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PHILOS-TYPE OSCILLATION CRITERIA FOR FRACTIONAL DIFFERENTIAL EQUATIONS WITH IMPULSIVE CONDITIONS

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Abstract: The present article is concerned with the oscillatory nature of the fractional differential equation of order $\alpha \in (2, 3)$ with impulsive effects. By employing a generalized Riccati transformation, we derive several oscillation criteria of Philos type, which are either new or improve several recent results in the literature. Also, we show the stability of the considered problem. To obtain the results, we transform the fractional differential equation into a second-order ordinary differential equation. In addition, we provide examples to show the effectiveness of the results.

Keywords and Phrases: Fractional differential equations, Impulsive conditions, Oscillation, Differential inequality method.

2020 Mathematics Subject Classification: 26A33, 34A08, 34D20, 34K11, 35R12.

1. Introduction

The concept of fractional derivatives was originally established in Leibniz's letter to L'Hospital on September 30, 1695 [17], when he raised the meaning of derivative of order $\frac{1}{2}$. The issue raised by Leibniz attracted many well-known mathematicians, including Liouville, Grünwald, Riemann, Euler, Lagrange, Heaviside, Fourier, Abel, Letnikov, and many others. Since the 19th century, the theory of fractional calculus originated rapidly and was the foundation for several disciplines such as fractional differential equations, noninteger order geometry, and fractional dynamics. Today, there are numerous applications in various branches, including optimal control, porous media, fractional filters, signal and image processing, fractals, rheology, electrochemistry, fluid mechanics, polymer physics, etc. [12, 13, 19]. For basics of fractional calculus and applications, see [22, 31].

The generalization of integer order classical differential equations to noninteger order are termed as fractional differential equations. Due to the wide applicability of these differential equations in many areas of science and engineering, fractional calculus deserves an independent study parallel to the well-known theory of ordinary differential equations. Nowadays, many scientific and engineering problems, including fractional derivatives, are already huge and still growing. Fractional order differential equations came into existence because ordinary differential equations cannot formulate many physical issues. Also, fractional differentials and integrals provide a more accurate model under consideration.

Many authors have studied the existence, uniqueness, and approximations of solutions of fractional differential equations. Some of these works can be seen in the papers [4, 6, 12, 19, 30, 35]. The fundamental results and definitions of fractional differential equations are discussed in [15, 23]. Many authors have studied various types of models based on noninteger order derivatives [29, 32, 36]. In recent years, many researchers have shown keen interest in the study of oscillating and non-oscillating behavior of solutions [3, 21, 32, 38, 41]. In papers [6, 25, 38], authors studied the oscillatory nature of different classes of fractional differential equations without impulses. In [33], S. Salahshour et al. studied the analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem. In [8], A. A. El-Sayed et al. examined numerical solutions of multi-term variable-order fractional differential equations via Jacobi operational matrix.

It is well known that many real-life phenomena are affected by the sudden change in their states at certain moments, such as heartbeats and blood flow in the human body [27]. These phenomena are discussed in the form of impulses whose duration is negligible compared to the whole process. Impulsive differential equations are used to simulate those discontinuous processes in which impulses occur. For example, disturbances in cellular neural networks [7], fluctuations of pendulum system in the case of external impulsive effects [2], relaxational oscillations of the electromechanical system [24], and so on. As a result, it becomes an important tool to handle the real process of mathematical models and phenomena. We refer to [16] for an introduction to the theory of impulsive differential equations. It has a wide range of applications, including drug diffusion in the human body, frequency-modulated systems, population dynamics, chemical technology, electric circuits, fractals, viscoelasticity, etc. For more details on impulsive differential equations and their applications, refer to [34, 35]. One of the main advantages of the impulses can be seen in the paper of Sugie and Ishihara [36], they provide the model in which the mass point might oscillate in the presence of impulsive effect and in the absence of impulsive effect, the mass point didn't oscillate.

Some researchers began to investigate the oscillatory nature of differential equations with impulses in 1989, and it is still in its early stages of development. Authors later extended the oscillation study to parabolic and hyperbolic impulsive partial differential equations in papers [11, 26]. In recent years, many researchers have shown a great interest in studying the oscillatory nature of noninteger order evolution equations with impulses. We refer to [29, 36] and references cited therein for these studies. Sadhasivam and Deepa [32] investigated a hybrid evolution system with impulsive conditions.

In the literature, we noticed Riccati techniques are widely used to obtain Kamenev and Philos type oscillation criteria [9, 10, 14, 18, 28, 36, 37, 39, 40]. The authors used Riccati techniques to investigate the oscillation of solutions of even-order differential equations with neutral terms in the papers [1, 20]. On the other hand, O. Bazighifan et al. [5] studied the fourth-order delay differential equation and discussed the oscillatory and asymptotic properties of solutions. We refer to [18, 40] for more works on second-order differential equations with damping.

In [29], Raheem and Maqbul studied the oscillatory behavior of solutions of the following fractional impulsive differential equation:

$$\begin{cases} \mathfrak{D}^{\beta}_{+,t}u(x,t) + a(t)\mathfrak{D}^{\beta-1}_{+,t}u(x,t) = b(t)\Delta u(x,t) + \sum\limits_{k=1}^{m} c_k(t)\Delta u(x,t-\tau_k) - F(x,t), \ t \neq t_j, \\ \mathfrak{D}^{\beta-1}_{+,t}u(x,t_j^+) - \mathfrak{D}^{\beta-1}_{+,t}u(x,t_j^-) = \sigma(x,t_j)\mathfrak{D}^{\beta-1}_{+,t}u(x,t_j), j = 1,2,\ldots,(x,t) \in \Omega \times R_+ = G, \end{cases}$$

where $a,\ b,\ c_k \in PC[R_+, R_+]$ and forcing term $F \in PC[\overline{\Omega} \times R_+, R_+]$, where PC denotes the class of functions which are piecewise continuous functions in t with discontinuities of first kind only at $t = t_j$, $j = 1, 2, \ldots$ and left continuous at $t = t_j$, $\beta \in (1, 2)$ is a constant, Δ is the Laplacian in R^n, Ω is a bounded domain in R^n with a smooth boundary $\partial\Omega$ and $\overline{\Omega} = \Omega \cup \partial\Omega$.

