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SOME TRANSFORMATIONS FOR GOURSAT'S FUNCTION ${}_2F_2[2z]$

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Abstract: The objective of this paper is to obtain six new explicit forms of quadratic transformations for Goursat's hypergeometric function ${}_2F_2[a, m+d; 2a \pm j, d; 2z]$ with suitable convergence conditions.

Keywords and Phrases: Kummer's first and second transformations, Kummer's confluent hypergeometric function, Goursat's hypergeometric function.

2020 Mathematics Subject Classification: Primary 33C05, Secondary 33C20.

1. Introduction and Preliminaries

In our present investigation, we shall make use of the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}$; $\mathbb{N}_0 := \mathbb{N} \bigcup \{0\}$; $\mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.
Also as usual, the symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers, respectively.
The Pochhammer's symbol (or the shifted factorial) $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by

In loving memory of Prof. R. Y. Denis (DDU University, Gorakhpur, U.P.), this paper is dedicated.

(see, for example, [13, 17])

$$(\alpha)_p := \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)} = \begin{cases} 1; & (p=0; \alpha \in \mathbb{C}\setminus\{0\}) \\ \prod_{j=0}^{n-1}(\alpha+j); & (p=n\in\mathbb{N}; \alpha\in\mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!}; & (\alpha=-n; p=k; k,n\in\mathbb{N}_0; 0\leqq k\leqq n) \\ 0; & (\alpha=-n; p=k; k,n\in\mathbb{N}_0; k>n) \\ \frac{(-1)^n}{(1-\alpha)_n}; & (p=-n; n\in\mathbb{N}; \alpha\in\mathbb{C}\setminus\mathbb{Z}), \end{cases} \tag{1.1}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The generalized ordinary hypergeometric function of one variable is defined by the series (see [17, p. 42, Equation (1)])

$${}_AF_B\left[\begin{matrix}(a_A); \\ \\ (b_B);\end{matrix}\; z\right] \equiv {}_AF_B\left[\begin{matrix}a_1,\, a_2,\, \ldots,\, a_A; \\ \\ b_1,\, b_2,\, \ldots,\, b_B;\end{matrix}\; z\right] = \sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_A)_k z^k}{(b_1)_k(b_2)_k\cdots(b_B)_k k!}, \tag{1.2}$$

where $A,\ B \in \mathbb{N}_0$; A numerator parameters $a_j \in \mathbb{C}$ $(j = 1, 2, \ldots, A)$ and B denominator parameters $b_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(j = 1, 2, \ldots, B)$. For details as regards convergence of the function ${}_AF_B$, we refer to the works ([13] and [17]).

It is worth noting here that the findings are very useful from an application standpoint whenever hypergeometric functions with specific argument and parameter values are represented in terms of gamma functions. Thus, the classical summation theorems such as Gauss, Kummer and Bailey for the series ${}_2F_1$ and of Watson, Dixon, Whipple and Saalschütz for the series ${}_3F_2$ and their generalizations (see, for example, [7-10]) play an important role in the theory of hypergeometric and generalized hypergeometric functions and in other applicable sciences. For the applications of the above-mentioned summation theorems, we refer to the research papers ([1, 3, 6, 10, 13, 14]).

From the theory of differential equations, Kummer has obtained the following two very useful and interesting results, known in the literature as Kummer's linear transformation [13, p. 125, Equation (2)], viz

$${}_1F_1[a; b; z] = e^z\, {}_1F_1[b-a; b; -z]; \quad \left(b \in \mathbb{C}\setminus\mathbb{Z}_0^-\right) \tag{1.3}$$

and Kummer quadratic transformation [13, p. 126, Equation (9)]:

$${}_1F_1\left[\begin{matrix}a; \\ \\ 2a;\end{matrix}\; 2z\right] = e^z\, {}_0F_1\left[\begin{matrix}\overline{}; \\ \\ a+\frac{1}{2};\end{matrix}\; \frac{z^2}{4}\right]; \quad \left(2a \neq -1, -3, -5, -7, \ldots\right). \tag{1.4}$$

