. *Let $\alpha \in (a, b)$ be a simple zero of sufficiently differentiable function $f : (a, b) \subset R \to R$ for an open interval (a, b). If x_0 adequately close to α, then the iterative method defined in (2.8) has order of convergence 5 and satisfies the error equation*

$$e_{n+1} = \left(\frac{463}{3}c_2^4 + \frac{77}{3}c_2 c_4 - 270c_2^2 c_3 - c_5\right) e_n^5 + O(e_n^4)$$

where $c_k = \frac{1}{k!}\frac{f^k(\alpha)}{f'(\alpha)}$, $k = 1, 2, 3, ...,$ $e_n = x_n - \alpha$ *and* $a, b \in R$.

Proof.

$$z_n = x_n - \frac{f(x_n)}{3f'(x_n)} = \alpha + \frac{2}{3}e_n + \frac{c_2}{3}e_n^2 + \frac{2(c_3 - c_2^2)}{3}e_n^3 + O(e_n^4)$$

Applying Taylor series expansion, we have

$$f''(z_n) = f'(\alpha)[2c_2 + 4c_3e_n + p_1e_n^2 + p_2e_n^3 + O(e_n^4)] \tag{3.2.1}$$

where $p_1 = 2c_2c_3 + \frac{16c_4}{3}, p_2 = 4c_3(c_3 - c_2^2) + \frac{16c_2c_4}{3} + \frac{160c_5}{27}$

$$f''(z_n)f(x_n) = [f'(\alpha)]^2[2c_2e_n + (4c_3 + 2c_2^2)e_n^2 + \left(8c_2c_3 + \frac{16c_4}{3}\right)e_n^3 + O(e_n^4)] \tag{3.2.2}$$

$$[f'(x_n)]^2 = [f'(\alpha)]^2[1 + 4c_2e_n + q_1e_n^2 + q_2e_n^3 + q_3e_n^4 + O(e_n^5)] \tag{3.2.3}$$

where $q_1 = 4c_2^2 + 6c_3, q_2 = 12c_2c_3 + 8c_4, q_3 = 9c_3^2 + 16c_2c_4$

$$\frac{1}{[f'(x_n)]^2} = \frac{1}{[f'(\alpha)]^2}[1 - 4c_2e_n + (12c_2^2 - 6c_3)e_n^2 + (36c_2c_3 - 32c_2^3 - 8c_4)e_n^3 + O(e_n^4)]$$

$$K_f(x_n) = \frac{f''(x_n)f(x_n)}{[f'(x_n)]^2} = w_1e_n^2 + w_2e_n^2 + w_3e_n^3 + w_4e_n^4 + O(e_n^5) \tag{3.2.4}$$

Here $w_1 = 2c_2$, $w_2 = 4c_3 - 6c_2^2$, $w_3 = 16c_2^3 - 20c_2c_3 + \frac{16c_4}{3}$, $w_4 = 76c_2^2c_3 - 40c_2^4 - \frac{112}{3}c_2c_4$.

$$\frac{1}{1 - K_f(x_n)} = 1 + 2c_2e_n + (4c_3 - 2c_2^2)e_n^2 + \left(\frac{16c_4}{3} - 4c_2c_3\right)e_n^3 + O(e_n^4) \tag{3.2.5}$$

$$[K_f(x_n)]^2 = z_1e_n^2 + z_2e_n^3 + z_3e_n^4 + O(e_n^5) \tag{3.2.6}$$

Where $z_1 = 4c_2^2$, $z_2 = 16c_2c_3 - 24c_2^3$, $z_3 = 100c_2^4 + 16c_3^2 - 128c_3c_2^2 + \frac{64}{3}c_2c_4$.