Moreover, in [10], L. Feng et al. investigated Philos type oscillation theorem for impulsive Riemann-Liouville's fractional differential equations. K. Wen et al. [39] studied a second-order linear impulsive differential equation with damping term and obtained several results on oscillations. For earlier works on oscillatory behavior of differential equations, refer to [18, 25, 26, 32, 37, 41] and the references cited therein.

After motivating from the above works, we extend the applications of Riccati transformation to noninteger order differential equations and study the oscillatory nature of the problem. In this paper, we have proved several Philos type oscillation theorems. The abstract form of the considered problem is as follows:

$$\begin{cases} \mathfrak{D}^{\alpha}_{+,t}u(t) + a_1(t)\mathfrak{D}^{\alpha-1}_{+,t}u(t) + a_2(t)\mathfrak{D}^{\alpha-2}_{+,t}u(t) = 0, \quad t \geq t_0 > 0, \quad t \neq t_j, \\ \Delta\mathfrak{D}^{\alpha-1}_{+,t}u(t_j) + d_j\mathfrak{D}^{\alpha-2}_{+,t}u(t_j) = 0, \quad j = 1, 2, 3, \ldots, \end{cases} \tag{1.1}$$

where a_1, a_2 are piecewise continuous functions defined from $[t_0, \infty)$ into $\mathbb{R}_+$ with discontinuities at $t = t_j$, $j = 1, 2, \ldots$, $\mathfrak{D}^{\alpha}_{+,t}$ is the fractional derivative of Riemann-Liouville type, where $\alpha \in (2, 3)$, and $\{d_j\}$ is a sequence of real numbers.

The rest of this paper is organized as follows. Section 2 contains some basic lemmas and assumptions which are required for the next sections. In section 3, Philos-type oscillation results and stability conditions are obtained for the problem (1.1) using Riccati transformation. In section 4, examples are given to show the effectuality of the results. In the end, a conclusion is added for future work.

2. Preliminaries and Assumptions

Throughout the paper, we assume the following assumptions:

(H1) The given numbers

$$0 < t_1 < \cdots < t_j < \cdots$$

are such that

$$\lim_{j\to\infty} t_j = +\infty.$$

(H2) $\mathfrak{D}^{\alpha-2}_{+,t}u(z, t)$ is continuous, i.e.,

$$\mathfrak{D}^{\alpha-2}u(z, t_j^+) = \mathfrak{D}^{\alpha-2}u(z, t_j^-) = \mathfrak{D}^{\alpha-2}u(z, t_j).$$

Lemma 2.1. [29] *For any function $u : \mathbb{R}_+ \to \mathbb{R}$, $(\mathfrak{D}^{\alpha}_{+,t}u)(t) = (\mathfrak{D}^{\alpha-1}_{+,t}u)'(t)$ and $(\mathfrak{D}^{\alpha}_{+,t}u)(t) = (\mathfrak{D}^{\alpha-2}_{+,t}u)''(t)$.*

Lemma 2.2. [21] *For any function $u : \mathbb{R}_+ \to \mathbb{R}$, let*

$$G(t) = \int_0^t (t-\tau)^{-(\alpha-2)}u(\tau)d\tau, \quad \textit{where} \;\; \alpha \in (2, 3) \;\; \textit{and} \;\; t \geq 0, \;\textit{then}$$

$$G'(t) = \Gamma(3-\alpha)(\mathfrak{D}^{\alpha-2}_{+,t}u)(t).$$

Lemma 2.3. *For an eventually positive solution $u(t)$ of (1.1) such that $(\mathfrak{D}^{\alpha}_{+,t}u)(t) > 0$ for $t \geq \tau > 0$, and*

$$\int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right) ds = \infty,$$

then $(\mathfrak{D}^{\alpha-2}_{+,t}u)(t) > 0$ *for* $t \geq \tau$.

Proof. If we take $U(t) = \exp\left(\int_{t_0}^{t} \frac{a_2(\sigma)}{a_1(\sigma)} d\sigma\right)$, then

$$\begin{aligned}
[(\mathfrak{D}^{\alpha-2}_{+,t}u)(t)U(t)]' &= (\mathfrak{D}^{\alpha-1}_{+,t}u)(t)U(t) + \frac{a_2(t)}{a_1(t)}(\mathfrak{D}^{\alpha-2}_{+,t}u)(t)U(t)\\
&= \frac{1}{a_1(t)}[a_1(t)(\mathfrak{D}^{\alpha-1}_{+,t}u)(t) + a_2(t)(\mathfrak{D}^{\alpha-2}_{+,t}u)(t)]U(t)\\
&\leq -\frac{1}{a_1(t)}[(\mathfrak{D}^{\alpha}_{+,t}u)(t)]U(t) < 0.
\end{aligned}$$

If $(\mathfrak{D}^{\alpha-2}_{+,t}u)(t) < 0$ for $t \geq \tau$, it follows that

$$(\mathfrak{D}^{\alpha-2}_{+,t}u)(t)U(t) < (\mathfrak{D}^{\alpha-2}_{+,t}u)(\tau)U(\tau) = C < 0, \quad t \geq \tau.$$

From Lemma 2.2, we get

$$\frac{G'(t)}{\Gamma(3-\alpha)} = (\mathfrak{D}^{\alpha-2}_{+,t}U)(t) \quad < \quad C\exp\left(-\int_{t_0}^{t} \frac{a_2(\sigma)}{a_1(\sigma)} d\sigma\right), \quad t \geq \tau.$$

Integrating from τ to t, we get

$$G(t) \quad < \quad G(\tau) + C\Gamma(3-\alpha)\int_{\tau}^{t} \exp\left(-\int_{t_0}^{s} \frac{a_2(\sigma)}{a_1(\sigma)} d\sigma\right) ds.$$

By taking limit, we obtain $\lim_{t\to\infty} G(t) = -\infty$. This contradiction gives $(\mathfrak{D}^{\alpha-2}_{+,t}u)(t) > 0$ for $t \geq \tau$.