In 1928, Bailey [1] obtained above result (1.4) by employing Kummer's second summation theorem [13, p. 69, Q. No. 2]. In 1998, Rathie-Choi [15]derived the same quadratic transformation (1.4) by employing Gauss's classical summation theorem [13, p. 49, Theorem 18].
In 1883, Goursat defined higher order hypergeometric function in his original notation in the form of double integral [4, p. 286]. So we have

$$G\left(\begin{array}{cc}\alpha,\, \beta; & \\ & z \\ \gamma,\, \delta; & \end{array}\right) = \frac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\delta-\beta)} \times$$
$$\times \int_0^1 \int_0^1 u^{\alpha-1} w^{\beta-1} (1-u)^{\gamma-\alpha-1} (1-w)^{\delta-\beta-1} e^{zuw}\, du dw, \quad (1.5)$$

where $\Re(\gamma) > \Re(\alpha) > 0$, $\Re(\delta) > \Re(\beta) > 0$,
and

$$G\left(\begin{array}{cc}\alpha,\, \beta; & \\ & z \\ \gamma,\, \delta; & \end{array}\right) = 1 + \sum_{n=1}^{\infty} \frac{(\alpha)_n (\beta)_n z^n}{(\gamma)_n (\delta)_n n!}$$
$$= {}_2F_2\left[\begin{array}{cc}\alpha,\, \beta; & \\ & z \\ \gamma,\, \delta; & \end{array}\right] \quad (1.6)$$

It is also well known that, under certain conditions, the Goursat's function [4, p. 286] ${}_2F_2(\alpha, \beta; \gamma, \delta; z)$ is defined by

$${}_2F_2\left[\begin{array}{cc}\alpha,\, \beta; & \\ & z \\ \gamma,\, \delta; & \end{array}\right] = \frac{\Gamma(\delta)}{\Gamma(\alpha)\Gamma(\delta-\alpha)} \int_0^1 v^{\alpha-1}(1-v)^{\delta-\alpha-1}\, {}_1F_1\left[\begin{array}{cc}\beta; & \\ & zv \\ \gamma; & \end{array}\right] dv, \quad (1.7)$$

where $\Re(\delta) > \Re(\alpha) > 0$ and ${}_1F_1(.)$ is Kummer's confluent hypergeometric function. Motivated by the transformations (1.3), (1.4) of Kummer, following linear transformation of Paris [11, p. 381, Equation (4)]:

$${}_2F_2\left[\begin{array}{cc}a,\, c+1; & \\ & z \\ b,\, c; & \end{array}\right] = e^z\, {}_2F_2\left[\begin{array}{cc}b-a-1,\, f+1; & \\ & -z \\ b,\, f; & \end{array}\right], \quad (1.8)$$

where $f = \frac{c(1+a-b)}{a-c}$ $(a \neq c,\ 1+a-b \neq 0)$, and a quadratic transformation of Rathie-Pogány ([16, p. 65, Equation (11)]; see also [5, p. 626, Equation (2.2)])

$$ {}_2F_2\left[\begin{matrix} a,\ 1+d; \\ 2a+1,\ d; \end{matrix}\ z\right] $$

$$ = \quad exp\left(\frac{z}{2}\right)\left\{{}_0F_1\left[\begin{matrix} \text{———}; \\ a+\frac{1}{2}; \end{matrix}\ \frac{z^2}{16}\right] - \frac{z(1-\frac{2a}{d})}{2(2a+1)}{}_0F_1\left[\begin{matrix} \text{———}; \\ a+\frac{3}{2}; \end{matrix}\ \frac{z^2}{16}\right]\right\}, \quad (1.9) $$

we have given six new quadratic transformations for Goursat's hypergeometric function ${}_2F_2[a, m+d; 2a\pm j, d; 2z]$ in section 2. In section 3, we provide the derivation of all six quadratic transformations by making use of series manipulation technique. The series ${}_AF_B[z]$ is always convergent for all z when $A \leqq B$.

