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AN EFFICIENT COMPUTATIONAL TECHNIQUE FOR SOLVING GENERALIZED TIME-FRACTIONAL BIOLOGICAL POPULATION MODEL

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Abstract: In this paper, we apply an efficient computational technique based on the coupling of the Sumudu transform method and the iterative method to solve the generalized time-fractional biological population model within the Caputo fractional derivative. This method is termed as Sumudu transform iterative method (STIM). The series form approximate analytical solutions are obtained in a closed form, having components of the converging behavior towards the exact solution. Furthermore, the outcomes of this investigation are illustrated graphically using the mathematical software Maple, and the solution graphs demonstrate that the approximate solution is closely related to the exact solution.

Keywords and Phrases: Sumudu transform, Generalized time-fractional biological population model, Iterative method, Caputo fractional derivative, Mittag-Leffler function, Fractional differential equations.

2020 Mathematics Subject Classification: 26A33, 33E12, 35R11, 44A20.

1. Introduction

Various physical phenomena can be explained effectively in the natural sciences and engineering by designing models based on the principle of fractional calculus. In last few years, the use of fractional differentiation for mathematical modeling of real-world physical problems such as earthquake modeling, traffic flow models with fractional derivatives, measurement of viscoelasticity material properties and so on has become widespread. Fractional partial differential equations have attained a lot of popularity because the fractional order system eventually converges to the equations of the integer order. The theory of fractional differential equations contributes to a more accurate and systematic translation of nature's reality.

In the last decade, many researchers have studied fractional partial differential equations (FPDEs) by different techniques such as the adomian decomposition method (ADM) [11], the homotopy perturbation method (HPM) [19, 24], the homotopy perturbation transform method (HPTM) [16], the homotopy analysis method (HAM) [1, 18], the q-homotopy analysis transform method (q-HATM) [3], the iterative Laplace transform method (ILTM) [4, 25], the Laplace decomposition method (LDM) [14], the modified generalized Taylor fractional series method (MGTFSM) [15] etc. There are several integral transforms available for solving PDEs, including Laplace, Sumudu, Fourier, Mellin, and Elzaki. The Laplace and Sumudu transforms are the most widely accepted.

Watugala proposed the Sumudu transform method (STM) [27] to solve engineering problems. Weerakoon [28] used this method to solve partial differential equations. The inverse formula of this transform was later discovered by Weerakoon [29]. Demiray *et al.* [10] used the Sumudu transform method (STM) to obtain exact solutions to fractional differential equations.

Recently, Wang and Liu developed the Sumudu transform iterative method (STIM) [26] by combining the Sumudu transform with an iterative technique to find approximate analytical solutions to time-fractional Cauchy reaction-diffusion equations. The Sumudu transform iterative method was used to successfully solve a variety of time and space fractional partial differential equations as well as their systems [17] and fractional Fokker-Planck equations [2]. The proposed technique is straightforward to implement and very efficient computationally.

2. Mathematical Formulation of the Problem

According to biological scientists the population of the species may be regulated through dispersal or emigration. The diffusion of a biological species within a region C is defined by three functions of position $\overrightarrow{x} = (x, y)$ and time t [13] namely population density $u(\overrightarrow{x}, t)$, diffusion velocity $v(\overrightarrow{x}, t)$, and the population supply, $g(\overrightarrow{x}, t)$. The population density $u(\overrightarrow{x}, t)$ indicates the number of individuals per unit volume, at a given position $\overrightarrow{x}$ and time t. At time t, its integral over any sub region D of region C yield the total population of D in that sub region. The entity $g(\overrightarrow{x}, t)$ represents the rate which individuals are added to the population at position $\overrightarrow{x}$ per unit volume via births and deaths. The diffusion velocity $v(\overrightarrow{x}, t)$ is the average velocity of all individuals who occupy position $\overrightarrow{x}$ at time t, and it depicts the population flow from point to point. For any regular sub region D of C and for all time t, the entities u, v and g must be compatible with the following population balance law

$$\frac{d^\alpha}{dt^\alpha}\int_D udV + \int_{\partial D} u\overrightarrow{v}.\hat{n}\ dA = \int_D gdV, \tag{1}$$

where $\hat{n}$ is the outward unit normal to the boundary ∂D of D. In equation (1), the derivative has been defined in the Caputo sense. It follows from equation (1), the rate of population change in D plus the rate at which individuals depart D over its boundary must equal the rate at which individuals are supplied directly to D. By assuming the aforementioned assumptions [20], we can see that

$$g = g(u), \qquad \overrightarrow{v} = -r(u)\nabla u, \tag{2}$$

where $r(u) > 0$ for $u > 0$, and ∇ is the Laplace operator.
For population density u, the following two-dimensional nonlinear degenerate parabolic partial differential equation may be derived as

