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COMMON FIXED POINT THEOREMS OF ALMOST SUZUKI TYPE CONTRACTIONS IN BI COMPLEX VALUED b-METRIC SPACES

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Abstract: We study common fixed point theorems of Suzuki type contractions employing alpha admissible function for two selfmaps in bicomplex valued b-metric space rendered by rational expressions. These results are enhanced through examples. Also, as a consequence, we obtain common fixed point theorems for bi complex valued b-metric space endowed with a partial order.

Keywords and Phrases: Bicomplex valued b-metric spaces, α-admissible function, Suzuki type contractions, common fixed points.

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1. Introduction

The notion of complex valued metric spaces was introduced by Azam [3]. Latter, the generalization of complex valued metric spaces namely complex valued b-metric spaces, complex valued rectangular metric spaces, extended complex valued metric spaces considered by several authors, for example we refer [1, 2, 5, 8, 14, 16, 17, 19, 20]. Recently, the generalized version of complex valued metric spaces namely bicomplex valued metric spaces were introduced by Cho et. al., [6].

In the present paper, we denote $\mathbb{R}^+$, set of positive real numbers and $\mathbb{C}_1$ set of complex numbers.

The set of bicomplex numbers introduced in [15, 20] in the following way.

Bicomplex Numbers

The set of bicomplex numbers denoted by $\mathbb{C}_2$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C}_1$.

$$\mathbb{C}_2 = \{w = c_0 + i_1c_1 + i_2c_2 + i_1i_2c_3 : c_p \in \mathbb{R}, (p = 0, 1, 2, 3)\}.$$

We can also express $\mathbb{C}_2$ as

$$\mathbb{C}_2 = \{z_1 + i_2z_2 : z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = c_0 + i_1c_1,\ z_2 = c_2 + i_1c_3$, i_1 and i_2 are imaginary independent units such that $i_1^2 = -1 = i_2^2$. The product of $i_1i_2 = j$ such that $j^2 = 1$. The product of units is commutative and is defined as $i_1j = -i_2$, $i_2j = -i_1$, with the addition and multiplication of two bicomplex numbers defined in the obvious way.

For a bicomplex number $w = z_1 + i_2z_2$, the norm is denoted by $\| w \|$ and is defined

$$\| w \| = \| z_1 + i_2z_2 \| = (|z_1|^2 + |z_2|^2)^{\frac{1}{2}}.$$

By choosing, $w = c_0 + i_1c_1 + i_2c_2 + i_1i_2c_3, c_p \in \mathbb{R}, (p = 0, 1, 2, 3)$ then

$$\| w \| = (c_0^2 + c_1^2 + c_2^2 + c_3^2)^{\frac{1}{2}}.$$

A bicomplex number $w = c_0 + i_1c_1 + i_2c_2 + i_1i_2c_3$ is degenerated [20] if the matrix$\begin{pmatrix} c_0 & c_1 \\ c_2 & c_3 \end{pmatrix}$ is degenerated.

Further, for any two bicomplex numbers $\delta, \theta \in \mathbb{C}_2$, we can show that
(i) $0 \prec_{i_2} \delta \prec_{i_2} \theta$ implies $\| \delta \| \leq \| \theta \|$
(ii) $\| \delta + \theta \| \leq \| \delta \| + \| \theta \|$
(iii) $\| \alpha\delta \| \leq |\alpha| \| \delta \|$
Also, for any two complex numbers $\delta, \theta \in \mathbb{C}_2$, we have
(i) $\| \delta\theta \| \leq \| \delta \| \| \theta \|$.
(ii) $\| \delta\theta \| = \| \delta \| \| \theta \|$ whenever at least one of δ and θ is degenerated [20].
(iii) The partial order relation on $\preceq_{i_2}$ defined in [6] as follows:
Let $\delta = \delta_1 + i_2\delta_2 \in \mathbb{C}_2$ and $\| \delta^{-1} \| = \| \delta \|^{-1}$ holds for any degenerated bicomplex number.

$\theta = \theta_1 + i_2\theta_2 \in \mathbb{C}_2$, we define a partial order relation on $\mathbb{C}_2$ as $\delta \preceq_{i_2} \theta$ if and only if $\delta_1 \preceq_{i_1} \theta_1$ and $\delta_2 \preceq_{i_1} \theta_2$, where $\preceq_{i_1}$ is a partial order relation in $\mathbb{C}_1$. Then
(1) $\Re e(\delta_1) = \Re e(\theta_1)$ and $\Im m(\delta_1) = \Im m(\theta_1)$

$\Re e(\delta_2) = \Re e(\theta_2)$ and $\Im m(\delta_2) = \Im m(\theta_2)$
(2) $\Re e(\delta_1) < \Re e(\theta_1)$ and $\Im m(\delta_1) < \Im m(\theta_1)$

$\Re e(\delta_2) = \Re e(\theta_2)$ and $\Im m(\delta_2) = \Im m(\theta_2)$
(3) $\Re e(\delta_1) = \Re e(\theta_1)$ and $\Im m(\delta_1) = \Im m(\theta_1)$

$\Re e(\delta_2) < \Re e(\theta_2)$ and $\Im m(\delta_2) < \Im m(\theta_2)$
(4) $\Re e(\delta_1) < \Re e(\theta_1)$ and $\Im m(\delta_1) < \Im m(\theta_1)$

$\Re e(\delta_2) < \Re e(\theta_2)$ and $\Im m(\delta_2) < \Im m(\theta_2)$.

We write $\delta \precnsim_{i_2} \theta$ if $\delta \preceq_{i_2} \theta$ and $\delta \neq \theta$ if any one of (1), (2) and (3) is satisfied and $\delta \prec_{i_2} \theta$ if condition (4) is satisfied.
The definition of the bicomplex valued metric space is introduced in [6] as follows.

Definition 1.1. *Let X be a nonempty set. A function $\Xi : X \times X \to \mathbb{C}_2$ is called a bicomplex valued metric on X if for all $x, y, z \in X$, the following conditions are satisfied:*
(1) $0 \preceq_{i_2} \Xi(x, y)$
(2) $\Xi(x, y) = 0$ *iff* $x = y$;
(3) $\Xi(x, y) = \Xi(y, x)$;
(4) $\Xi(x, y) \preceq_{i_2} \Xi(x, z) + \Xi(y, z)$
The pair (X, Ξ) is called a bicomplex valued metric space.
In this connection many researchers obtained fixed point results in bi complex valued metric spaces, we refer [5, 6, 12, 15, 20].