Any parameter and variable settings that result in nonsensical results in the following sections are implicitly excluded.

2. Six Quadratic Transformations for Goursat Function

This section will establish the following six quadratic transformations for Goursat's hypergeometric function ${}_2F_2[2z]$.

Theorem 2.1. *The following transformations of Goursat's function ${}_2F_2[2z]$ in terms of finite sums of ${}_1F_2[\frac{z^2}{4}]$ hold true:*

$$ {}_2F_2\left[\begin{matrix} a,\ 1+d; \\ 2a+j,\ d; \end{matrix}\ 2z\right] $$

$$ = \frac{\Gamma(1-a)e^z}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\left\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}{}_1F_2\left[\begin{matrix} \frac{1-r+j}{2}; \\ \frac{1+2a-r+j}{2},\ \frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right] + \right. $$

$$ \left. + \frac{z\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j}{2})}{}_1F_2\left[\begin{matrix} \frac{2-r+j}{2}; \\ \frac{2+2a+j-r}{2},\ \frac{3}{2}; \end{matrix}\ \frac{z^2}{4}\right]\right\} + \frac{aze^z\Gamma(1-a)}{2^{2a+j}(a)_j d\Gamma(1-2a-j)} \times $$

$$ \times\sum_{r=0}^{j-1}\binom{j-1}{r}(-1)^r\left\{\frac{\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j+2}{2})}{}_1F_2\left[\begin{matrix} \frac{-r+j}{2}; \\ \frac{-r+2a+j+2}{2},\ \frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right] + \frac{z\Gamma(\frac{r-2a-j-1}{2})}{\Gamma(\frac{r-j+1}{2})}\times\right. $$

$$\times {}_1F_2\left[\begin{matrix} \frac{-r+j+1}{2}; \\ \frac{-r+j+2a+3}{2},\ \frac{3}{2}; \end{matrix}\ \frac{z^2}{4}\right]\Bigg\}, \tag{2.1}$$

$$\Big(|z| < \infty; 2a, d, 1-a, 1-2a-j \in \mathbb{C}\backslash\mathbb{Z}_0^-; j \in \mathbb{N}\Big),$$

$$ {}_2F_2\left[\begin{matrix} a,\ 2+d; \\ 2a+j,\ d; \end{matrix}\ 2z\right]$$

$$= \frac{\Gamma(1-a)e^z}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}{}_1F_2\left[\begin{matrix} \frac{1-r+j}{2}; \\ \frac{1+2a-r+j}{2},\ \frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right]+$$

$$+\frac{z\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j}{2})}{}_1F_2\left[\begin{matrix} \frac{2-r+j}{2}; \\ \frac{2+2a+j-r}{2},\ \frac{3}{2}; \end{matrix}\ \frac{z^2}{4}\right]\Bigg\} + \frac{aze^z\Gamma(1-a)}{2^{2a+j-1}(a)_j d\Gamma(1-2a-j)}\times$$

$$\times\sum_{r=0}^{j-1}\binom{j-1}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j+2}{2})}{}_1F_2\left[\begin{matrix} \frac{-r+j}{2}; \\ \frac{-r+2a+j+2}{2},\ \frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a-j-1}{2})}{\Gamma(\frac{r-j+1}{2})}\times$$

$$\times\ {}_1F_2\left[\begin{matrix} \frac{-r+j+1}{2}; \\ \frac{-r+j+2a+3}{2},\ \frac{3}{2}; \end{matrix}\ \frac{z^2}{4}\right]\Bigg\} + \frac{(a)_2z^2e^z\Gamma(1-a)}{2^{2a+j}(a)_j(d)_2\Gamma(1-2a-j)}\times$$

