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BAILEY'S TRANSFORM AND KARLSSON–MINTON FORMULA

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Abstract: In this paper, making use of Bailey's transform and Karlsson-Minton summation formula, certain transformation formulas have been established. Interesting special cases have also been deduced.

Keywords and Phrases: Bailey's transform, Karlsson-Minton summation formula, basic hypergeometric series, q-binomial theorem, transformation formula.

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1. Introduction, Notations and Definitions

Throughout the paper we adopt the standard notations and terminology for q-series from [1] due to Gasper and Rahman. The $q-$shifted factorial for complex variable α with the base $q:\ |q|<1$ are given below.

$$(\alpha;q)_\infty=\prod_{n=0}^{\infty}(1-\alpha q^n)$$

and

$$(\alpha;q)_n = \frac{(\alpha;q)_\infty}{(\alpha q^n;q)_\infty}$$

for all integers n. For integer $m \geq 1$, we use the notation,

$$(a_1, a_2, ..., a_m; q)_n = (a_1;q)_n (a_2;q)_n ... (a_m;q)_n.$$

The unilateral basic hypergeometric series is defined as [1; (1.2.22) p.4],

$$ {}_r\Phi_s \left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-1)^n q^{n(n-1)/2}\right\}^{1+s-r} z^n, \quad (1.1)$$

where $r \leq 1+s$ and $|z| < 1$.
The theorem to be considered now was first stated explicity by W. N. Bailey in 1944 which states as,
If

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \quad (1.2)$$

and

$$\gamma_n = \sum_{r=n}^{\infty} \delta_r u_{r-n} v_{r+n} = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \quad (1.3)$$

then under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n, \quad (1.4)$$

where α_r, u_r, v_r and δ_r are any function of r only and the infinite series in (1.3) and (1.4) are convergent.
The classical $q-$binomial theorem to be used in this paper is given by

$$\sum_{n=0}^{\infty} \frac{(a;q)_n z^n}{(q;q)_n} = \frac{(az;q)_\infty}{(z;q)_\infty}. \quad (1.5)$$

[1; App.II (II.3) p. 354]

We shall also make use of Karlsson-Minton summation formula

$$ {}_{k+2}\Phi_{k+1} \left[\begin{matrix} a, b, b_1 q^{m_1}, b_2 q^{m_2}, ..., b_k q^{m_k}; q; a^{-1} q^{1-m_1-...-m_k} \\ bq, b_1, b_2, ..., b_k \end{matrix} \right]$$

$$= \frac{(a, bq/a; q)_\infty (b_1/b; q)_{m_1} ... (b_k/b; q)_{m_k} b^{m_1+...+m_k}}{(q/a, bq; q)_\infty (b_1; q)_{m_1} ... (b_k; q)_{m_k}}, \tag{1.6}$$

provided $|a^{-1}q^{1-m_1-m_2-...-m_k}| < 1$.

[1; (1.9.6) p. 16]

Taking $a = q^{-n}$ in (1.6) we get,

$$\sum_{r=0}^{n} \frac{(q^{-n}, b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q)_r}{(q, bq, b_1, b_2, .., b_k; q)_r} q^{1+n-m_1-m_2-...-m_k}$$

$$= \frac{(q; q)_n}{(bq; q)_n} \frac{(b_1/b; q)_{m_1} ... (b_k/b; q)_{m_k}}{(b_1; q)_{m_1} ... (b_k; q)_{m_k}} b^{m_1+m_2+...+m_k}. \tag{1.7}$$