$$\frac{[K_f(x_n)]^2}{2(1 - K_f(x_n))} = 2c_2^2e_n^2 + (8c_2c_3 - 8c_2^3)e_n^3 + (-40c_2^2c_3 + 22c_2^4 + 8c_3^2 + \frac{32}{3}c_2c_4)e_n^4 + O(e_n^5) \tag{3.2.7}$$

$$\therefore \quad Y_n = 1 + \frac{1}{2}K_f(x_n) + \frac{[K_f(x_n)]^2}{2(1 - K_f(x_n))}$$

$$= 1 + c_2e_n + (2c_3 - c_2^2)e_n^2 + \left(\frac{8c_4}{3} - 2c_2c_3\right)e_n^3 + (2c_2^4 + 8c_3^2 - 2c_2^2c_3 - 8c_2c_4)e_n^4 + O(e_n^5)$$

Now

$$Y_n \frac{f(x_n)}{f'(x_n)} = e_n + \left(3c_2c_3 - 2c_2^3 - \frac{c_4}{3}\right) e_n^4 + O(e_n^5)$$
$$x_{n+1}^* = x_n - Y_n \frac{f(x_n)}{f'(x_n)} = \alpha + \left(2c_2^3 + \frac{c_4}{3} - 3c_2c_3\right) e_n^4 + O(e_n^5)$$

Assuming

$$y_n^* = x_{n+1}^* = \alpha + \left(2c_2^3 + \frac{c_4}{3} - 3c_2c_3\right) e_n^4 + O(e_n^5)$$

Using Taylor series, we get

$$f'(y_n^*) = f'(\alpha)\left[1 + 2c_2\left(2c_2^3 + \frac{c_4}{3} - 3c_2c_3\right) e_n^4 + O(e_n^5)\right] \tag{3.2.8}$$
$$a_n = \frac{3x_n + y_n^*}{4} = \alpha + \frac{3}{4}e_n + \frac{\left(2c_2^3 + \frac{c_4}{3}\right)}{4} e_n^4 + O(e_n^5)$$

Applying Taylor series, we have

$$f'(a_n) = f'(\alpha)\left[1 + \frac{3c_2}{2}e_n + \frac{27}{16}c_3e_n^2 + \frac{27}{16}c_4e_n^3 + \left(c_2^4 + \frac{c_2c_4}{6} - \frac{3c_2^2c_3}{2}\right) e_n^4 + O(e_n^5)\right] \tag{3.2.9}$$
$$b_n = \frac{x_n + y_n}{2} = \alpha + \frac{1}{2}e_n + \left(c_2^3 + \frac{c_4}{6} - \frac{3c_2c_3}{2}\right) e_n^4 + O(e_n^5)$$

Similarly,

$$f'(b_n) = f'(\alpha)\left[1 + c_2e_n + \frac{3c_3}{4}e_n^2 + \frac{c_4}{2}e_n^3 + \left(2c_2^4 + \frac{c_2c_4}{3} - 3c_2^2c_3\right) e_n^4 + O(e_n^5)\right] \tag{3.2.10}$$
$$g_n = \frac{x_n + 3y_n}{4} = \alpha + \frac{1}{4}e_n + \frac{(6c_2^3 + c_4 - 9c_2c_3)}{4} e_n^4 + O(e_n^5)$$
$$f'(g_n) = f'(\alpha)\left[1 + \frac{c_2}{2}e_n + \frac{3c_3}{16}e_n^2 + \frac{c_4}{16}e_n^3 + \frac{(6c_2^4 + c_2c_4 - 9c_2^2c_3)}{4} e_n^4 + O(e_n^5)\right] \tag{3.2.11}$$
$$\begin{aligned} M_n &= 7f'(x_n) + 32f'(a_n) + 12f'(b_n) + 32f'(g_n) + 7f'(y_n^*) \\ &= 90f'(\alpha)[1 + c_2e_n + c_3e_n^2 + c_4e_n^3 + v_1e_n^4 + O(e_n^5)]. \end{aligned}$$

Where $v_1 = \frac{460}{3}c_2^4 - 270c_2^2c_3 + \frac{74}{3}c_2c_4$.
Hence, from (2.8) we have

$$\begin{aligned} x_{n+1} &= x_n - \frac{90f(x_n)}{M_n} \\ &= (e_n+\alpha) - \left[e_n + \left(\frac{-463}{3}c_2^4 - \frac{77}{3}c_2c_4 + 270c_2^2c_3 + c_5\right)e_n^5 + O(e_n^6)\right] \end{aligned}$$

So

$$e_{n+1} = \left(\frac{463}{3}c_2^4 + \frac{77}{3}c_2c_4 - 270c_2^2c_3 - c_5\right)e_n^5 + O(e_n^6)$$

Hence, algorithm-2 has order of convergence 5.