$$\times\sum_{r=0}^{j-2}\binom{j-2}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{-2a-j+r-1}{2})}{\Gamma(\frac{r-j+3}{2})}{}_1F_2\left[\begin{matrix} \frac{-r+j-1}{2}; \\ \frac{-r+2a+j+3}{2},\ \frac{1}{2}; \end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a-j-2}{2})}{\Gamma(\frac{r-j+2}{2})}\times$$

$$\times\ {}_1F_2\left[\begin{matrix} \frac{-r+j}{2}; \\ \frac{-r+j+2a+4}{2},\ \frac{3}{2}; \end{matrix}\ \frac{z^2}{4}\right]\Bigg\}, \tag{2.2}$$

$$\Big(|z| < \infty; 2a, d, 1-a, 1-2a-j \in \mathbb{C}\backslash\mathbb{Z}_0^-; j \in \mathbb{N}\backslash\{1\}\Big),$$

.

$$ {}_2F_2\left[\begin{matrix} a,\ 3+d; \\ 2a+j,\ d; \end{matrix}\ 2z\right]$$

$$= \frac{\Gamma(1-a)e^z}{2^{2a+j}(a)_j\Gamma(1-2a-j)}\sum_{r=0}^{j}\binom{j}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{1-2a-j+r}{2})}{\Gamma(\frac{1+r-j}{2})}{}_1F_2\left[\begin{matrix}\frac{1-r+j}{2};\\ \frac{1+2a-r+j}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+$$

$$+\frac{z\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j}{2})}{}_1F_2\left[\begin{matrix}\frac{2-r+j}{2};\\ \frac{2+2a+j-r}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}+\frac{3aze^z\Gamma(1-a)}{2^{2a+j}(a)_j d\Gamma(1-2a-j)}\times$$

$$\times\sum_{r=0}^{j-1}\binom{j-1}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{-2a-j+r}{2})}{\Gamma(\frac{r-j+2}{2})}{}_1F_2\left[\begin{matrix}\frac{-r+j}{2};\\ \frac{-r+2a+j+2}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a-j-1}{2})}{\Gamma(\frac{r-j+1}{2})}\times$$

$$\times{}_1F_2\left[\begin{matrix}\frac{-r+j+1}{2};\\ \frac{-r+j+2a+3}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}+\frac{3(a)_2z^2e^z\Gamma(1-a)}{2^{2a+j}(a)_j(d)_2\Gamma(1-2a-j)}\times$$

$$\times\sum_{r=0}^{j-2}\binom{j-2}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{-2a-j+r-1}{2})}{\Gamma(\frac{r-j+3}{2})}{}_1F_2\left[\begin{matrix}\frac{-r+j-1}{2};\\ \frac{-r+2a+j+3}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a-j-2}{2})}{\Gamma(\frac{r-j+2}{2})}\times$$

$$\times{}_1F_2\left[\begin{matrix}\frac{-r+j}{2};\\ \frac{-r+j+2a+4}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}+\frac{(a)_3z^3e^z\Gamma(1-a)}{2^{2a+j}(a)_j(d)_3\Gamma(1-2a-j)}\times$$

$$\times\sum_{r=0}^{j-3}\binom{j-3}{r}(-1)^r\Bigg\{\frac{\Gamma(\frac{-2a-j+r-2}{2})}{\Gamma(\frac{r-j+4}{2})}{}_1F_2\left[\begin{matrix}\frac{-r+j-2}{2};\\ \frac{-r+2a+j+4}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a-j-3}{2})}{\Gamma(\frac{r-j+3}{2})}\times$$

$$\times{}_1F_2\left[\begin{matrix}\frac{-r+j-1}{2};\\ \frac{-r+j+2a+5}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}, \tag{2.3}$$

$$\left(|z|<\infty; 2a, d, 1-a, 1-2a-j\in\mathbb{C}\backslash\mathbb{Z}_0^-; j\in\mathbb{N}\backslash\{1,2\}\right),$$