The notion of bicomplex valued b-metric spaces defined by S. K. Datta et. al., [13, 9-11] as:

Definition 1.2. *Let X be a nonempty set and $s \geq 1$. A function $\Xi : X \times X \to \mathbb{C}_2$ is called a bicomplex valued b-metric on X if for all $x, y, z \in X$, the following conditions are satisfied:*
(i) $0 \preceq_{i_2} \Xi_{\mathbb{BC}}(x, y)$
(ii) $\Xi(x, y) = 0$ *iff* $x = y$;
(iii) $\Xi(x, y) = \Xi(y, x)$;
(iv) $\Xi(x, y) \preceq_{i_2} s[\Xi(x, z) + \Xi(y, z)]$
The pair (X, Ξ) is called a bicomplex valued b- metric space.
Here we give the examples of bicomplex valued b-metric spaces.

Example 1.3. Let $X = [0, +\infty)$. We define $\Xi : X \times X \to \mathbb{C}_2$ by $\Xi(x, y) = (1 + i_1)(1 + i_2)|x - y|^2$, for all $x, y \in X$. Then (X, Ξ) is a bi-complex valued b-

metric space with $s = 2$, for $x, y \in X$,
$\Xi(x,y) = (1+i_1)(1+i_2)|x-y|^2$
$\preceq_{i_2} (1+i_1)(1+i_2)|x-z+z-y|^2$
$\preceq_{i_2} (1+i_1)(1+i_2)[|x-z|^2+|z-y|^2+2|x-z||z-y|]$
$\preceq_{i_2} (1+i_1)(1+i_2)[2|x-z|^2+2|z-y|^2]$
$\preceq_{i_2} 2[(1+i_1)(1+i_2)|x-z|^2+(1+i_1)(1+i_2)|z-y|^2]$
$= 2[\Xi(x,z)+\Xi(z,y)]$.

Example 1.4. Let $X = [0, +\infty)$. We define $\Xi : X \times X \to \mathbb{C}_2$ by

$$\Xi(x,y) = \begin{cases} 0 & if x = y \\ \\ (1+i_1+i_2+i_1i_2)(x+y)^2 & if\ x \neq y. \end{cases}$$

Then (X, Ξ) is a bi-complex valued b-metric space with $s = 2$.

Definition 1.5. [11] *Let (X, Ξ) be a bicomplex valued b-metric space and $\{x_n\}$ be a sequence in X. We say that:*
(i) The sequence $\{x_n\}$ converges to $x \in X$ if for each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$ there is a $n_0 \in N$ such that for all $n > n_0$, $\Xi(x_n, x) \prec_{i_2} c$. We denote this by $lim_{n\to+\infty} x_n = x$.
(ii) The sequence $\{x_n\}$ is a Cauchy sequence if for each $c \in \mathbb{C}_2$ with $0 \prec_{i_2} c$ there is $n_0 \in N$ such that for all $n > n_0$, $\Xi(x_n, x_{n+m}) \prec_{i_2} c$, where $m \in N$.
(iii) (X, Ξ) is said to be complete bicomplex valued b-metric space if every Cauchy sequence in X is convergent to a point in X.

Lemma 1.6. [11] *Let (X, Ξ) be a bicomplex valued b-metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $\| \Xi(x_n, x) \| \to 0$ as $n \to +\infty$.*

Lemma 1.7. [11] *Let (X, Ξ) be a generalized bicomplex valued b-metric space and $\{x_n\}$ be a sequence in X. If $lim_{n\to+\infty} \| \Xi(x_n, x_{n+m}) \| \to 0$ then $\{x_n\}$ is a Cauchy sequence.*

Lemma 1.8. [11] *Let (X, Ξ) be a generalized bicomplex valued b-metric space and let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x then for any $a \in X$,*
$\lim_{n\to+\infty} \| \Xi(x_n, a) \| \to \| \Xi(x, a) \|$.

Definition 1.9. [21] *Let P be a self map on a nonempty space X and $\alpha : X \times X \to [0, +\infty)$. We say that P is α admissible if, for all $x, y \in X$, we have*

$$\alpha(x,y) \geq 1 \text{ implies } \alpha(Px, Py) \geq 1.$$

Definition 1.10. [4] *Let P, g be self maps on a nonempty space X and $\alpha : X \times X \to [0, +\infty)$. We say that P is $g - \alpha$ admissible if, for all $x, y \in X$, we have*

$$\alpha(gx, gy) \geq 1 \text{ implies } \alpha(Px, Py) \geq 1.$$

If $g = I$, then P is called $g - \alpha$ admissible.
We denote $C(P, g)$, the set of fixed points of P and g *i.e.*,

$$C(P, g) = \{z \in X : Pz = gz = z\}.$$

We study common fixed point theorems of Suzuki type contractions employing alpha admissible function for two maps in bicomplex valued metric b-space rendered by rational expressions. These results are enhanced through examples. As a consequence, we obtain common fixed point theorems for bi complex valued b-metric spaces endowed with a partial order.

2. Main Results

In this section, first we prove the existence common fixed points for almost Suzuki type contractions in bi complex valued b-metric spaces.

Theorem 2.1. *Let (X, Ξ) be a complete bicomplex valued b-metric space with $s \geq 1$ and $1 + \Xi(x, y) + \Xi(u, v)$ degenerated for all $x, y, u, v \in X$. Assume that $\alpha : X \times X \to \mathbb{R}^+$ is a mapping and P and g are selfmaps on X satisfying the following conditions:*
(i) $PX \subseteq gX$.
(ii)

$$\frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} \leq max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}$$

implies

$$\alpha(gx, gy)\Xi(Px, Py) \preceq_{i_2} a\Xi(gx, gy) + b\Xi(gy, Py) + c\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s} + d\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + e\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} \tag{2.1.1}$$

for all $x, y \in X$, where $a, b, c, d, e \geq 0$ and $a + sb + 2c + d + e < 1$
(iii) P is α- admissible with respect to g
(iv) there exists $x_0 \in X$ such that $\alpha(gx_0, Px_0) \geq 1$
(v) if $\{gx_n\}$ is a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in gX$ as $n \to +\infty$ then there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $\alpha(gx_{n(k)}, gz) \geq 1$ for all k
(vi) gX is closed.
Then P and g have a unique coincidence point in X.
Proof. In view of condition (iv), let $x_0 \in X$ be such that $\alpha(gx_0, Px_0) \geq 1$. Since