Following the process of the above Lemma 2.3, we have:

Lemma 2.4. *For an eventually negative solution of (1.1) such that* $(\mathfrak{D}^{\alpha}_{+,t}u)(t) < 0$ *for* $t \geq \tau > 0$, *and*

$$\int_{\tau}^{\infty} \exp\left(-\int_{t_0}^{s} \frac{a_2(\sigma)}{a_1(\sigma)} d\sigma\right) ds = \infty,$$

then $(\mathfrak{D}^{\alpha-2}_{+,t}u)(t) < 0$ *for* $t \geq \tau$.

Lemma 2.5. *Let all the conditions of Lemma 2.3 and Lemma 2.4 are satisfied, then if all non-zero solutions of the following system:*

$$\begin{cases} z''(t) + a_1(t)z'(t) + a_2(t)z(t) = 0, \quad t \geq t_0 > 0, \quad t \neq t_j, \\ \Delta z'(t_j) + d_j z(t_j) = 0, \quad j = 1, 2, 3, \ldots, \end{cases} \tag{2.1}$$

are oscillatory, then each non-zero solution of the system (1.1) is oscillatory.

Proof. Let on contrary, system (1.1) has an eventually positive solution $u(t)$. Putting $z(t) = \mathfrak{D}^{\alpha-2}u(t)$, (1.1), and using Lemma 2.1, we get

$$\begin{cases} z''(t) + a_1(t)z'(t) + a_2(t)z(t) = 0, \quad t \geq t_0 > 0, \quad t \neq t_j, \\ \Delta z'(t_j) + d_j z(t_j) = 0, \quad j = 1, 2, 3, \ldots. \end{cases}$$

From Lemma 2.3, it follows that the above system has an eventually positive solution.

Throughout the rest paper, we assume that all the conditions of Lemma2.3 and Lemma 2.4 are satisfied.

3. Main Results

Theorem 3.1. *Assume that there exist real valued continuously differentiable functions $\Psi(t,s), \phi(t,s)$ with domain $D_1 = \{(t,s)|t \geq s \geq t_0 > 0\}$ such that*

(A1) $\Psi(t,t) = 0$ *for* $t \geq t_0$ *and* $\Psi(t,s) > 0$ *for* $t > s \geq t_0$;

(A2) $\frac{\partial}{\partial t}\Psi(t,s) \geq 0$, $\frac{\partial}{\partial s}\Psi(t,s) \leq 0$;

(A3) $-\frac{\partial \Psi(t,s)}{\partial s} = \phi(t,s)\sqrt{\Psi(t,s)}, \quad (t,s) \in D_1$.

Then every non-zero solution of (1.1) is oscillatory if

$$\limsup_{t\to\infty}\left[\frac{1}{\Psi(t,t_0)}\int_{t_0}^{t}\left[\Psi(t,s)a_2(s) - \frac{a_1(s)}{4}R^2(t,s)\right]ds + \sum_{j=1}^{n}\Psi(t,t_j)d_j\right] = \infty,$$

where

$$R(t,s) = \left(\frac{\phi(t,s)}{\sqrt{a_1(s)}} + \sqrt{a_1(s)}\sqrt{\Psi(t,s)}\right).$$

Proof. If (2.1) has a nonoscillatory solution $z = z(t)$, then there exists a $T \geq t_0$ such that $z(t) \neq 0$ for $t \geq T$. Thus, we define the Riccati transformation as:

$$v(t) = \frac{z'(t)}{z(t)} \quad for \quad t \geq T.$$

It follows from (2.1) that

$$\begin{aligned} v'(t) &= \frac{z''(t)z(t) - (z'(t))^2}{z^2(t)} \\ &= \frac{z''(t)}{z(t)} - \left(\frac{z'(t)}{z(t)}\right)^2 \\ &= \frac{-a_1(t)z'(t) - a_2(t)z(t)}{z(t)} - v^2(t) \end{aligned}$$

$$v'(t) \quad = \quad -a_1(t)v(t) - a_2(t) - v^2(t),$$

or,

$$v'(t) + v^2(t) + a_1(t)v(t) + a_2(t) = 0, \quad t \neq t_j.$$

Since $z(t)$ is continuous on $[T, \infty)$, we have

$$\begin{aligned}
\Delta v(t_j) &= v(t_j^+) - v(t_j^-) \\
&= \frac{z'(t_j^+)}{z(t_j^+)} - \frac{z'(t_j^-)}{z(t_j^-)} \\
&= \frac{\Delta z'(t_j)}{z(t_j)} \\
\Delta v(t_j) &= -d_j.
\end{aligned}$$

Therefore, the function $v(t)$ satisfies

$$\begin{cases} v'(t) + v^2(t) + a_1(t)v(t) + a_2(t) = 0, & t \neq t_j, \\ \Delta v(t_j) + d_j = 0, & j = 1, 2, 3, \ldots. \end{cases} \tag{3.1}$$