$${}_2F_2\left[\begin{matrix}a,\ 1+d;\\ 2a-j,\ d;\end{matrix}\ 2z\right]$$

$$= \frac{\Gamma(1-a)e^z}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\Bigg\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}{}_1F_2\left[\begin{matrix}\frac{1-r+j}{2};\\ \frac{1+2a-r-j}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+$$

$$+\frac{z\Gamma(\frac{r-2a+j}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{matrix}\frac{2-r+j}{2};\\ \frac{2+2a-j-r}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}+\frac{ze^z\Gamma(1-a)}{2^{2a-j}d\Gamma(1-2a+j)}\times$$

$$\times\sum_{r=0}^{j+1}\binom{j+1}{r}\Bigg\{\frac{\Gamma(\frac{-2a+j+r}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{matrix}\frac{-r+j+2}{2};\\ \frac{-r+2a-j+2}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a+j-1}{2})}{\Gamma(\frac{r-j-1}{2})}\times$$

$$\times\,{}_1F_2\left[\begin{matrix}\frac{-r+j+3}{2};\\ \frac{-r-j+2a+3}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}, \tag{2.4}$$

$$\left(|z|<\infty; a, d, 2a-j, 1-a, 1-2a\in\mathbb{C}\backslash\mathbb{Z}_0^-; j\in\mathbb{N}_0\right),$$

$${}_2F_2\left[\begin{matrix}a,\ 2+d;\\ 2a-j,\ d;\end{matrix}\ 2z\right]$$

$$=\frac{\Gamma(1-a)e^z}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\Bigg\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}\,{}_1F_2\left[\begin{matrix}\frac{1-r+j}{2};\\ \frac{1+2a-r-j}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+$$

$$+\frac{z\Gamma(\frac{r-2a+j}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{matrix}\frac{2-r+j}{2};\\ \frac{2+2a-j-r}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}+\frac{ze^z\Gamma(1-a)}{2^{2a-j-1}d\Gamma(1-2a+j)}\times$$

$$\times\sum_{r=0}^{j+1}\binom{j+1}{r}\Bigg\{\frac{\Gamma(\frac{-2a+j+r}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{matrix}\frac{-r+j+2}{2};\\ \frac{-r+2a-j+2}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a+j-1}{2})}{\Gamma(\frac{r-j-1}{2})}\times$$

$$\times\,{}_1F_2\left[\begin{matrix}\frac{-r+j+3}{2};\\ \frac{-r-j+2a+3}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}+\frac{z^2e^z\Gamma(1-a)}{2^{2a-j}(d)_2\Gamma(1-2a+j)}\times$$

$$\times\sum_{r=0}^{j+2}\binom{j+2}{r}\Bigg\{\frac{\Gamma(\frac{-2a+j+r-1}{2})}{\Gamma(\frac{r-j-1}{2})}\,{}_1F_2\left[\begin{matrix}\frac{-r+j+3}{2};\\ \frac{-r+2a-j+3}{2},\ \frac{1}{2};\end{matrix}\ \frac{z^2}{4}\right]+\frac{z\Gamma(\frac{r-2a+j-2}{2})}{\Gamma(\frac{r-j-2}{2})}\times$$

$$\times\,{}_1F_2\left[\begin{matrix}\frac{-r+j+4}{2};\\ \frac{-r-j+2a+4}{2},\ \frac{3}{2};\end{matrix}\ \frac{z^2}{4}\right]\Bigg\}, \tag{2.5}$$

$$\left(|z| < \infty; a, d, 2a-j, 1-a, 1-2a \in \mathbb{C}\backslash\mathbb{Z}_0^-; j \in \mathbb{N}_0\right)$$

and

$$
{}_2F_2\left[\begin{array}{c} a,\, 3+d; \\ \\ 2a-j,\, d; \end{array} 2z\right]
$$