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 \phi(u)}{\partial x^2} + \frac{\partial^2 \phi(u)}{\partial y^2} + g(u), \quad t \geq 0, x, y \in \mathbb{R}. \tag{3}$$

The model described above in equation (3) is referred to as the time fractional biological population model. For modeling of the population of animal, Gurney and Nisbet [12] employed $\phi(u)$ as a special case.
This model leads to equation (3) with $\phi(u) = u^2$, to the following equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + g(u), \quad t \geq 0, x, y \in \mathbb{R}. \tag{4}$$

with the given initial condition u(x,y,0). Various properties of equation (4) were investigated in [20], including Holder estimates and its solutions.

The following are three examples of constitutive equations for $g(u)$ that may be expressed as
(a) $g(u) = cu,\ c =$ constant, Malthusian Law [13].
(b) $g(u) = c_1 u - c_2 u^2,\ c_1, c_2 =$ positive constants, Verhulst Law [20].
(c) $g(u) = cu^\theta,\ (c > 0, 0 < \theta < 1)$, Porous media [5, 22].
Consider a more general expression for $g(u)$ as $g(u) = hu^a(1 - ru^b)$, which converts equation (4) to the generalized time fractional biological population model represented by

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^a(1 - ru^b),\ t \geq 0,\ x, y, h, a, r, b \in \mathbb{R},\ 0 < \alpha \leq 1, \tag{5}$$

with subject to the initial condition $u(x, y, 0) = f_0(x, y)$, where u denotes the population density and $\frac{\partial^\alpha u}{\partial t^\alpha}$ is the Caputo fractional derivative of order α. If $h = c, a = 1, r = 0$ and $h = c_1, a = b = 1, r = c_2/c_1$ where $c_1 > 0, c_2 > 0$, c are constants, then equation (5) leads to Malthusian law and Verhulst law, respectively.

The purpose of this study is to apply the Sumudu transform iterative method to solve the generalized time-fractional biological population model because it is more precise and computationally efficient than other existing methods.

3. Preliminaries and Basic Definitions

In this section, we provide some fundamental definitions, notations, and properties of fractional calculus using Sumudu transform theory, which will be used later in this paper.

Definition 1. *In Caputo's sense, the fractional derivative of a function $u(x, t)$ is defined as* [8]

$$\begin{aligned}\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} &= \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\xi)^{m-\alpha-1}u^{(m)}(x,\xi)d\xi,\ m-1<\alpha\le m\ ,\ m\in\mathbb{N},\\ &= I_t^{m-\alpha}D^m u(x,t). \qquad (6)\end{aligned}$$

Here $D^m \equiv \frac{d^m}{dt^m}$ and I_t^α stands for the Riemann-Liouville fractional integral operator of order $\alpha > 0$, defined as [21]

$$I_t^\alpha u(x,t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}u^{(m)}(x,\xi)d\xi,\ \xi>0. \qquad (7)$$

Definition 2. *The Sumudu transform over the set of functions $\{f(t)|\exists M, \rho_1 > 0, \rho_2 > 0$ such that $|f(t)| < Me^{|t|/\rho_j}$ if $t \in (-1)^j \times [0, \infty)\}$ by the following formula* [6, 27]

$$S[f(t)] = F(\omega) = \int_0^\infty e^{-t}f(\omega t)dt\ ,\ \omega\in(-\rho_1,\rho_2). \qquad (8)$$

Definition 3. *The Sumudu transform of Caputo fractional derivative is presented in following manner* [10, 26]

$$\begin{aligned}S[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}] = \omega^{-\alpha}S[u(x,t)] - \sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}u^{(k)}(x,0)\Big],\\ m-1<\alpha\le m\ ,\ m\in\mathbb{N}. \qquad (9)\end{aligned}$$

where $u^{(k)}(x, 0)$ is the k-order derivative of $u(x, t)$ with respect to t at $t = 0$.