$PX \subseteq gX$, we can choose a point $x_1 \in X$ such that $Px_0 = gx_1$, on continuing this process, we can choose sequence $\{x_n\}$ in X such that

$$Px_n = gx_{n+1} \;\; \text{for} \;\; n = 0, 1, 2, 3, ... \tag{2.1.2}$$

Further, P is α-admissible with respect to g, we have
$\alpha(gx_0, Px_0) = \alpha(gx_0, gx_1) \geq 1$ implies $\alpha(Px_0, Px_1) = \alpha(gx_1, gx_2) \geq 1$.
Using mathematical induction, we get

$$\alpha(gx_n, gx_{n+1}) \geq 1 \tag{2.1.3}$$

for all $n = 0, 1, 2, 3, ...$
If $gx_{n+1} = gx_{n+2}$ for some $n \in \mathbb{N}$, for some n, then by (2.1.2), we have $gx_{n+1} = Px_{n+1}$, so that x_{n+1} is a coincidence point of P and g and the proof is completed. Thus, with out loss of generality, suppose that $\Xi(Px_n, Px_{n+1}) > 0$, for all n. Since,

$$\frac{1}{2s} min\{\| \Xi(gx_n, Px_n) \|, \| \Xi(Px_{n+1}, gx_{n+1}) \|\}$$
$$\leq max\{\| \Xi(gx_n, gx_{n+1}) \|, \| \Xi(Px_n, Px_{n+1}) \|\}$$

implies from (2.1.1), we have

$$\begin{aligned}
&\Xi(Px_n, Px_{n+1}) \preceq_{i_2} \alpha(gx_n, gx_{n+1})\Xi(Px_n, Px_{n+1}) \\
&\preceq_{i_2} a\Xi(gx_n, gx_{n+1}) + b\Xi(gx_{n+1}, Px_{n+1}) + c\frac{\Xi(gx_n, Px_{n+1}) + \Xi(gx_{n+1}, Px_n)}{s} \\
&+ d\frac{\Xi(gx_n, Px_n)\Xi(gx_{n+1}, Px_{n+1})}{1 + \Xi(gx_n, gx_{n+1}) + \Xi(Px_n, Px_{n+1})} + e\frac{\Xi(gx_n, Px_{n+1})\Xi(gx_{n+1}, Px_n)}{1 + \Xi(gx_n, gx_{n+1}) + \Xi(Px_n, Px_{n+1})} \\
&\preceq_{i_2} a\Xi(Px_{n-1}, Px_n) + b\Xi(Px_n, Px_{n+1}) + c[\Xi(Px_{n-1}, Px_n) + \Xi(Px_n, Px_{n+1})] \\
&+ d\frac{\Xi(Px_{n-1}, Px_n)\Xi(Px_n, Px_{n+1})}{1 + \Xi(Px_{n-1}, Px_n) + \Xi(Px_n, Px_{n+1})}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\| \Xi(Px_n, Px_{n+1}) \| \leq a \| \Xi(Px_{n-1}, Px_n) \| + b \| \Xi(Px_n, Px_{n+1}) \| \\
&+ c \| \Xi(Px_{n-1}, Px_n) \| + c \| \Xi(Px_{n+1}, Px_n) \| \\
&+ d\frac{\| \Xi(Px_{n+1}, Px_n) \|}{\| 1 + \Xi(Px_n, Px_{n+1}) + \Xi(Px_{n-1}, Px_n) \|} \| \Xi(Px_n, Px_{n-1}) \|,
\end{aligned} \tag{2.1.4}$$

since $\| \Xi(Px_{n+1}, Px_n) \| \leq \| 1 + \Xi(Px_n, Px_{n+1}) + \Xi(Px_{n-1}, Px_n) \|$, from (2.1.4), we have

$$(1 - c - b) \| \Xi(Px_{n+1}, Px_n) \| \leq (a + c + d) \| \Xi(Px_{n+1}, Px_n) \|,$$

therefore

$$\parallel \Xi(Px_{n+1}, Px_n) \parallel \leq \frac{a+c+d}{1-b-c} \parallel \Xi(Px_n, Px_{n-1}) \parallel . \quad (2.1.5)$$

Similarly, we can show that

$$\parallel \Xi(Px_n, Px_{n-1}) \parallel \leq \frac{a+b+c+d}{1-c} \parallel \Xi(Px_{n-1}, Px_{n-2}) \parallel . \quad (2.1.6)$$

Let $\beta = max\{\frac{a+c+d}{1-b-c}, \frac{a+b+c+d}{1-c}\}$.
Combining (2.1.5) and (2.1.6), we get

$$\parallel \Xi(Px_n, Px_{n+1}) \parallel \leq \beta \parallel \Xi(Px_n, Px_{n-1}) \parallel \quad (2.1.7)$$

for all $n = 1, 2, 3, ...$
Therefore, from (2.1.7), we have

$$\parallel \Xi(Px_n, Px_{n+1}) \parallel \leq \beta \parallel \Xi(Px_n, Px_{n-1}) \parallel \leq .. \leq \beta^n \parallel \Xi(Px_1, Px_0) \parallel . \quad (2.1.8)$$

We now show that $\{Px_n\} = \{gx_{n+1}\}$ is a Cauchy sequence in X.
In view of triangle inequality, we have