$$
\begin{aligned}
&= \frac{\Gamma(1-a)e^z}{2^{2a-j}\Gamma(1-2a+j)}\sum_{r=0}^{j}\binom{j}{r}\left\{\frac{\Gamma(\frac{1-2a+j+r}{2})}{\Gamma(\frac{1+r-j}{2})}{}_1F_2\left[\begin{array}{c} \frac{1-r+j}{2}; \\ \\ \frac{1+2a-r-j}{2},\ \frac{1}{2}; \end{array} \frac{z^2}{4}\right] +\right.\\
&\left. + \frac{z\Gamma(\frac{r-2a+j}{2})}{\Gamma(\frac{r-j}{2})}\,{}_1F_2\left[\begin{array}{c} \frac{2-r+j}{2}; \\ \\ \frac{2+2a-j-r}{2},\ \frac{3}{2}; \end{array} \frac{z^2}{4}\right]\right\} + \frac{3ze^z\Gamma(1-a)}{2^{2a-j}d\Gamma(1-2a+j)}\times\\
&\times \sum_{r=0}^{j+1}\binom{j+1}{r}\left\{\frac{\Gamma(\frac{-2a+j+r}{2})}{\Gamma(\frac{r-j}{2})}{}_1F_2\left[\begin{array}{c} \frac{-r+j+2}{2}; \\ \\ \frac{-r+2a-j+2}{2},\ \frac{1}{2}; \end{array} \frac{z^2}{4}\right] + \frac{z\Gamma(\frac{r-2a+j-1}{2})}{\Gamma(\frac{r-j-1}{2})}\times\right.\\
&\left.\times\,{}_1F_2\left[\begin{array}{c} \frac{-r+j+3}{2}; \\ \\ \frac{-r-j+2a+3}{2},\ \frac{3}{2}; \end{array} \frac{z^2}{4}\right]\right\} + \frac{3z^2e^z\Gamma(1-a)}{2^{2a-j}(d)_2\Gamma(1-2a+j)}\times\\
&\times \sum_{r=0}^{j+2}\binom{j+2}{r}\left\{\frac{\Gamma(\frac{-2a+j+r-1}{2})}{\Gamma(\frac{r-j-1}{2})}{}_1F_2\left[\begin{array}{c} \frac{-r+j+3}{2}; \\ \\ \frac{-r+2a-j+3}{2},\ \frac{1}{2}; \end{array} \frac{z^2}{4}\right] + \frac{z\Gamma(\frac{r-2a+j-2}{2})}{\Gamma(\frac{r-j-2}{2})}\times\right.\\
&\left.\times\,{}_1F_2\left[\begin{array}{c} \frac{-r+j+4}{2}; \\ \\ \frac{-r-j+2a+4}{2},\ \frac{3}{2}; \end{array} \frac{z^2}{4}\right]\right\} + \frac{z^3e^z\Gamma(1-a)}{2^{2a-j}(d)_3\Gamma(1-2a+j)}\times\\
&\times \sum_{r=0}^{j+3}\binom{j+3}{r}\left\{\frac{\Gamma(\frac{-2a+j+r-2}{2})}{\Gamma(\frac{r-j-2}{2})}{}_1F_2\left[\begin{array}{c} \frac{-r+j+4}{2}; \\ \\ \frac{-r+2a-j+4}{2},\ \frac{1}{2}; \end{array} \frac{z^2}{4}\right] + \frac{z\Gamma(\frac{r-2a+j-3}{2})}{\Gamma(\frac{r-j-3}{2})}\times\right.\\
&\left.\times\,{}_1F_2\left[\begin{array}{c} \frac{-r+j+5}{2}; \\ \\ \frac{-r-j+2a+5}{2},\ \frac{3}{2}; \end{array} \frac{z^2}{4}\right]\right\}, \qquad (2.6)
\end{aligned}
$$

$$\left(|z| < \infty; a, d, 2a-j, 1-a, 1-2a \in \mathbb{C}\backslash\mathbb{Z}_0^-; j \in \mathbb{N}_0\right).$$