Definition 4. *The Mittag-Leffler function, which is a generalization of exponential function is defined as* [21, 23]

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)}\ , \quad \alpha \in \mathbb{C}, Re(\alpha) > 0. \tag{10}$$

4. Basic Concept of Sumudu Transform Iterative Method

In order to illustrate the basic idea of this method [26], we consider the general fractional partial differential equation with the initial conditions of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + Ru(x,t) + Nu(x,t) = g(x,t),\ m-1 < \alpha \leq m,\ m \in \mathbb{N}, \tag{11}$$

$$u^{(k)}(x,0) = h_k(x), \quad k = 0,1,2,...,m-1, \tag{12}$$

where $\dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order α, $m-1 < \alpha \leq m$, $m \in \mathbb{N}$, defined by equation (6), R is a linear operator and may include other fractional derivatives of order less than α, N is a non-linear operator which may include other fractional derivatives of order less than α and g(x,t) is a known function.
Applying the Sumudu transform on both sides of equation (11), we have

$$S\Big[\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\Big] + S[Ru(x,t) + Nu(x,t)] = S[g(x,t)]. \tag{13}$$

By using the equation (9), we get

$$S[u(x,t)] = \omega^\alpha \sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + \omega^\alpha S[g(x,t)] - \omega^\alpha S[Ru(x,t) + Nu(x,t)]. \tag{14}$$

On taking inverse Sumudu transform on equation (14), we have

$$u(x,t) = S^{-1}\Big[\omega^\alpha\Big(\sum_{k=0}^{m-1}[\omega^{-\alpha+k}u^{(k)}(x,0) + S[g(x,t)]]\Big)\Big] - S^{-1}\Big[\omega^\alpha S[Ru(x,t) + Nu(x,t)]\Big]. \tag{15}$$

Further, we apply the iterative method introduced by Daftardar-Gejji and Jafari [7], which represents a solution u(x,t) in infinite series of components

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t). \tag{16}$$

As R is a linear operator, so we have

$$R\Big(\sum_{i=0}^{\infty} u_i(x,t)\Big) = \sum_{i=0}^{\infty} R[u_i(x,t)], \tag{17}$$

and the non-linear operator N is decomposed as follows

$$N\Big(\sum_{i=0}^{\infty} u_i(x,t)\Big) = N[u_0(x,t)] + \sum_{i=0}^{\infty}\Big[N\Big(\sum_{j=0}^{i} u_j(x,t)\Big) - N\Big(\sum_{j=0}^{i-1} u_j(x,t)\Big)\Big]. \tag{18}$$

Substituting the results given by equations from (16) to (18) in the equation (15), we get

$$\sum_{i=0}^{\infty} u_i(x,t) = S^{-1}\Big[\omega^{\alpha}\Big(\sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + S[g(x,t)]\Big)\Big]$$
$$- S^{-1}\Big[\omega^{\alpha} S\Big\{\sum_{i=0}^{\infty} R[u_i(x,t)] + N[u_0(x,t)] + \sum_{i=1}^{\infty}\Big[N\Big(\sum_{j=0}^{i} u_j(x,t)\Big) - N\Big(\sum_{j=0}^{i-1} u_j(x,t)\Big)\Big]\Big\}\Big]. \tag{19}$$

We have defined the recurrence formulae as

$$\left.\begin{aligned} u_0(x,t) &= S^{-1}\Big[\omega^{\alpha}\Big(\sum_{k=0}^{m-1} \omega^{-\alpha+k} u^{(k)}(x,0) + S[g(x,t)]\Big)\Big],\\ u_1(x,t) &= -S^{-1}\Big[\omega^{\alpha} S[R(u_0(x,t)) + N(u_0(x,t))]\Big],\\ u_{m+1}(x,t) &= -S^{-1}\Big[\omega^{\alpha} S\Big\{R[u_m(x,t)] - \Big[N\Big(\sum_{j=o}^{m} u_j(x,t)\Big) - N\Big(\sum_{j=0}^{m-1} u_j(x,t)\Big)\Big]\Big\}\Big],\ m \geq 1. \end{aligned}\right\} \tag{20}$$

Therefore, the approximate analytical solution of equations (11) and (12) in truncated series form is given by

$$u(x,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,t). \tag{21}$$

In general, the solutions in the above series converge quickly. The classical approach to convergence of this type of series has been presented by Bhalekar and Daftardar-Gejji [7] and Daftardar-Gejji and Jafari [9].

5. Solution of Generalized Time Fractional Biological Population Model

In this section, we employ the Sumudu transform iterative approach to solve generalized time fractional biological population models.