Let m be a positive integer such that $t_{m-1} \leq T < t_m$. For sufficiently large t, we can choose a positive integer n, which satisfies $t_n \leq t < t_{n+1}$. Let J exclude the points $t_m, t_{m+1}, \ldots, t_n$ from $[T, t]$. For $t \neq t_j$, from the first equation of (3.1), we have

$$\int_T^t \Psi(t,s)a_2(s)ds = \quad -\int_J \Psi(t,s)v'(s)ds - \int_J \Psi(t,s)v^2(s)ds - \int_J \Psi(t,s)a_1(s)v(s)ds. \tag{3.2}$$

We have

$$\begin{aligned}
\int_J \Psi(t,s)v'(s)ds &= -\sum_{j=m}^{n} (\Psi(t,t_j^+)v(t_j^+) - \Psi(t,t_j^-)v(t_j^-)) - \Psi(t,T)v(T) - \int_J v(s)\frac{\partial \Psi}{\partial s}ds \\
&= -\sum_{j=m}^{n} \Psi(t,t_j)\Delta v(t_j) - \Psi(t,T)v(T) - \int_J v(s)\frac{\partial \Psi}{\partial s}ds.
\end{aligned}$$

Using second equation of (3.1), we get

$$\int_J \Psi(t,s)v'(s)ds \quad = \quad \sum_{j=m}^{n} \Psi(t,t_j)d_j - \Psi(t,T)v(T) - \int_J v(s)\frac{\partial \Psi}{\partial s}ds.$$

From (3.2), we have

$$\int_T^t \Psi(t,s)a_2(s)ds$$

$$\begin{aligned}
&= \Psi(t,T)v(T) - \sum_{j=m}^{n} \Psi(t,t_j)d_j - \int_J a_1(s)\left\{R(t,s)\frac{\sqrt{\Psi(t,s)}}{\sqrt{a_1(s)}}v(s) + \frac{\Psi(t,s)}{a_1(s)}v^2(s)\right\}ds \\
&= \Psi(t,T)v(T) - \sum_{j=m}^{n} \Psi(t,t_j)d_j - \int_J a_1(s)\left[\left\{\frac{R(t,s)}{2} + \frac{\sqrt{\Psi(t,s)}}{\sqrt{a_1(s)}}v(s)\right\}^2 - \frac{R^2(t,s)}{4}\right]ds \\
&\leq \Psi(t,T)v(T) - \sum_{j=m}^{n} \Psi(t,t_j)d_j + \frac{1}{4}\int_J a_1(s)R^2(t,s)ds.
\end{aligned}$$

The above inequality implies that

$$\int_T^t \left[\Psi(t,s)a_2(s) - \frac{1}{4}a_1(s)R^2(t,s)\right]ds + \sum_{j=m}^{n} \Psi(t,t_j)d_j \leq \Psi(t,T)v(T).$$

Using the above inequality, we get

$$\begin{aligned}
&\int_{t_0}^t \left[\Psi(t,s)a_2(s) - \frac{1}{4}a_1(s)R^2(t,s)\right]ds + \sum_{j=1}^{n} \Psi(t,t_j)d_j \\
&\qquad \leq \int_{t_0}^T \left[\Psi(t,s)a_2(s) - \frac{1}{4}a_1(s)R^2(t,s)\right]ds + \sum_{j=1}^{m-1} \Psi(t,t_j)d_j + \Psi(t,T)v(T) \\
&\qquad \leq \int_{t_0}^T \Psi(t,s)a_2(s)ds + \sum_{j=1}^{m-1} \Psi(t,t_j)d_j + \Psi(t,T)v(T).
\end{aligned}$$

As $\Psi(t,s)$ is decreasing in s, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\left[\frac{1}{\Psi(t,t_0)}\int_{t_0}^t \left\{\Psi(t,s)a_2(s) - \frac{1}{4}a_1(s)R^2(t,s)\right\}ds + \sum_{j=1}^{n} \Psi(t,t_j)d_j\right] \\
&\qquad \leq \left[v(T) + \int_{t_0}^T a_2(s)ds + \sum_{j=1}^{m-1} d_j\right] < \infty.
\end{aligned}$$

This contradiction completes the proof.

Theorem 3.2. *Let Ψ and ϕ be functions as defined in Theorem (3.1). Further, we assume that there is a constant $\rho > 0$ such that*

$$\inf_{s>t_0}\left[\liminf_{t\to\infty}\frac{\Psi(t,s)}{\Psi(t,t_0)}\right] > \rho, \tag{3.3}$$

and

$$\limsup_{t\to\infty}\frac{1}{\Psi(t,t_0)}\int_T^t a_1(s)R^2(t,s)ds < \infty.$$

Then every non-zero solution of (1.1) oscillates if there exists a positive continuous function F on $[t_0,\infty)$ with

$$\int_{t_0}^{\infty} F^2(\sigma)d\sigma = \infty, \tag{3.4}$$

and

$$\limsup_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_T^t\left\{\Psi(t,s)a_2(s)-\frac{1}{4}a_1(s)R^2(t,s)\right\}ds+\sum_{j=m}^{n}\Psi(t,t_j)d_j\right] \geq F(T),\ T\geq t_0. \tag{3.5}$$

Proof. Following the proof of previous theorem, we have

$$\limsup_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_T^t\left\{\Psi(t,s)a_2(s)-\frac{1}{4}a_1(s)R^2(t,s)\right\}ds+\sum_{j=m}^{n}\Psi(t,t_j)d_j\right]$$
$$\leq v(T)-\liminf_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_J a_1(s)\left\{\frac{1}{2}R(t,s)+\frac{\sqrt{\Psi(t,s)}}{\sqrt{a_1(s)}}v(s)\right\}^2 ds\right].$$

Consequently using the condition 3.5, we obtain

$$v(T)\geq F(T)+\liminf_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_J a_1(s)\left\{\frac{1}{2}R(t,s)+\frac{\sqrt{\Psi(t,s)}}{\sqrt{a_1(s)}}v(s)\right\}^2 ds\right]. \tag{3.6}$$

