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COMMON FIXED POINT THEOREMS FOR NON COMPATIBLE MAPPINGS IN G- METRIC SPACE

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Abstract: In this paper, we prove some common fixed point theorems for a pair of non- compatible faintly compatible self-mappings in G-metric spaces. Our results extend and unify some fixed points theorems in literature.

Keywords and Phrases: G-metric space, common fixed point, non-compatible mappings, faintly compatible mappings.

2020 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Fixed point theory is an important, productive, and powerful mathematical tool due to its application in areas such as variation and linear inequalities. The fixed point theorems in metric spaces are playing a significant role to construct methods in mathematics to solve problems in applied science, economics, physics and engineering. So over the past few decades, the metric fixed point theory has become an important field of research in both pure and applied science. Some of these works should be noted in [1, 3, 8-13]. In fact it has become one of the most essential tool in non-linear functional analysis, optimization, mathematical model, economy and medicine. The concept of G-metric spaces was introduced by Mustafa and Sims [17] in the year 2006 as a generalization of the metric spaces. In these type of spaces a non – negative real number is assigned to every triplet of element. In [22] the celebrated Banach contraction mapping principle was also established and a fixed point result has been proved. It ensures the existence and uniqueness of

a fixed point under some contractive conditions. After that many researchers have contributed with different concepts in this space and several fixed point results relevant to metric spaces are being extended to G-metric spaces. Also one can note that fixed point results in G-metric spaces have been applied to proving the existence of solutions for a class of integral equations.
Now, we recall some preliminaries and basic definitions which are given below and will be used in our subsequent discussion.
In 2006, The concept of G-metric spaces was introduced by Mustafa and Sims as follows:

Definition 1.1. [17] *Let X be a nonempty set, and let $G : X \times X \times X \to \mathbb{R}^+$ be a function satisfying the following:*

(G1) $G(x, y, z) = 0$ *if* $x = y = z$,

(G2) $0 < G(x, x, y)$ *for all* x, y *in* X *with* $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ *for all* x, y, z *in* X *with* $z \neq y$,

(G4) $(x, z, y) = G(x, y, z) = G(y, z, x) = \ldots$ *(symmetry in all three variables),*

(G5) $G(x, y, z) \leq G(x, a, a)+G(a, y, z)$ *for all* x, y, z, a *in* X *(rectangle inequality).*

Then the function G is called a generalized metric or, more specifically, a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. [20] *Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence of points in X. We can say that $\{x_n\}$ is G-convergent to x if $\lim_{n\to\infty} G(x, x_n, x_m) = 0$, this implies that for each $\epsilon > 0$ there exists a positive integer N such that $G(x, x_n, x_n) < \epsilon \ \forall m, n \geq N$. We can say that x is the limit of the sequence and can write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$.*

Proposition 1.1. [20] *Let (X, G) be a G-metric space then the following are equivalent:*

1. $\{x_n\}$ *is G-convergent to* x,
2. $G(x_n, x_n, x) \to 0$ *as* $n \to \infty$,
3. $G(x_n, x, x) \to 0$ *as* $n \to \infty$,
4. $G(x_m, x_n, x) \to 0$ *as* $n \to \infty$.

Proposition 1.2. [20] *Let (X, G) be a G-metric space then the function $G(x, y, z)$ is jointly continuous in all three variables.*

Definition 1.3. [20] *A G-metric space (X, G) is called a symmetric G-metric if $G(x, y, y) = G(y, x, x)$ $\forall x, y \in X$.*

Proposition 1.3. [20] *A G-metric space (X, G) is called a G-complete if and only if (X, d_G) is a complete metric space.*

Proposition 1.4. [20] *Let (X, G) be a G-metric space. Then, for any $x, y, z, a \in X$ it follows that*

1. *If $G(x, y, z) = 0$ then $x = y = z$,*
2. $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
3. $G(x, y, y) \leq 2G(y, xx)$
4. $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
5. $G(x, y, z) \leq \frac{2}{3}(G(x, a, a) + G(y, a, a) + G(z, a, a))$.