$$\Xi(Px_n, Px_m) \preceq s[\Xi(Px_n, Px_{n+1}) + \Xi(Px_{n+1}, Px_m)],$$

which implies

$$\begin{aligned}
&\parallel \Xi(Px_n, Px_m) \parallel \leq s \parallel \Xi(Px_n, Px_{n+1}) \parallel +s \parallel \Xi(Px_{n+1}, Px_m) \parallel \\
&\leq s \parallel \Xi(Px_n, Px_{n+1}) \parallel +s^2 \parallel \Xi(Px_{n+1}, Px_{n+2}) \parallel \\
&+ s^3 \parallel \Xi(Px_{n+2}, Px_{n+3}) \parallel +.... + s^{m-n-1} \parallel \Xi(Px_{m+1}, Px_m) \parallel \\
&\leq s \parallel \Xi(Px_n, Px_{n+1}) \parallel +s^2 \parallel \Xi(Px_{n+1}, Px_{n+2}) \parallel \\
&+ s^3 \parallel \Xi(Px_{n+2}, Px_{n+3}) \parallel +.... + s^{m-n} \parallel \Xi(Px_{m+1}, Px_m) \parallel \\
&(\text{since } s \geq 1) \\
&\leq s\beta^n \parallel \Xi(Px_0, Px_1) \parallel +s^2\beta^{n+1} \parallel \Xi(Px_0, Px_1) \parallel +........ \\
&+ s^{m-n}\beta^{m-1} \parallel \Xi(Px_0, Px_1) \parallel \\
&\leq \Sigma_{i=1}^{m-n} s^i \beta^{i+n-1} \parallel \Xi(Px_0, Px_1) \parallel \\
&\leq s\beta^n \Sigma_{i=1}^{m-n} s^{i-1} \beta^{i-1} \parallel \Xi(Px_0, Px_1) \parallel \\
&\parallel \Xi(Px_n, Px_m) \parallel \leq s\beta^n \Sigma_{i=1}^{m-n} s^{i-1} \beta^{i-1} \parallel \Xi(Px_0, Px_1) \parallel \\
&\leq s\beta^n \Sigma_{i=1}^{\infty} s^{i-1} \beta^{i-1} \parallel \Xi(Px_0, Px_1) \parallel \\
&\leq \frac{s(\beta)^n}{1-s\beta} \parallel \Xi(Px_0, Px_1) \parallel \to 0 \text{ as } n \to +\infty.
\end{aligned}$$

Therefore $\{Px_n\} = \{gx_{n+1}\}$ is a Cauchy sequence in X.
Since gX is closed there exists $z \in X$ such that

$$lim_{n\to+\infty} gx_n = lim_{n\to+\infty} Px_{n+1} = gz \tag{2.1.9}$$

We now show that z is a coincidence point of P and g. If not there exists $0 \prec_{i_2} \theta \in \mathbb{C}_2$ such that $\Xi(Pz, gz) = \theta$.
Again by condition (v) of our assumptions, we have $\alpha(gx_{n(k)}, gz) \geq 1$ and $\alpha(gz, gx_{n(k)}) \geq 1$.
Suppose that

$$\begin{aligned}&\frac{1}{2s} min\{\| \Xi(Px_{n(k)}, gx_{n(k)}) \|, \| \Xi(Pz, gz) \|\}\\ &> max\{\| \Xi(gx_{n(k)}, gz) \|, \| \Xi(Px_{n(k)}, Pz) \|\}.\end{aligned}$$

Letting $n \to +\infty$, using (2.1.9), we get $0 \geq\| \Xi(Pz, gz) \|$, which is a contradiction to our assumption. Therefore

$$\begin{aligned}&\frac{1}{2s} min\{\| \Xi(Px_{n(k)}, gx_{n(k)}) \|, \| \Xi(Pz, gz) \|\}\\ &\leq max\{\| \Xi(gx_{n(k)}, gz) \|, \| \Xi(Px_{n(k)}, Pz) \|\}\end{aligned}$$

which implies from (2.1.1), we have

$$\begin{aligned}\theta = \Xi(Pz, gz) &\preceq_{i_2} s\Xi(gz, Px_{n(k)}) + s\Xi(Px_{n(k)}, Pz)\\ &\preceq_{i_2} s\Xi(gz, Px_{n(k)}) + s\alpha(gx_{n(k)}, gz)\Xi(Px_{n(k)}, Pz)\\ &\preceq_{i_2} s\Xi(gz, Px_{n(k)}) + as\Xi(gx_{n(k)}, gz) + bs\Xi(gz, Pz)]\\ &+ sc\frac{\Xi(gx_{n(k)}, Pz) + \Xi(gz, Px_{n(k)})}{s} + sd\frac{\Xi(gx_{n(k)}, Px_{n(k)})\Xi(Pz, gz)}{1 + \Xi(gx_{n(k)}, gz) + \Xi(Px_{n(k)}, Pz)}\\ &+ se\frac{\Xi(gx_{n(k)}, Pz)\Xi(gz, Px_{n(k)})}{1 + \Xi(gx_{n(k)}, gz) + \Xi(gz, Px_{n(k)}}\end{aligned}$$

which implies

$$\begin{aligned}&\| \theta \|\leq s \| \Xi(gz, Px_{n(k)}) \| + as \| \Xi(gx_{n(k)}, gz) \| + sb \| \Xi(gz, Pz) \|\\ &+ sc\frac{\| \Xi(gx_{n(k)}, Pz) \| + \| \Xi(gz, Px_{n(k)}) \|}{s} + sd\frac{\| \Xi(gx_{n(k)}, Px_{n(k)}) \|\| \Xi(Pz, gz) \|}{\| 1 + \Xi(gx_{n(k)}, gz) + \Xi(Px_{n(k)}, Pz) \|}\\ &+ se\frac{\| \Xi(gx_{n(k)}, Pz) \|\| \Xi(gz, Px_{n(k)}) \|}{\| 1 + \Xi(gx_{n(k)}, gz) + \Xi(gz, Px_{n(k)} \|}.\end{aligned}$$

On taking limits as $k \to +\infty$, using (2.1.9), we get
$\| \theta \| \leq sb \| \Xi(gz, Pz) \|$
$\| \Xi(gz, Pz) \| \leq sb \| \Xi(gz, Pz) \|$
$(1 - sb) \| \Xi(Pz, gz) \| \leq 0$
$\| \Xi(Pz, gz) \| = 0.$
Hence $Pz = gz$. Thus, P and g have a common fixed point in X.

Theorem 2.2. *In addition to the hypotheses of Theorem 2.1, suppose that for $u, v \in C(P, g)$ if $\alpha(gu, gv) \geq 1$ and the pair (P, g) is weakly compatible, then P and g have a unique common fixed point in X.*

Proof. From the proof of Theorem 2.1, we have $\{gx_n\}$ is a non decreasing sequence and converges to gz and $Pz = gz$. Also, since P and g are weakly compatible, we have

$$Pz = Pgz = gPz = gz.$$

Hence $Px = gx = x$ so that P and g have a common fixed point. To prove uniqueness, let x and x' be two common fixed points of P and g i.e.,

$$Px = gx = x \text{ and } Px' = gx' = x'. \tag{2.2.1}$$

Since

$$\frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Px', gx') \|\} = 0$$
$$\leq max\{\| \Xi(gx, gx') \|, \| \Xi(Px, Px') \|\}$$