From the above inequality, it follows that

$$\liminf_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_J a_1(s)\left\{\frac{1}{2}R(t,s)+\frac{\sqrt{\Psi(t,s)}}{\sqrt{a_1(s)}}v(s)\right\}^2 ds\right] < \infty.$$
$$\Longrightarrow \liminf_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_J\left\{R(t,s)\sqrt{\Psi(t,s)}\sqrt{a_1(s)}v(s)+\Psi(t,s)v^2(s)\right\}ds\right] < \infty.$$

If we assume that

$$\xi_1(t) = \frac{1}{\Psi(t,T)} \int_J R(t,s)\sqrt{\Psi(t,s)}\sqrt{a_1(s)}v(s)ds, \quad t > T,$$

and

$$\xi_2(t) = \frac{1}{\Psi(t,T)} \int_J \Psi(t,s)v^2(s)ds, \quad t > T,$$

then we get

$$\liminf_{t\to\infty}[\xi_1(t) + \xi_2(t)] < \infty. \tag{3.7}$$

For $t > T$, we have

$$\begin{aligned}
\xi_2(t) &= \frac{1}{\Psi(t,T)} \int_J \Psi(t,s)d\Big[\int_T^s v^2(\sigma)d\sigma\Big] \\
&= \frac{1}{\Psi(t,T)}\Big[\Psi(t,s)\int_T^s v^2(\sigma)d\sigma\Big]_T^{t_m^-} + \frac{1}{\Psi(t,T)}\Big[\Psi(t,s)\int_{t_n^+}^s v^2(\sigma)d\sigma\Big]_{t_n^+}^{t} \\
&\quad + \frac{1}{\Psi(t,T)} \sum_{j=m+1}^{n} \Big[\Psi(t,s)\int_{t_{j-1}^+}^s v^2(\sigma)d\sigma\Big]_{t_{j-1}^+}^{t_j^-} \\
&\quad + \frac{1}{\Psi(t,T)} \int_J \Big[\frac{\partial}{\partial s}\big(-\Psi(t,s)\big)\Big]\Big[\int_T^s v^2(\sigma)d\sigma\Big]ds \\
&= \frac{\Psi(t,t_m)}{\Psi(t,T)} \int_T^{t_m^-} v^2(\sigma)d\sigma + \frac{1}{\Psi(t,T)} \sum_{j=m+1}^{n} \Psi(t,s)\int_{t_{j-1}^+}^{t_j^-} v^2(\sigma)d\sigma \\
&\quad + \frac{1}{\Psi(t,T)} \int_J \Big[\frac{\partial}{\partial s}\big(-\Psi(t,s)\big)\Big]\Big[\int_T^s v^2(\sigma)d\sigma\Big]ds.
\end{aligned}$$

Since $\Psi(t,s)$ is decreasing in s, we have

$$\begin{aligned}
&\frac{\Psi(t,t_m)}{\Psi(t,T)} \int_T^{t_m^-} v^2(\sigma)d\sigma + \frac{1}{\Psi(t,T)} \sum_{j=m+1}^{n} \Psi(t,s)\int_{t_{j-1}^+}^{t_j^-} v^2(\sigma)d\sigma \\
&\qquad \geq \frac{\Psi(t,t_n)}{\Psi(t,T)} \int_T^{t_m^-} v^2(\sigma)d\sigma + \frac{\Psi(t,t_n)}{\Psi(t,T)} \sum_{j=m+1}^{n} \int_{t_{j-1}^+}^{t_j^-} v^2(\sigma)d\sigma \\
&\qquad = \frac{\Psi(t,t_n)}{\Psi(t,T)} \int_J v^2(\sigma)d\sigma.
\end{aligned}$$

Using condition (3.4), for an arbitrary $\mu > 0$, there exists $T_1 > T$ such that

$$\int_J v^2(\sigma)d\sigma \geq \frac{\mu}{\rho}, \quad t \geq T_1.$$

and consequently, we have for $t \geq T_1$,

$$\begin{aligned} \xi_2(t) &\geq \frac{\mu}{\rho}\frac{\Psi(t,t_n)}{\Psi(t,T)} + \frac{\mu}{\rho}\frac{1}{\Psi(t,T)}\int_{T_1}^{t}\frac{\partial}{\partial s}\big(-\Psi(t,s)\big)ds \\ &= \frac{\mu}{\rho}\frac{\Psi(t,t_n)}{\Psi(t,T)} + \frac{\mu}{\rho}\frac{\Psi(t,T_1)}{\Psi(t,T)} \geq 2\frac{\mu}{\rho}\frac{\Psi(t,t_n)}{\Psi(t,T)}. \end{aligned}$$

Using condition (3.3), we can choose $T_1' \geq T_1$ such that

$$\frac{\Psi(t,t_n)}{\Psi(t,t_0)} \geq \rho, \quad t \geq T_1'. \tag{3.8}$$

So we have,

$$\xi_2(t) \geq 2\mu, \quad t \geq T_1'.$$

Since $\mu > 0$ is arbitrary, we have

$$\lim_{t\to\infty} \xi_2(t) = \infty.$$

Using (3.7), there exists a sequence $\{t_\nu\}$ converging to ∞ and a constant M such that

$$\xi_1(t_\nu) + \xi_2(t_\nu) \leq M, \quad \nu = 1, 2, 3 \ldots \tag{3.9}$$

$\lim_{\nu\to\infty} \xi_2(t_\nu) = \infty$ implies that

$$\lim_{\nu\to\infty} \xi_1(t_\nu) = -\infty. \tag{3.10}$$

From (3.9), we have

$$1 + \frac{\xi_1(t_\nu)}{\xi_2(t_\nu)} \leq \frac{M}{\xi_2(t_\nu)} < \frac{1}{3},$$