3. Derivations of Quadratic Transformations

To establish the transformation (2.1), we proceed as follows
Denoting the left-hand side of the transformation (2.1) by P, expressing ${}_2F_2$ as a series, we obtain

$$P = e^{-z} \sum_{m=0}^{\infty} \frac{(a)_m (2z)^m}{(2a+j)_m \, m!} \left(1 + \frac{m}{d}\right)$$
$$= e^{-z} \left\{ {}_1F_1 \left[\begin{matrix} a; \\ 2a+j; \end{matrix} \; 2z \right] + \frac{1}{d} \sum_{m=1}^{\infty} \frac{(a)_m (2z)^m}{(2a+j)_m \, (m-1)!} \right\}. \tag{3.1}$$

Now, replacing m by $m+1$ in the second member of right-hand side of (3.1), after straightforward algebra, we get

$$P = e^{-z} {}_1F_1 \left[\begin{matrix} a; \\ 2a+j; \end{matrix} \; 2z \right] + \frac{2az}{d(2a+j)} e^{-z} {}_1F_1 \left[\begin{matrix} a+1; \\ 2a+j+1; \end{matrix} \; 2z \right]. \tag{3.2}$$

Using the transformation formula [12, p. 83, Equation (29)] in the first and second expression on the right hand of equation (3.2) and multiplying both sides of equation (3.2) by e^z, after simplification we arrive at the right-hand side of transformation (2.1).

Now consider two more series ${}_2F_2$ in the following forms:

$${}_2F_2 \left[\begin{matrix} a, \, 2+d; \\ 2a+j, \, d; \end{matrix} \; 2z \right] = e^{-z} \sum_{m=0}^{\infty} \frac{(a)_m (2z)^m}{(2a+j)_m \, m!} \left\{ 1 + \frac{2m}{d} + \frac{m(m-1)}{d(d+1)} \right\} \tag{3.3}$$

and

$${}_2F_2 \left[\begin{matrix} a, \, 3+d; \\ 2a+j, \, d; \end{matrix} \; 2z \right] = e^{-z} \sum_{m=0}^{\infty} \frac{(a)_m (2z)^m}{(2a+j)_m \, m!} \times$$
$$\times \left\{ 1 + \frac{3m}{d} + \frac{3m(m-1)}{d(d+1)} + \frac{m(m-1)(m-2)}{d(d+1)(d+2)} \right\}. \tag{3.4}$$

Now, using the same result as the authors [12, p. 83, Equation (29)], we will proceed in a similar manner to the derivation of the transformation (2.1), and with a few simplifications, we get the transformations (2.2) and (2.3).

Similarly, by using another result of the same authors [12, p. 83, Equation (30)], the transformation formulas (2.4), (2.5) and (2.6) can be proved.

Remark 1. *Similarly, we can derive two more transformation formulas for Goursat's hypergeometric function* ${}_2F_2\left[\begin{matrix} a,\, 4+d; \\ \\ 2a \pm j,\, d; \end{matrix}\; 2z\right]$ *by the application of following result involving the quotient of Pochhammer symbols defined by (1.1):*

$$\frac{(c+4)_r}{(c)_r} = 1 + \frac{4}{c}r + \frac{6r(r-1)}{c(c+1)} + \frac{4r(r-1)(r-2)}{c(c+1)(c+2)} + \frac{r(r-1)(r-2)(r-3)}{c(c+1)(c+2)(c+3)}, \quad (3.5)$$

$$\left(c \in \mathbb{C}\backslash\mathbb{Z}_0^-;\, r \in \mathbb{N}_0\right).$$

4. Conclusion

We conclude our present investigation by observing that several further interesting hypergeometric quadratic transformation formulas can also be obtained in an analogous manner.

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