$\Rightarrow$ from (2.1.1), we have

$$\Xi(x, x') = \Xi(Px, Px') \preceq_{i_2} \alpha(gx, gx')\Xi(Px, Px')$$
$$\preceq_{i_2} a\Xi(gx, gx') + b\Xi(gx', Px') + c\frac{\Xi(gx, Px') + \Xi(gx', Px)}{s}$$
$$+ d\frac{\Xi(Px, gx)\Xi(gx', Px')}{1 + \Xi(gx, gx') + \Xi(Px, Px')} + e\frac{\Xi(gx, Px')\Xi(gx', Px)}{1 + \Xi(gx, gx') + \Xi(Px, Px')}$$
$$\| \Xi(x, x') \| \leq a \| \Xi(gx, gx') \| + b \| \Xi(gx', Px') \| + c\frac{\| \Xi(gx, Px') \| + \| \Xi(gx', Px) \|}{s}$$
$$+ d\frac{\| \Xi(Px, gx) \| \| \Xi(gx', Px') \|}{\| 1 + \Xi(gx, gx') + \Xi(Px, Px') \|} + e\frac{\| \Xi(gx, Px') \| \| \Xi(gx', Px) \|}{\| 1 + \Xi(gx, gx') + \Xi(Px, Px') \|},$$

which implies

$$\| \Xi(x, x') \| \leq (a + \frac{c}{s} + e) \| \Xi(x, x') \|,$$

this implies $\| \Xi(x, x') \| = 0$. Therefore P and g have a unique common fixed point in X.

3. Examples and Corollaries

The following examples are in support of Theorem 2.2.

Example 3.1. Let $X = [0, 5]$, we define $\Xi : X \times X \to \mathbb{C}_2$ by

$$\Xi(x, y) = \begin{cases} 0 \quad if\ x = y \\ (i_1 + i_2 + 2i_1 i_2)(x + y)^2 \quad if \quad x \neq y \end{cases}$$

Then (X, Ξ) is complete bi complex valued b-metric space with $s = 2$ and $1 + \Xi(x, y) + \Xi(u, v)$ degenerated for all $x, y, u, v \in X$.
We define $P, g : X \to X$ by

$$Px = \begin{cases} \frac{3x}{4} \quad if\ x \in [0, 1] \\ \frac{x}{4}\ if \quad x \in (1, 5] \end{cases} \quad \text{and} \quad gx = \begin{cases} 3x \quad if\ x \in [0, 1] \\ x\ if \quad x \in (1, 5]. \end{cases}$$

Clearly, $PX \subseteq gX$ and gX is closed set.
Define the function $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 2 \quad if\ x \in [0, 3] \\ 3 \quad otherwise. \end{cases}$$

We now verify inequality (2.1.1) with $a = \frac{3}{16}, b = \frac{1}{4}, c = \frac{1}{8}, d = 0 = e$.
Case (i): Let $x, y \in [0, 1]$ with $x \neq y$. Then $\alpha(gx, gy) = \alpha(3x, 3y) = 2$.
If $x > y$, then

$$\begin{aligned} \frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} = \frac{\sqrt{6}}{4}\frac{225}{16}y^2 \leq 9\sqrt{6}(x + y)^2 \\ = max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}. \end{aligned}$$

Then from (2.1.1), we have

$$\begin{aligned} \alpha(gx, gy)\Xi(Px, Py) &= 2(i_1 + i_2 + 2i_1 i_2)\frac{9}{16}(x + y)^2 = \frac{2}{16}(i_1 + i_2 + 2i_1 i_2)(3x + 3y)^2 \\ &\leq \frac{3}{16}(i_1 + i_2 + 2i_1 i_2)(3x + 3y)^2 = \frac{3}{16}\Xi(gx, gy) \\ &\preceq_{i_2} \frac{3}{16}\Xi(gx, gy) + \frac{1}{4}\Xi(gy, Py) + \frac{1}{8}\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s} \\ &+ 0\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + 0\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)}. \end{aligned}$$

Similarly, when $x < y$, condition (2.1.1) follows.
Case (ii): Let $x, y \in (1,5]$ with $x \neq y$. Then $\alpha(gx, gy) = 3$. Also,

$$\begin{aligned}\frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} &= \frac{\sqrt{6}}{4}\frac{25}{16}y^2 \leq \sqrt{6}(x+y)^2 \\ &= max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}\end{aligned}$$

$\Rightarrow$ from (2.1.1), we have

$$\begin{aligned}\alpha(gx, gy)\Xi(Px, Py) &= 3(i_1 + i_2 + 2i_1i_2)\frac{1}{16}(x+y)^2 \\ &= \frac{3}{16}(i_1 + i_2 + 2i_1i_2)(x+y)^2 = \frac{3}{16}\Xi(gx, gy) \\ &\preceq_{i_2} \frac{3}{16}\Xi(gx, gy) + \frac{1}{4}\Xi(gy, Py) + \frac{1}{8}\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s} \\ &+ 0\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + 0\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)}.\end{aligned}$$

Case (iii): Let $x \in [0,1]$ and $y \in (1,5]$. Then $\alpha(gx, gy) = 3$ and

$$\begin{aligned}\frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} &= \frac{\sqrt{6}}{4}\frac{225}{16}x^2 \leq \sqrt{6}(3x+y)^2 \\ &= max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}.\end{aligned}$$

This implies from (2.1.1), we have

$$\begin{aligned}\alpha(gx, gy)\Xi(Px, Py) &= 3(i_1 + i_2 + 2i_1i_2)\frac{1}{16}(3x+y)^2 \\ &= \frac{3}{16}(i_1 + i_2 + 2i_1i_2)(3x+y)^2 = \frac{3}{16}\Xi(gx, gy) \\ &\preceq_{i_2} \frac{3}{16}\Xi(gx, gy) + \frac{1}{4}\Xi(gy, Py) + \frac{1}{8}\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s} \\ &+ 0\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + 0\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)}.\end{aligned}$$

Case (iv): Let $x \in (1,5]$ and $y \in [0,1]$. Then $\alpha(gx, gy) = 3$. Also,

$$\begin{aligned}\frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} &= \frac{\sqrt{6}}{4}\frac{225}{16}y^2 \leq \sqrt{6}(x+3y)^2 \\ &= max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}.\end{aligned}$$