for sufficiently large ν. Thus,

$$\frac{\xi_1(t_\nu)}{\xi_2(t_\nu)} < -\frac{2}{3}.$$

Equation (3.10) ensures that

$$\lim_{\nu\to\infty}\frac{\xi_1^2(t_\nu)}{\xi_2(t_\nu)}=\infty. \tag{3.11}$$

On the other hand, by using the Cauchy inequality, for any positive integer ν, we have

$$\begin{aligned}\xi_1^2(t_\nu) &= \frac{1}{\Psi^2(t_\nu,T)}\int_T^{t_\nu} a_1(s)R^2(t,s)\Psi(t_\nu,s)v^2(s)ds\\ &\le \left[\frac{1}{\Psi(t_\nu,T)}\int_T^{t_\nu} a_1(s)R^2(t_\nu,s)ds\right]\left[\frac{1}{\Psi(t_\nu,T)}\int_T^{t_\nu}\Psi(t_\nu,s)v^2(s)ds\right]\\ &\le \left[\frac{1}{\Psi(t_\nu,T)}\int_T^{t_\nu} a_1(s)R^2(t_\nu,s)ds\right]\xi_2(t_\nu),\end{aligned}$$

and consequently,

$$\frac{\xi_1^2(t_\nu)}{\xi_2(t_\nu)}\le\frac{1}{\Psi(t_\nu,T)}\int_T^{t_\nu} a_1(s)R^2(t_\nu,s)ds.$$

Using (3.8), we get

$$\frac{\xi_1^2(t_\nu)}{\xi_2(t_\nu)}\le\frac{1}{\rho}\left[\frac{1}{\Psi(t_\nu,t_0)}\int_T^{t_\nu} a_1(s)R^2(t_\nu,s)ds\right].$$

Equation (3.11) gives

$$\limsup_{t\to\infty}\frac{1}{\Psi(t,t_0)}\int_T^{t} a_1(s)R^2(t,s)ds=\infty,$$

which is a contradiction.

Theorem 3.3. *Assume that the following conditions hold:*

(C1) $a_2(t)z(t)$ *is continuous in* $[t_0,\infty)$*, and* $a_2(t)z(t)/f(z(t))\ge q(t)$*, where* $f(\mu z)\ge \mu f(z)$ $(\mu>0)$*,* $f'(z)>0$*, and* $q(t)$ *is continuous on* $[t_0,\infty)$ *with* $q(t)\ge 0$*;*

(C2) There exist real numbers b_j, b_j^**, and* c_j *such that* $b_j^*<\frac{z(t_j)}{z'(t_j)}<b_j$ *and* $c_j=-b_jd_j$*;*

(C3) $\displaystyle\lim_{t\to\infty}\int_{t_0}^{t}\prod_{t_0<t_j<s}c_j\exp\left(\int_{t_0}^{s}a_1(\nu)d\nu\right)q(s)ds=\infty.$

Then every solution $z(t)$ of (2.1) satisfies $\liminf_{t\to\infty} |z(t)| = 0$.

Proof. Let $z(t)$ be a solution of (2.1) and let on contrary that $\liminf_{t\to\infty} z(t) > 0$. So $z(t)$ is nonoscillatory. Without loss of generality, we may assume that $z(t) > 0$ on $[T, \infty)$. Following the idea of ([14], Lemma 2), we can easily find that $z'(t) > 0$ for all $t \geq T$. We employ a Riccati transformation as:

$$V(t) = \frac{z'(t)}{f(z(t))}. \tag{3.12}$$

Differentiating it, we get

$$\begin{aligned} V'(t) &= \frac{z''(t)f(z(t)) - \big(z'(t)\big)^2 f'(z(t))}{f^2(z(t))} \\ &= \frac{-a_1(t)z'(t) - a_2(t)z(t)}{f(z(t))} - f'(z(t))V^2(t) \\ &\leq -a_1(t)V(t) - q(t). \end{aligned} \tag{3.13}$$

Using the continuity of $z(t)$ and (C2), we have

$$\begin{aligned} \Delta V(t_j) &= \frac{\Delta z'(t_j)}{f(z(t_j))} \\ &= -\frac{d_j z(t_j)}{f(z(t_j))} \\ &\leq -b_j d_j V(t_j) = c_j V(t_j). \end{aligned} \tag{3.14}$$

Integrating (3.13) with (3.14), we obtain

$$V(t) \leq V(t_0) \prod_{t_0<t_j<t} c_j \exp\left(\int_{t_0}^{t} -a_1(s)ds\right) - \int_{t_0}^{t} \prod_{s<t_j<t} c_j \exp\left(\int_{s}^{t} -a_1(\nu)d\nu\right) q(s)ds,$$

or,

$$V(t) \leq \prod_{t_0<t_j<t} c_j \exp\left(\int_{t_0}^{t} -a_1(s)ds\right)\left[V(t_0) - \int_{t_0}^{t} \prod_{t_0<t_j<s} c_j \exp\left(\int_{t_0}^{s} a_1(\nu)d\nu\right) q(s)ds\right].$$

From the condition (C3), the above inequality is impossible. This contradiction establishes the result.

Remark. *If every solution $z(t)$ of (2.1) satisfies* $\liminf_{t\to\infty} |z(t)| = 0$, *then every solution $u(t)$ of (1.1) also satisfies* $\liminf_{t\to\infty} |u(t)| = 0$.