2. Main Results

In this section we establish our main transformation formulas. Choosing $u_r = \frac{1}{(q; q)_r}$, $v_r = 1$ and

$$\alpha_r = \frac{(-1)^r q^{r(r+1)/2} (b, b_1q^{m_1}, b2q^{m_2}, ..., b_kq^{m_k}; q)_r}{(q, bq, b_1, b_2, ..., b_k; q)_r} q^{-r(m_1+m_2+...+m_k)}$$

in (1.2) and using (1.7) we get,

$$\beta_n = \frac{1}{(q; q)_n} \sum_{r=0}^{n} \frac{(q^{-n}, b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q)_r q^{r(1+n-m_1-m_2-...-m_k)}}{(q, bq, b_1, b_2, ..., b_k; q)_r}$$

$$= \frac{(b_1/b; q)_{m_1} ... (b_k/b; q)_{m_k}}{(b_1; q)_{m_1} ... (b_k; q)_{m_k}} b^{m_1+m_2+...+m_k} \frac{1}{(bq; q)_n}. \tag{2.1}$$

Again, taking $\delta_r = (\alpha; q)_r z^r$, where α is not 1 and $0 < z < 1$, in (1.3) and using (1.5) we get,

$$\gamma_n = \frac{(\alpha; q)_n z^n}{(\alpha z; q)_n} \frac{(\alpha z; q)_\infty}{(z; q)_\infty}. \tag{2.2}$$

Putting these values of β_n, γ_n, α_n and δ_n in (1.4) we get the following transformation formula.

$$(\alpha z; q)_\infty \sum_{n=0}^{\infty} \frac{(\alpha, b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q)_n}{(q, \alpha z, bq, b_1, b_2, ..., b_k; q)_n} (-1)^n q^{n(n-1)/2} q^{n(1-m_1-...-m_k)} z^n$$

$$= (z;q)_\infty \frac{(b_1/b;q)_{m_1}...(b_k/b;q)_{m_k}}{(b_1;q)_{m_1}...(b_k;q)_{m_k}} b^{m_1+m_2+...+m_k} \sum_{n=0}^{\infty} \frac{(\alpha;q)_n z^n}{(bq;q)_n}. \tag{2.3}$$

Now, using the definition (1.1) and replacing z by $zq^{m_1+m_2+...+m_k}$ in (2.3) we have,

$${}_{k+2}\Phi_{k+2}\left[\begin{array}{l} a, b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q; zq \\ \alpha zq^M, bq, b_1, b_2, ..., b_k \end{array}\right]$$

$$= \frac{(z;q)_\infty(\alpha z;q)_M(b_1/b;q)_{m_1}...(b_k/b;q)_{m_k}}{(\alpha z;q)_\infty(z;q)_M(b_1;q)_{m_1}...(b_k;q)_{m_k}} b^M \sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(bq;q)_n}(zq^M)^n, \tag{2.4}$$

where $M = m_1 + m_2 + ... + m_k$ and $|z| < 1, |q| < 1$.
Putting $\alpha = 0$ in (2.4) we get the transformation

$${}_{k+1}\Phi_{k+1}\left[\begin{array}{l} b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q; zq \\ bq, b_1, b_2, ..., b_k \end{array}\right]$$

$$= \frac{(z;q)_\infty(b_1/b;q)_{m_1}...(b_k/b;q)_{m_k}}{(z;q)_M(b_1;q)_{m_1}...(b_k;q)_{m_k}} b^M \sum_{n=0}^{\infty} \frac{(zq^M)^n}{(bq;q)_n} \tag{2.5}$$

(2.5) can also be expressed as,

$$\sum_{n=0}^{\infty} \frac{(b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k};q)_n(-1)^n q^{n(n+1)/2} z^n}{(q, b_1, b_2, ..., b_k;q)_n(1-bq^n)}$$

$$= \frac{(z;q)_\infty}{(z;q)_M} \frac{(b_1/b;q)_{m_1}...(b_k/b;q)_{m_k}}{(b_1;q)_{m_1}...(b_k;q)_{m_k}} b^M \sum_{n=0}^{\infty} \frac{(zq^M)^n}{(b;q)_{n+1}}. \tag{2.6}$$

3. Special Cases

In this section, certain interesting special cases of the results established in previous section have been deduced.