Then from (2.1.1), we have

$$\begin{aligned}\alpha(gx,gy)\Xi(Px,Py) &= 3(i_1+i_2+2i_1i_2)\frac{1}{16}(x+3y)^2\\ &= \frac{3}{16}(i_1+i_2+2i_1i_2)(x+3y)^2 = \frac{3}{16}\Xi(gx,gy)\\ &\preceq_{i_2} \frac{3}{16}\Xi(gx,gy)+\frac{1}{4}\Xi(gy,Py)+\frac{1}{8}\frac{\Xi(gx,Py)+\Xi(gy,Px)}{s}\\ &+0\frac{\Xi(Px,gx)\Xi(gy,Py)}{1+\Xi(gx,gy)+\Xi(Px,Py)}+0\frac{\Xi(gx,Py)\Xi(gy,Px)}{1+\Xi(gx,gy)+\Xi(Px,Py)}.\end{aligned}$$

Thus inequality (2.1.1) is satisfied with $a=\frac{3}{16}, b=\frac{1}{4}, c=\frac{1}{8}, d=0=e$. Also, we have $\alpha(Px_0,gx_0)\geq 1$ for any $x_0\in[0,2]$. Clearly, P is α- admissible with respect to g. Now, all the hypotheses of Theorem 2.1 are satisfied. Consequently, P and g have a coincidence point. Here, 0 is a coincidence point of P and g. Also, clearly all the hypotheses of Theorem 2.2 are satisfied. In this example, 0 is the unique common fixed point of P and g.

Example 3.2. Let $X=[0,2]$, we define $\Xi: X\times X\to\mathbb{C}_2$ by

$$\Xi(x,y) = \begin{cases} 0 \ \ if\ x=y \\ (3i_1+3i_2+18i_1i_2)max\{x,y\}^2 \quad if \quad x\neq y\end{cases}$$

Clearly, (X,Ξ) is complete bi complex valued b-metric space with $s=2$ and $1+\Xi(x,y)+\Xi(u,v)$ degenerated for all $x,y,u,v\in X$.
We define $P,g: X\to X$ by

$$Px = \begin{cases} 0 & if \quad x=0\\ \frac{1-x^2}{4} & if\ x\in(0,1]\\ \frac{1}{4} & if\ \ x\in(1,2]\end{cases} \quad \text{and} \quad gx = \begin{cases} 0 & if \quad x=0\\ 1-x^2 & if\ x\in(0,1]\\ \frac{1+x}{4} & if\ \ x\in(1,2]\end{cases}$$

Clearly, $PX\subseteq gX$ and gX is closed set.
Define the function $\alpha: X\times X\to[0,+\infty)$ by

$$\alpha(x,y) = \begin{cases} 2 & if\ x\in[0,1]\\ 1 & otherwise.\end{cases}$$

We now verify inequality (2.1.1).
Case (i): Let $x,y\in[0,1]$ with $x\neq y$, then $\alpha(gx,gy)=\alpha(1-x^2,1-y^2)=2$.

First we suppose that $x > y$, then we have

$$\begin{aligned}\frac{1}{2s}min\{\| \Xi(Px,gx) \|, \| \Xi(Py,gy) \|\} &= \frac{1}{4}\sqrt{342}(1-x^2)^2 \leq 2\sqrt{342}(1-y^2)^2 \\ &= max\{\| \Xi(gx,gy) \|, \| \Xi(Px,Py) \|\},\end{aligned}$$

which implies from (2.1.1), we have

$$\alpha(gx,gy)\Xi(Px,Py) = 2\rho max\{\frac{1-x^2}{4}, \frac{1-y^2}{4}\}^2$$

where $\rho = (3i_1 + 3i_2 + 18i_1i_2)$ then

$$\begin{aligned}\alpha(gx,gy)\Xi(Px,Py) &= 2\rho\frac{(1-y^2)^2}{16} = \rho\frac{1}{8}(1-y^2)^2 \preceq_{i_2} \rho\frac{1}{4}(1-y^2)^2 = \frac{1}{4}\Xi(gx,gy) \\ &\preceq_{i_2} \frac{1}{4}\Xi(gx,gy) + \frac{1}{4}\Xi(gy,Py) + \frac{1}{16}\frac{\Xi(gx,Py)+\Xi(gy,Px)}{s} \\ &+ 0\frac{\Xi(Px,gx)\Xi(gy,Py)}{1+\Xi(gx,gy)+\Xi(Px,Py)} + 0\frac{\Xi(gx,Py)\Xi(gy,Px)}{1+\Xi(gx,gy)+\Xi(Px,Py)}.\end{aligned}$$

Next, we suppose that $x < y$, then we have

$$\begin{aligned}\frac{1}{2s}min\{\| \Xi(Px,gx) \|, \| \Xi(Py,gy) \|\} &= \frac{1}{4}\sqrt{342}(1-y^2)^2 \leq \sqrt{342}(1-x^2)^2 \\ &= max\{\| \Xi(gx,gy) \|, \| \Xi(Px,Py) \|\},\end{aligned}$$

which implies from (2.1.1), we have

$$\alpha(gx,gy)\Xi(Px,Py) = 2\rho max\{\frac{1-x^2}{4}, \frac{1-y^2}{4}\}^2$$

where $\rho = (3i_1 + 3i_2 + 18i_1i_2)$ then

$$\begin{aligned}\alpha(gx,gy)\Xi(Px,Py) &= 2\rho\frac{(1-x^2)^2}{16} = \rho\frac{1}{8}(1-x^2)^2 \preceq_{i_2} \rho\frac{1}{4}(1-x^2)^2 = \frac{1}{4}\Xi(gx,gy) \\ &\preceq_{i_2} \frac{1}{4}\Xi(gx,gy) + \frac{1}{4}\Xi(gy,Py) + \frac{1}{16}\frac{\Xi(gx,Py)+\Xi(gy,Px)}{s} \\ &+ 0\frac{\Xi(Px,gx)\Xi(gy,Py)}{1+\Xi(gx,gy)+\Xi(Px,Py)} + 0\frac{\Xi(gx,Py)\Xi(gy,Px)}{1+\Xi(gx,gy)+\Xi(Px,Py)}.\end{aligned}$$

Case (ii): Let $x, y \in (1,2]$ with $x \neq y$. Then $\alpha(gx,gy) = \alpha(\frac{1+x}{2}, \frac{1+y}{2}) = 1$.
First we suppose that $x > y$, then we have

$$\frac{1}{2s}min\{\| \Xi(Px,gx) \|, \| \Xi(Py,gy) \|\} = \frac{1}{4}min\{\sqrt{342}(\frac{1+x}{2})^2, \sqrt{342}(\frac{1+y}{2})^2\}$$

$$= \frac{1}{4}\sqrt{342}(\frac{1+y}{2})^2 \leq \sqrt{342}(\frac{1+x}{2})^2 = max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}.$$