4. Application

In this section, we consider the following example to illustrate the main results:

Example 4.1. Consider the following system of fractional impulsive differential equations:

$$\begin{cases} \mathfrak{D}^{\frac{11}{5}}_{+,t}u(z,t) + \frac{1}{t}\mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t) + \frac{1}{t^2}\mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t) = 0, \quad t \neq t_j, \\ \Delta\mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t_j) + j^{-3}\mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t_j) = 0, j = 1,2,3,\ldots, \quad (z,t) \in (0,\frac{\pi}{2}) \times \mathbb{R}_+ = D. \end{cases} \tag{4.1}$$

Here $a_1(t) = \frac{1}{t}, a_2(t) = \frac{1}{t^2}$ and $d_j = j^{-3}$. We can easily see that

$$\int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right)ds = \int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{1}{\sigma}d\sigma\right)ds = \int_\tau^\infty \frac{t_0}{s}ds = \infty.$$

Furthermore, we assume that all other conditions of Lemma 2.3 and Lemma 2.4.

Let $\Psi(t,s) = (t-s)^2$, $\phi(t,s) = 2$, then

$$R(t,s) = 2\sqrt{s} + \frac{(t-s)}{\sqrt{s}},$$

and

$$\limsup_{t\to\infty}\left[\frac{1}{(t-t_0)^2}\int_{t_0}^t \left\{\frac{(t-s)^2}{s^2} - \frac{1}{4s}R^2(t,s)\right\}ds + \sum_{j=1}^n (t-t_j)^2 j^{-3}\right] = \infty.$$

Thus all the assumptions of Theorem 3.1 are satisfied. Therefore all non-zero solutions of 4.1 are oscillatory.

Example 4.2.

$$\begin{cases} \mathfrak{D}^{\frac{11}{5}}_{+,t}u(z,t) + \frac{1}{t^2}\mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t) + \frac{1}{t^3}\mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t) = 0, \quad t \neq t_j, \\ \Delta\mathfrak{D}^{\frac{6}{5}}_{+,t}u(z,t_j) + j^{-4}\mathfrak{D}^{\frac{1}{5}}_{+,t}u(z,t_j) = 0, j = 1,2,3,\ldots, \quad (z,t) \in (0,\frac{\pi}{2}) \times \mathbb{R}_+ = D. \end{cases} \tag{4.2}$$

Here $a_1(t) = \frac{1}{t^2}$, $a_2(t) = \frac{1}{t^3}$. Obviously,

$$\int_\tau^\infty \exp\left(-\int_{t_0}^s \frac{a_2(\sigma)}{a_1(\sigma)}d\sigma\right)ds = \infty.$$

Let $\Psi(t,s) = (t-s)^2$, $\phi(t,s) = 2$, then

$$R(t,s) = 2s + \frac{t-s}{s},$$

and

$$\int_{t_0}^{t} a_1(s)R^2(t,s)ds = \int_{t_0}^{t} \frac{1}{s^2}\left[2s + \frac{(t-s)^2}{s}\right]ds.$$

After some simplification, we have

$$\limsup_{t\to\infty} \frac{1}{(t-t_0)^2}\int_{t_0}^{t} a_1(s)R^2(t,s)ds = \frac{3}{t_0^3} < \infty.$$

Further, we take $F(\sigma) = \frac{5}{4\sigma^2} - 1$. We see that

$$\int_{t_0}^{\infty} F^2(\sigma)d\sigma = \int_{t_0}^{\infty}\left(\frac{5}{4\sigma^2} - 1\right)^2 d\sigma = \infty.$$

Here $t_j = j$, $d_j = j^{-4}$ and it can be easily verified

$$\limsup_{t\to\infty}\left[\frac{1}{\Psi(t,T)}\int_{T}^{t}\left[\Psi(t,s)a_2(s) - \frac{1}{4}a_1(s)R^2(t,s)\right]ds + \sum_{j=m}^{n}\Psi(t,t_j)d_j\right]$$
$$\geq \frac{5}{4T^2} - 1 = F(T).$$

Thus all the conditions of Theorem 3.2 are satisfied. Therefore every non-zero solution of (4.2) is oscillatory.

5. Conclusion

We have studied the fractional-order differential equation by transforming it into the second-order ordinary differential equation. The considered problem is investigated with impulsive effects. We have established some results based on Philos type oscillation criteria. Moreover, we proved the stability condition. Finally, examples are given to validate the results. In future, one can extend the above work to a higher order multi-term time-fractional system.

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References

[1] Althubiti, S., Aldawish, I., Awrejcewicz, J., Bazighifan, O., New oscillation results of even-order Emden-Fowler neutral differential equations, Symmetry, 13 (2021), 2177.

[2] Babitskii, V., Krupenin, V., Vibration in strongly nonlinear systems, Moskow: Nauka, 1985.

[3] Bainov, D. D., Mishev, D. P., Oscillation theory for neutral differential equations with delay, IOP Publishing Ltd, New York, 1991.

[4] Bayram, M., Adiguzel, H., Ogrekci, S., Oscillation of fractional order functional differential equations with nonlinear damping, Open Phys., 13 (2015), 377-382.

[5] Bazighifan, O., Al-Kandari, M., Al-Ghafri, K. S., Ghanim, F., Askar, S., Oros, G. I., Delay differential equations of fourth-order: oscillation and asymptotic properties of solutions, Symmetry, 13 (2021), 2015.

[6] Chen, D. X., Oscillatory behavior of a class of fractional differential equations with damping, U. P. B. Sci. Buli. Series A., 75 (1) (2013), 107-117.

[7] Chua, L., Yang, L., Cellular neural networks: applications, IEEE Trans Circuits Syst CAS, 35 (1988), 1273–90.

[8] El-Sayed, A. A., Baleanu, D., Agarwal, P., A novel Jacobi operational matrix for numerical solution of multi-term variable-order fractional differential equations, J. Taibah Univ. Sci., 14 (1) (2020), 963–974.

[9] Erbe, L. H., Kong, Q., Ruan, S., Kamenev type theorems for second order matrix differential systems, Proc. Am. Math. Soc., 117 (4) (1993), 957-962.