Now, we give example of non-symmetric G-metric spaces.

Example 1.1. [20] Let $X = \{a, b\}$, and $G(a, a, a) = G(a, a, a) = 0$, $G(a, a, b) = 1$, $G(a, b, b) = 2$ and extend G to all of $X \times X \times X$ by symmetry in the variables. Then G is a G-metric space. It is non symmetric since $G(a, b, b) \neq G(a, a, b)$.
There has been a considerable interest to study common fixed point for a pair of mappings satisfying some contractive conditions in metric spaces. The notion of commutativity was introduced by G. Jungck [6] to find common fixed point theorems. In 1986 Jungck introduced the compatibility. Some theorems on non-compatibility are also notable in [7].

Definition 1.4. [2] *Let f and g be two self mappings on G-metric space (X, G). The mappings f and g are said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$ for some $z \in X$.*

Definition 1.5. [2] *Two maps f and g are said to be weakly compatible if they commute at coincidence points.*

Example 1.2. [2] Let $X = [-1, 1]$ and let G be the G-metric on $X \times X \times X$ defined as follows; $G(x, y, z) = |x - y| + |y - z| + |z - x|$ $\forall x, y, z \in X$.
Then (X, G) be a G-metric space. Let us define $fx = x$, $gx = \frac{x}{4}$. Consider the sequence $\{x_n\}$, where $x_n = \frac{1}{n}$, n is a natural number. It is clearly seen that the

mappings f, g are compatible, since $G(fgx_n, gfx_n, gfx_n) = 0$, here the sequence $\{x_n\}$ in X such that $x_n = \frac{1}{n}$ and $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = 0$ for $0 \in X$.

Example 1.3. [2] Let $X = [0, 3]$ and let G be a G-metric on $X \times X \times X$ defined as follows : $G(x, y, z) = |x - y| + |y - z| + |z - x|$ $\forall x, y, z \in X$ Now f, g are defined as follows :$fx = \left\{ \begin{array}{ll} 0 & \text{if } x \in [0,1), \\ 3 & \text{if } x \in [1,3] \end{array} \right\}$ and $gx = \left\{ \begin{array}{ll} 3-x & \text{if } x \in [0,1), \\ 3 & \text{if } x \in [1,3] \end{array} \right\}$.
Then for any $x \in [1, 3]$, x is a coincidence point and $fgx = gfx$, showing that f, g are compatible mapping.

2. Main Results

Definition 2.1. [2] *Two self-mappings f and g of a G-metric space (X, G) are said to be faintly compatible iff f and g are conditionally compatible and they commute on a nonempty subset of coincidence points whenever the set of coincidences is nonempty.*

Theorem 2.1. *Let f and g be non- compatible faintly compatible self- mappings of a G-metric space (X, G) satisfying the following conditions.*

(2.1) $fX \subseteq gX$.

(2.2) $G(fx, fy, fz) \le kG(gx, gy, gz)$, $0 \le k < 1$ $\forall x, y, z \in X$.

(2.3) Either f or g is continuous

Then f and g have a unique common fixed point.

Proof. Non-compatibility of f and g implies that there exist some sequence $\{x_n\}$ in X such that $f(x_n) \to t$ and $g(x_n) \to t$ for some $t \in X$, but $\lim_{n\to\infty} G(f(g(x_n)), g(f(x_n)), g(f(x_n)))$ is either non - zero or non - existent. Since f and g are faintly compatible and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$, there exist a sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = u$ (say) such that $\lim_{n\to\infty} G(f(g(z_n)), g(f(z_n)), g(f(z_n))) = 0$. Further, since f is continuous, then $\lim_{n\to\infty} f(f(z_n)) = f(u)$ and $\lim_{n\to\infty} f(g(z_n)) = f(u)$. From last three limits together implies that $\lim_{n\to\infty} g(f(z_n)) = f(u)$. Since $fX \subseteq gX$ implies $f(u) = g(v)$ for some $v \in X$ and $f(f(z_n)) \to g(v), g(f(z_n)) \to g(v)$, by using condition (2.2) of theorem 2.2 we get that $G(f(v), f(f(z_n)), f(f(z_n))) \le kG(g(v), g(f(z_n))g, g(f((z_n))))$. On letting $n \to \infty$, we get that $f(v) = g(v)$. Thus v is a coincidence point of f and g. Further, faint compatibility off and g imply that $f(g(v)) = g(f(v))$, and hence $f(g(v)) = g(f(v)) = f(f(v)) = g(g(v))$. Now we claim that $f(v) = f(f(v))$. If $f(v) \neq f(f(v))$, then by using condition (2,2) of theorem 2.2 we get