1. Taking $\alpha = bq$ in (2.4) we get the summation formula for

$${}_{k+1}\Phi_{k+1}\left[\begin{array}{l} b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q; zq \\ bzq^{1+M}, b_1, b_2, ..., b_k \end{array}\right]$$

$$= \frac{(z;q)_\infty(bzq;q)_M(b_1/b;q)_{m_1}(b_2/b;q)_{m_2}...(b_k/b;q)_{m_k}}{(bzq;q)_\infty(z;q)_{M+1}(b_1;q)_{m_1}(b_2;q)_{m_2}...(b_k;q)_{m_k}} b^M, \tag{3.1}$$

where $|z| < 1$, $|q| < 1$ and $M = m_1 + m_2 + ... + m_k$.
As $z \to 1$, (3.1) yields,

$${}_{k+1}\Phi_{k+1}\left[\begin{array}{l} b, b_1q^{m_1}, b_2q^{m_2}, ..., b_kq^{m_k}; q; q \\ bq^{1+M}, b_1, b_2, ..., b_k \end{array}\right]$$

$$= \frac{(q;q)_\infty (bq;q)_M (b_1/b;q)_{m_1} (b_2/b;q)_{m_2} ... (b_k/b;q)_{m_k}}{(q;q)_{M+1} (bq;q)_\infty (b_1;q)_{m_1} (b_2;q)_{m_2} ... (b_k;q)_{m_k}} b^M, \tag{3.2}$$

where $M = m_1 + m_2 + ... + m_k$.
Taking z/α for z then $\alpha \to \infty$ in (2.3) we find

$$(z;q)_\infty \sum_{n=0}^{\infty} \frac{(b_1 q^{m_1}, b_2 q^{m_2}, ..., b_k q^{m_k}; q)_n}{(q, z, b_1, b_2, ..., b_k; q)_n (1 - bq^n)} q^{n(n-1)} q^{n(1-M)} z^n$$

$$= \frac{(b_1/b;q)_{m_1} ... (b_k/b;q)_{m_k}}{(b_1;q)_{m_1} ... (b_k;q)_{m_k}} b^M \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} z^n}{(b;q)_{n+1}}. \tag{3.3}$$

Taking $b = q$ in (3.3) we get

$$\sum_{n=0}^{\infty} \frac{(b_1 q^{m_1}, b_2 q^{m_2}, ..., b_k q^{m_k}; q)_n}{(q, z, b_1, b_2, ..., b_k; q)_n (1 - q^{n+1})} q^{n^2 - nM} z^n$$

$$= \frac{(q - b_1)(q - b_2) ... (q - b_k)}{(1 - b_1 q^{m_1 - 1})(1 - b_2 q^{m_2 - 1}) ... (1 - b_k q^{m_k - 1})} \sum_{n=0}^{\infty} \frac{(-z)^n q^{n(n-1)/2}}{(q;q)_{n+1}}. \tag{3.4}$$

Taking $\alpha = b$ in (2.4) we have

$${}_{k+2}\Phi_{k+2} \left[\begin{matrix} b, b, b_1 q^{m_1}, b_2 q^{m_2}, ..., b_k q^{m_k}; q; zq \\ bzq^M, bq, b_1, b_2, ..., b_k \end{matrix} \right]$$

$$= \frac{(z;q)_\infty (bz;q)_M (b_1/b;q)_{m_1} ... (b_k/b;q)_{m_k} (1 - b) b^M}{(bz;q)_\infty (z;q)_M (b_1;q)_{m_1} ... (b_k;q)_{m_k}} \sum_{n=0}^{\infty} \frac{(zq^M)^n}{(1 - bq^n)}. \tag{3.5}$$

Similar other results can also be deduced.

References

[1] Gasper G. and Rahman M., Basic Hypergeometric Series, Second Edition, Cambridge University Press, 2004.