4. Numerical Verification

Comparative study of algorithms 2.1 and 2.2 is given in the following table-4.1. All computations are carried out with the help of Python 3.8. The tolerance during computation is taken as $\in = 10^{-16}$. We take $|x_{n+1} - x_n| < \in$ and $|f(x_n)| < \in$.

Table 4.1

Equations	Root (α^*) obtained by both Alg-1 and alg-2.	Initial Value	No. of iterations required to get root(α^*) (n)	
			By Alg-2.1	By Alg-2.2
$f_1(x) = x^2 - 5x + 2 = 0$	4.561552812808831	$x_0 = 4.0$	3	2
$f_2(x) = e^{-x} + \cos x = 0$	1.7461395304080125	$x_0 = 2.0$	3	2
$f_3(x) = e^x + x^2 - x - 4 = 0$	1.2886779668238684	$x_0 = 1.0$	3	2
$f_4(x) = sin^2x - x^2 + 1 = 0$	1.4044916482153411	$x_0 = 1.6$	3	2
$f_5(x) = \ln x - \cos x = 0$	1.3029640012160126	$x_0 = 1.3$	3	2
$f_6(x) = x^3 + 4x^2 - 15 = 0$	1.6319808055660636	$x_0 = 2.0$	4	3
$f_7(x) = \cos x - xe^x = 0$	0.5177573636824583	$x_0 = 1.0$	4	3
$f_8(x) = xe^{x^2} - sin^2x + 3cosx + 5 = 0$	-1.2076478271309188	$x_0 = -1.0$	5	4
$f_9(x) = e^x - 4x^2 = 0$	0.7148059123627778	$x_0 = 2.0$	4	3
$f_{10}(x) = e^{-x} + \cos x = 0$	1.7461395304080125	$x_0 = 1.5$	3	2

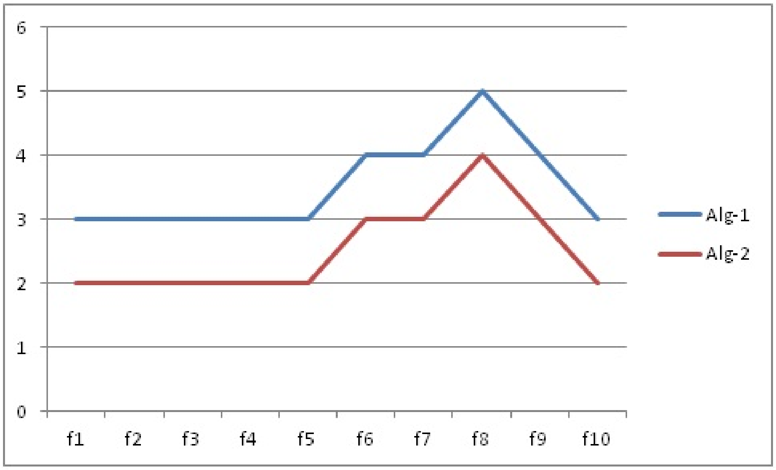

Figure 1. Graphical comparison of Alg-1 and Alg-2

6. Conclusion

We observed from table-4.1 that the algorithm-2.2 yields more accurate result in comparison to algorithm-2.1. This shows that the second method is more efficient than the first one. However, we like to mention that the newly developed two iterative methods in this paper, is much more competent and yielding encouraging results than those in case of other earlier established iterative methods.

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