$$G(f(v), f(f(v)), f(f(v))) \le kG(g(v), g(f(v))g, g(f((v)))) = G(f(v), f(f(v)), f(f(v))).$$

A contradiction. Hence $f(v)$ is a common fixed point of f and g. The same conclusion is obtained when g is assumed to be continuous, since the continuity of g implies the continuity of f.

Uniqueness: We assume that $z_1 \neq z$ be another common fixed point of f and g. Then we have $G(z, z_1, z_1) > 0$ and

$$G(z, z_1, z_1) = G(f(z), f(z_1), f(z_1)) \leq kG(g(z), g(z_1), g(z_1)) < G(z, z_1, z_1)$$

acontradiction, therefore $z = z_1$. Hence uniqueness follows.

Theorem 2.2. *Let f and g be two non compatible faintly compatible self-mappings of a G-metric space (X, G) satisfying the condition (2.1) of Theorem 2.2 and (2.4) $G(fx, fy, fz) \leq G(gx, gy, gz)$ whenever $gx \neq gy = gz$. If either f or g is continuous, then f and g have a unique common fixed point.*

Proof. Non-compatibility of f and g implies that there exist some sequence $\{x_n\}$ in X such that $f(x_n) \to t$ and $g(x_n) \to t$ for some $t \in X$, but $\lim_{n\to\infty} G(f(g(x_n)), g(f(x_n)), g(f(x_n)))$ is either non - zero or non - existent. Since f and g are faintly compatible and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$, there exist a sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = u$ (say) such that $\lim_{n\to\infty} G(f(g(z_n)), g(f(z_n)), g(f(z_n))) = 0$. Further, since f is continuous, then $\lim_{n\to\infty} f(f(z_n)) = f(u)$ and $\lim_{n\to\infty} f(g(z_n)) = f(u)$. From last three limits together implies that $\lim_{n\to\infty} g(f(z_n)) = f(u)$. Since $fX \subseteq gX$ implies $f(u) = g(v)$ for some $v \in X$ and $f(f(z_n)) \to g(v), g(f(z_n)) \to g(v)$, by using condition (2.4) of Theorem 2.3 we get that $G(f(v), f(f(z_n)), f(f(z_n))) \leq G(g(v), g(f(z_n))g, g(f((z_n))))$. On letting $n \to \infty$, we get that $f(v) = g(v)$. Thus v is a coincidence point of f and g. Further, faint compatibility off and g imply that $f(g(v)) = g(f(v))$, and hence $f(g(v)) = g(f(v)) = f(f(v)) = g(g(v))$. Now we claim that $f(v) = f(f(v))$. If $f(v) \neq f(f(v))$, then by using condition (2.4) of Theorem 2.2, we get

$$G(f(v), f(f(v)), f(f(v))) \leq G(g(v), g(f(v))g, g(f((v)))) = G(f(v), f(f(v)), f(f(v))),$$

a contradiction. Hence $f(v)$ is a common fixed point of f and g. The same conclusion is obtained when g is assumed to be continuous, since the continuity of g implies the continuity of f. The uniqueness of the common fixed point follows from Theorem 2.2.