Thus from (2.1.1), we have

$$\alpha(gx, gy)\Xi(Px, Py) = 0 \preceq_{i_2} \frac{1}{4}\Xi(gx, gy) + \frac{1}{4}\Xi(gy, Py) + \frac{1}{16}\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s}$$

$$+0\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + 0\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)}.$$

Similarly, when $x < y$, we have

$$\frac{1}{2s}min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} = \frac{1}{4}min\{\sqrt{342}(\frac{1+x}{2})^2, \sqrt{342}(\frac{1+y}{2})^2\}$$

$$\leq \sqrt{342}(\frac{1+y}{2})^2 = max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}.$$

Thus from (2.1.1), we have

$$\alpha(gx, gy)\Xi(Px, Py) = 0 \preceq_{i_2} \frac{1}{4}\Xi(gx, gy) + \frac{1}{4}\Xi(gy, Py) + \frac{1}{16}\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s}$$

$$+0\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + 0\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)}.$$

Case (iii): Let $x \in [0, 1]$ and $y \in (1, 2]$. Then $\alpha(gx, gy) = 1$. Also,

$$\frac{1}{2s}min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} = \frac{1}{4}min\{\sqrt{342}(\frac{1+y}{2})^2, \sqrt{342}(1 - x^2)^2\}$$

$$= \frac{1}{4}\sqrt{342}(1 - x^2)^2 \leq \sqrt{342}(\frac{1+y}{2})^2 = max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}$$

$\Rightarrow$ from (2.1.1), we have

$$\alpha(gx, gy)\Xi(Px, Py) = \rho(\frac{1}{4})^2 = \frac{\rho}{16} \preceq_{i_2} \frac{1}{4}\rho\frac{(y+1)^2}{4} = \frac{1}{4}\Xi(gx, gy)$$

$$\preceq_{i_2} \frac{1}{4}\Xi(gx, gy) + \frac{1}{4}\Xi(gy, Py) + \frac{1}{16}\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s}$$

$$+0\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + 0\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)}.$$

Case (iv): Let $x \in (1,2]$ and $y \in [0,1]$. Then $\alpha(gx,gy)=1$. Also,

$$\frac{1}{2s}min\{\| \Xi(Px,gx) \|, \| \Xi(Py,gy) \|\} = \frac{1}{4}min\{\sqrt{342}(\frac{1+x}{2})^2, \sqrt{342}(1-y^2)^2\}$$

$$= \frac{1}{4}\sqrt{342}(1-y^2)^2 \leq \sqrt{342}(\frac{1+x}{2})^2 = max\{\| \Xi(gx,gy) \|, \| \Xi(Px,Py) \|\}$$

$\Rightarrow$ from (2.1.1), we have

$$\alpha(gx,gy)\Xi(Px,Py) = \rho(\frac{1}{4})^2 = \frac{\rho}{16} \precsim_{i_2} \frac{1}{4}\rho\frac{(x+1)^2}{4} = a\Xi(gx,gy)$$

$$\precsim_{i_2} \frac{1}{4}\Xi(gx,gy) + \frac{1}{4}\Xi(gy,Py) + \frac{1}{16}\frac{\Xi(gx,Py)+\Xi(gy,Px)}{s}$$
$$+0\frac{\Xi(Px,gx)\Xi(gy,Py)}{1+\Xi(gx,gy)+\Xi(Px,Py)} + 0\frac{\Xi(gx,Py)\Xi(gy,Px)}{1+\Xi(gx,gy)+\Xi(Px,Py)}.$$

Thus condition (2.1.1) is satisfied with $a=\frac{1}{4}, b=\frac{1}{4}, c=\frac{1}{16}, d=0=e$. Also, we have $\alpha(Px_0,gx_0) \geq 1$ for any $x_0 \in [0,2]$. Clearly, P is α- admissible with respect to g.

Now, all the hypotheses of Theorem 2.1 are satisfied. Consequently, P and g have a coincidence point. Here, 0 is a coincidence point of P and g. Also, clearly all the hypotheses of Theorem 2.2 are satisfied. In this example, 0 is the unique common fixed point of P and g.

By choosing $s=1$, we have the following corollary.

Corollary 3.3. *Let (X,Ξ) be a complete bicomplex valued metric space and $1+\Xi(x,y)+\Xi(u,v)$ degenerated for all $x,y,u,v \in X$. Assume that $\alpha : X \times X \to \mathbb{R}^+$ be a mapping and P and g are selfmaps on X satisfying the following conditions:*
(i) $PX \subseteq gX$.
(ii)

$$\frac{1}{2s}min\{\| \Xi(Px,gx) \|, \| \Xi(Py,gy) \|\} \leq max\{\| \Xi(gx,gy) \|, \| \Xi(Px,Py) \|\}$$

$$\Longrightarrow \alpha(gx,gy)\Xi(Px,Py) \precsim_{i_2} a\Xi(gx,gy) + b\Xi(gy,Py) + c\frac{\Xi(gx,Py)+\Xi(gy,Px)}{s}$$
$$+\Xi\frac{\Xi(Px,gx)\Xi(gy,Py)}{1+\Xi(gx,gy)+\Xi(Px,Py)} + e\frac{\Xi(gx,Py)\Xi(gy,Px)}{1+\Xi(gx,gy)+\Xi(Px,Py)}$$

for all $x,y \in X$, where $a,b,c,d,e \geq 0$ and $a+sb+2c+d+e<1$
(iii) P is α- admissible with respect to g

(iv) there exists $x_0 \in X$ *such that* $\alpha(gx_0, Px_0) \geq 1$
(v) If $\{gx_n\}$ *is a sequence in* X *such that* $\alpha(gx_n, gx_{n+1}) \geq 1$ *for all* n *and* $gx_n \to gz \in gX$ *as* $n \to +\infty$ *then there exists a subsequence* $\{gx_{n(k)}\}$ *of* $\{gx_n\}$ *such that* $\alpha(gx_{n(k)}, gz) \geq 1$ *for all* k
(vi) gX *is closed.*
Then P *and* g *have a unique coincidence point in* X.
By choosing $g = I$, the identity map, we have the following corollary.