[10] Feng, L., Sun, Y., Han, Z., Philos-type oscillation criteria for impulsive fractional differential equations, J. Appl. Math. and Comp., 62 (2020), 361-376.

[11] Fu, X., Liu, X., Sivalogannathan, S., Oscillation criteria for impulsive parabolic systems, Applicable Analysis, 79 (2007), 239-255.

[12] Gaul, L., Klein, P., Kempfle, S., Damping description involving fractional operators, Mech. Syst. Signal Process, 5 (1991), 81-88.

[13] Hilfer, R., Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[14] Jiab, J., Chen, L., Li, L., Asymptotic behavior of solutions of second-order nonlinear impulsive differential equations, J. Math. Anal. Appl., 337 (2008), 458–463.

[15] Kilbas, A. A., Srivastava, H. M., Trujilo, J. J., Theory and applications of fractional differential equations, Elsevier Science, Amsterdam, 204, (2006).

[16] Lakshmikantham, V., Bainov, D. D., Simeonov, P. S., Theory of impulsive differential equations, World Scientific, Singapore, 1989.

[17] Leibniz, G. W., Letter from Hanover, Germany to G. F. A. L'Hopital, September 30, 1695, in Mathematische Schriften, 1849, reprinted 1962, Olms verlag, Hidesheim, Germany, 2 (1965), 301-302.

[18] Martin, B., TongXing, L. I., Kamenev-type criteria for nonlinear damped dynamic equations, Sci. China Math., 58 (7) (2015), 1445-1452.

[19] Metzler, F., Schick, W., Kilian, H. G., Nonnenmacher, T. F., Relaxation in filled polymers: A fractional calculus approach, J. Chem. Phys, 103 (1995), 7180-7186.

[20] Mofarreh, F., Almutairi, A., Bazighifan, O., Aiyashi, M. A., Vîlcu, A.-D., On the oscillation of solutions of differential equations with neutral term, Mathematics, 9 (2021), 2709.

[21] Nian Li, W, Sheng, W., Oscillation properties for solutions of a kind of partial fractional differential equations with damping term, J. Nonlinear Sci. Appl., 9 (2016), 1600-1608.

[22] Ortigueira, M. D., Fractional Calculus for Scientists and Engineers, Springer Netherlands, 2011.

[23] Podlubny, I., Fractional Differential Equations, Academic Press, San Diego, 1999.

[24] Popov, E., The dynamics of automatic control systems, Moskow: Gostehizdat, 1964.

[25] Prakash, P., Harikrishnan, S., Nieto, J. J., Kim, J. H., Oscillation of a time fractional differential equation, Electron. J. Qual. Theory Differ. Equ., 15 (2014), 1-10.

[26] Prakash, P., Harikrishnan, S., Oscillation of solutions of impulsive vector hyperbolic differential equations with delays, Applicable Analysis, 91 (3) (2012), 459-473.

[27] Qureshi, S., Yusuf, A., Shaikh, A. A., et al., Fractional modeling of blood ethanol concentration system with real data application, Chaos: Interdiscip J Nonlinear Sci., 29 (1) (2019), 013143.

[28] Raheem, A., Afreen, A., Khatoon, A., On oscillatory and asymptotic behavior of higher order neutral differential equations with impulsive conditions, Int. J. Appl. Comput. Math, 7 (2021), 155.

[29] Raheem, A., Maqbul, Md., Oscillation criteria for impulsive partial fractional differential equations, Comput. Math. with Appl., 73 (2017), 1781-1788.

[30] Raheem, A., Kumar, M., An approximate solution to a class of impulsive fractional differential equations in a reflexive Banach space, Int. J. Appl. Comput. Math., 111 (5) (2019), 1-16.

[31] Sabatier, J., Agrawal, O., Machado, J., (eds.), In advances in Fractional Calculus, theoretical developments and applications in Physics and Engineering, Springer Netherlands, 2007.

[32] Sadhasivam, V., Deepa, M., Oscillation criteria for fractional impulsive hybrid partial differential equations, Probl. Anal. Issues Anal., 8(26)(2) (2019), 73-91.

[33] Salahshour, S., Ahmadian A., et al., On analytical solutions of the fractional differential equation with uncertainty: application to the Basset problem, Entropy, 17 (2015), 885-902.

[34] Shah, K., Ali, A., Bushnaq, S., Hyers-Ulam stability analysis to implicit Cauchy problem of fractional differential equations with impulsive conditions, Math. Methods Appl. Sci., 41(17) (2018), 8329-8343.

[35] Shah, K., Khalil, H., Khan, R. A., Investigation of positive solution to a coupled system of impulsive boundary value problems for nonlinear fractional order differential equations, Chaos Solitons Fractals, 77 (2015), 240-246.

[36] Sugie, J., Ishihara, K., Philos-type oscillation criteria for linear differential equations with impulsive effects, J. Math. Anal Appl., 470 (2019), 911-930.

[37] Sugie, J., Interval criteria for oscillation of second order self adjoint impulsive differential equations, Proc. Am. Math. Soc., 148(3) (2020), 1095-1108.

[38] Wang, J., Meng, F., Oscillatory behavior of a fractional partial differential equation, J. Appl. Anal. Comput., 8(3) (2018), 1011-1020.

[39] Wen, K., Zeng, Y., Peng, H., Huang, L., Philos-type oscillation criteria for second-order linear impulsive differential equation with damping, Boundary Value Problems, 111 (2019), 1-16.

[40] Wong, J. S. W., On Kamenev-type oscillation theorems for second order differential equations with damping, J. Math. Anal. and Appl., 258 (2001), 244-257.

[41] Zheng, B., Feng, Q., Some new oscillation criteria for a class of nonlinear fractional differential equations with damping Term, J. Appl. Math., 2013 (2013), 1-11.