Theorem 2.3. *Let f and g be two noncompatible faintly compatible self mappings of a G-metric space (X, G) satisfying the condition (2.1) of theorem 2.2 and (2.5) $G(fx, fy, fz) \leq kG(gx, gy, gz), k \geq 0$;*
(2.6) $G(fx, f(fx), f(fx)) \neq \max\{G(fx, g(fx), g(fx), G(g(fx), f(fx), f(fx)))\}$, whenever the right is non zero. Then the mappings have a common fixed point.

Proof. Non-compatibility of f and g implies that there exist some sequence $\{x_n\}$

in X such that $f(x_n) \to t$ and $g(x_n) \to t$ for some $t \in X$, but $\lim_{n\to\infty} G(f(g(x_n)), g(f(x_n)), g(f(x_n)))$ is either non - zero or non - existent. Since f and g are faintly compatible and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$, there exist a sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = u$ (say) such that $\lim_{n\to\infty} G(f(g(z_n)), g(f(z_n)), g(f(z_n))) = 0$. Further, since f is continuous, then $\lim_{n\to\infty} f(f(z_n)) = f(u)$ and $\lim_{n\to\infty} f(g(z_n)) = f(u)$. From last three limits together implies that $\lim_{n\to\infty} g(f(z_n)) = f(u)$. Since $fX \subseteq gX$ implies $f(u) = g(v)$ for some $v \in X$ and $f(f(z_n)) \to g(v), g(f(z_n)) \to g(v)$, by using condition (2.1) of Theorem 2.2 we get that $G(f(v), f(f(z_n)), f(f(z_n))) \leq kG(g(v), g(f(z_n)), g(f((z_n))))$. On letting $n \to \infty$, we get that $f(v) = g(v)$. Thus v is a coincidence point of f and g. Further, faint compatibility of f and g imply that $f(g(v)) = g(f(v))$, and hence $f(g(v)) = g(f(v)) = f(f(v)) = g(g(v))$. Now we claim that $f(v) = f(f(v))$. If $f(v) \neq f(f(v))$, then by using condition (2.5) of Theorem 2.3, we get

$$G(f(v), f(f(v)), f(f(v))) \leq G(g(v), g(f(v))g, g(f((v)))) = G(f(v), f(f(v)), f(f(v))),$$

Hence $f(v)$ is a common fixed point of f and g. The same conclusion is obtained when g is assumed to be continuous, since the continuity of g implies the continuity of f.

Theorem 2.4. *Let f and g be two noncompatible faintly compatible self mappings of a G-metric space (X, G) satisfying the conditions as follows.*
(2.7) $G(fx, f(fx), f(fx)) \neq \max\{G(fx, g(fx), g(fx), G(g(fx), f(fx), f(fx)))\}$,
whenever the right-hand side is non zero. Suppose f and g are continuous, then the two mappings have a common fixed point.

Proof. Non-compatibility of f and g implies that there exist some sequence $\{x_n\}$ in X such that $f(x_n) \to t$ and $g(x_n) \to t$ for some $t \in X$, but $\lim_{n\to\infty} G(f(g(x_n)), g(f(x_n)), g(f(x_n)))$ is either non - zero or non - existent. Since f and g are faintly compatible and $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = t$, there exist a sequence $\{z_n\}$ in X satisfying $\lim_{n\to\infty} f(z_n) = \lim_{n\to\infty} g(z_n) = u$ (say) such that $\lim_{n\to\infty} G(f(g(z_n)), g(f(z_n)), g(f(z_n))) = 0$. Further, since f and g are continuous, then $\lim_{n\to\infty} f(g(x_n)) = f(t)$ and $\lim_{n\to\infty} g(f(x_n)) = f(t)$. By faint compatibility and continuous of f and g, we can easily obtain a common fixed point as it has been proved in the corresponding part of Theorem 2.4. Hence the proof.

3. Conclusion

In this paper, we prove some common fixed point theorems for a pair of non-compatible faintly compatible self-mappings in G-metric spaces. Our results extend and unify some fixed points theorems in literature.

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