Corollary 3.4. *Let* (X, Ξ) *be a complete bicomplex valued metric space with* $s \geq 1$ *and* $1+\Xi(x,y)+\Xi(u,v)$ *degenerated for all* $x, y, u, v \in X$. *Assume that* $\alpha : X \times X \to \mathbb{R}^+$ *be a mapping and* P *is selfmap on* X *satisfying the following conditions:*
(i)

$$\frac{1}{2s} min\{\| \Xi(Px, x) \|, \| \Xi(Py, y) \|\} \leq max\{\| \Xi(x, y) \|, \| \Xi(Px, Py) \|\}$$

$$\Longrightarrow \alpha(x, y)\Xi(Px, Py) \preceq_{i_2} a\Xi(x, y) + b\Xi(y, Py) + c\frac{\Xi(x, Py) + \Xi(y, Px)}{s}$$

$$+\Xi\frac{\Xi(Px, x)\Xi(y, Py)}{1 + \Xi(x, y) + \Xi(Px, Py)} + e\frac{\Xi(x, Py)\Xi(y, Px)}{1 + \Xi(x, y) + \Xi(Px, Py)} \quad (3.4.1)$$

for all $x, y \in X$, *where* $a, b, c, d, e \geq 0$ *and* $a + sb + 2c + d + e < 1$
(ii) P *is* α*- admissible*
(iii) there exists $x_0 \in X$ *such that* $\alpha(x_0, Px_0) \geq 1$
(iv) If $\{x_n\}$ *is a sequence in* X *such that* $\alpha(x_n, x_{n+1}) \geq 1$ *for all* n *and* $x_n \to z \in X$ *as* $n \to +\infty$ *then there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $\alpha(x_{n(k)}, z) \geq 1$ *for all* k.
Then P *has a fixed point in* X.

4. Fixed point theorems on bicomplex valued metric space endowed with a partial order

Definition 4.1. [7] *Let* (X, Ξ) *be a partially ordered set and* $P : X \to X$ *be a given mapping. We say that* P *is nondecreasing with respect to* $\preceq$ *if for all* $x, y \in X$, $x \preceq y$ *implies* $Px \preceq Py$.

Definition 4.2. [7] *Let* $(X, \preceq)$ *be a partially ordered set. A sequence* $\{x_n\} \subseteq X$ *is said to be nondecreasing with respect to* $\preceq$ *if* $x_n \preceq x_{n+1}$ *for all* n.

Definition 4.3. [12] *Let* $(X, \preceq)$ *be a partially ordered set and* Ξ *be a metric on* X. *We say that* $(X, \preceq, \Xi)$ *is regular if for every nondecreasing sequence* $\{x_n\} \in X$ *such that* $x_n \subseteq X$ *such that* $x_n \to x \in X$ *as* $n \to +\infty$, *there exists a subsequence* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *of* $\{x_n\}$ *such that* $x_{n(k)} \preceq x$ *for all* k.

Definition 4.4. [19] *Let $(X, \preceq)$ be a partially ordered set and $P, g : X \to X$ be a given mappings. We say that P is $g-$nondecreasing if for all $x, y \in X$, $gx \preceq gy$ implies $Px \preceq Py$.*

Definition 4.5. *Let $(X, \preceq)$ be a partially ordered set and d be a metric on X. We say that is $g-$regular if for every nondecreasing sequence $\{gx_n\} \in X$ such that $gx_n \to gz \in X$ as $n \to +\infty$, there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $gx_{n(k)} \preceq gz$ for all k.*

Corollary 4.6. *Let $(X, \preceq)$ be a poset and Ξ is a complete bicomplex valued metric space with $s \geq 1$ and $1+\Xi(x, y)+\Xi(u, v)$ degenerated for all $x, y, u, v \in X$. Assume that P and g are selfmaps on X satisfying the following conditions:*
(i) $PX \subseteq gX$.
(ii)

$$\frac{1}{2s} min\{\| \Xi(Px, gx) \|, \| \Xi(Py, gy) \|\} \leq max\{\| \Xi(gx, gy) \|, \| \Xi(Px, Py) \|\}$$

$$\Longrightarrow \alpha(gx, gy)\Xi(Px, Py) \preceq_{i_2} a\Xi(gx, gy) + b\Xi(gy, Py) + c\frac{\Xi(gx, Py) + \Xi(gy, Px)}{s}$$
$$+d\frac{\Xi(Px, gx)\Xi(gy, Py)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} + e\frac{\Xi(gx, Py)\Xi(gy, Px)}{1 + \Xi(gx, gy) + \Xi(Px, Py)} \tag{4.6.1}$$

for all $x, y \in X$, with $gx \preceq gy$ and $a, b, c, d, e \geq 0$ and $a + sb + 2c + d + e < 1$
(iii) P is g-nondecreasing with respect to $\preceq$
(iv) $(X, \preceq, \Xi)$ is g-regular.
(v) gX is closed.
Then P and g have a unique coincidence point in X. Moreover, for $u, v \in C(P, g)$ such that $u \preceq v$ and if P and g commute at their coincidence points then P and g have a unique common fixed point.
Proof. Define the mapping $\alpha : X \times X \to [0, +\infty)$ by

$$\alpha(x, y) = \begin{cases} 1 & if\ x \preceq y\ or\ y \preceq x \\ 0 & otherwise. \end{cases}$$

For any $x, y \in X$, we have $\alpha(x, y) = 1$ if and only if $x \preceq y$ or $x \succeq y$, so condition (4.6.1) follows. In view of condition (iii), i.e., P is g-nondecreasing with respect to $\preceq$, then we have $\alpha(gx, gy) \geq 1 \Rightarrow gx \preceq gy$ or $gx \succeq gy \Rightarrow Px \preceq Py$ or $Px \succeq Py \Rightarrow \alpha(Px, Py) \geq 1$, which implies P is α-admissible with respect to g. Let $\{gx_n\}$ be a sequence in X such that $\alpha(gx_n, gx_{n+1}) \geq 1$ for all n and $gx_n \to gz \in X$ as $n \to +\infty$.

From condition (iv) of our hypotheses there exists a subsequence $\{gx_{n(k)}\}$ of $\{gx_n\}$ such that $gx_{n(k)} \preceq gz$ for all k which amounts $\alpha(gx_{n(k)}, gz) \geq 1$. Also, by condition (iii), we have $\alpha(gx_0, Px_0) \geq 1$. Thus all the conditions of Theorem 2.1 are satisfied. Hence f and g have a coincidence point. Moreover, by the hypotheses if for all $u, v \in C(P, g)$ with $u \preceq v$ then by definition of α we have $\alpha(gx, gy) \geq 1$. Hence we infer that the existence and uniqueness of common fixed point by Theorem 2.2.

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