Example 1. Consider the following generalized time-fractional biological population model [11]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u(1 - ru), \quad t \geq 0, r \in \mathbb{R}, 0 < \alpha \leq 1, \tag{22}$$

with the initial condition

$$u(x, y, 0) = exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\Big]. \tag{23}$$

Taking the Sumudu transform on the both sides of equation (22), and making use of the result given by equation (23), we have

$$S[u(x, y, t)] = exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\Big] + \omega^\alpha S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u(1 - ru)\Big]. \tag{24}$$

Operating with the inverse Sumudu transform on both sides of equation (24) gives

$$u(x, y, t) = exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\Big] + S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u(1 - ru)\Big]\Big]. \tag{25}$$

Substituting the results from equations (16) to (18) in the equation (25) and applying the equation (20), we determine the components of the STIM solution as follows

$$u_0(x, y, t) = u(x, y, 0) = exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\Big], \tag{26}$$

$$\begin{aligned} u_1(x, y, t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + u_0(1 - ru_0)\Big]\Big] \\ &= \frac{t^\alpha}{\Gamma(\alpha + 1)} exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x + y)\Big], \end{aligned} \tag{27}$$

$$\begin{aligned} u_2(x, y, t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0 + u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0 + u_1)^2}{\partial y^2} + (u_0 + u_1)(1 - r(u_0 + u_1))\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + u_0(1 - ru_0)\Big]\Big] \end{aligned}$$

$$= \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big], \tag{28}$$

$$\begin{aligned} u_3(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2(u_0+u_1+u_2)^2}{\partial x^2} + \frac{\partial^2(u_0+u_1+u_2)^2}{\partial y^2} \\ &+ (u_0+u_1+u_2)(1-r(u_0+u_1+u_2))\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2(u_0+u_1)^2}{\partial x^2} + \frac{\partial^2(u_0+u_1)^2}{\partial y^2} + (u_0+u_1)(1-r(u_0+u_1))\Big]\Big] \\ &= \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big], \end{aligned} \tag{29}$$

and so on. The other components can be found accordingly.
Thus, the approximate analytical solution in the series form can be obtained as

$$u(x,y,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+, ...,$$

$$\begin{aligned} u(x,y,t) &= exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big] + \frac{t^{\alpha}}{\Gamma(\alpha+1)} exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big] \\ &+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big] + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big] +, ..., \\ &= exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big] \sum_{n=0}^{\infty} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ &= exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y)\Big] E_\alpha(t^\alpha). \end{aligned} \tag{30}$$

which is the same result was obtained by El-Sayed *et al*. [11] using ADM.

Remark 1. *For* $\alpha = 1$ *the result in equation (30) reduces to the following form*

$$u(x,y,t) = exp\Big[\frac{1}{2}\sqrt{\frac{r}{2}}(x+y) + t\Big]. \tag{31}$$

This exact solution for the standard form of the biological population model was obtained earlier by Roul [24] using the HPM method.

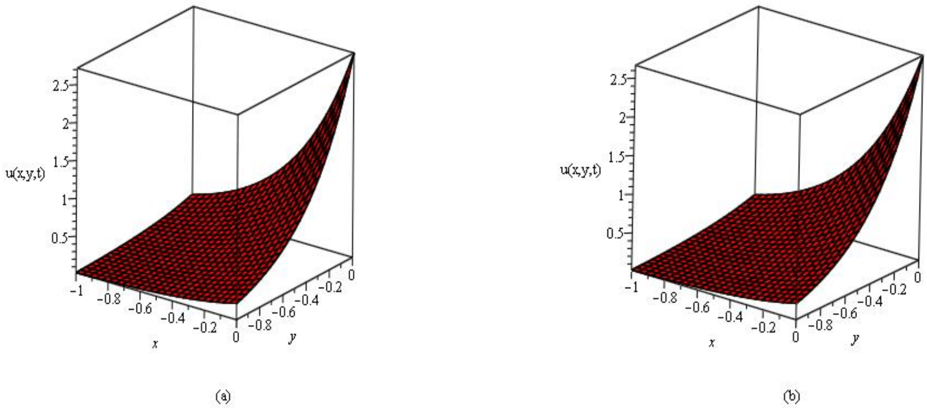


Figure 1. Solution graph for Example 1: (a) Exact Solution (b) approximate solution at $\alpha = 1,\ r = 48,\ t = 1$

Example 2. Consider the following generalized time-fractional biological population model [11]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu, \quad t \geq 0, h \in \mathbb{R}, 0 < \alpha \leq 1, \tag{32}$$

with the initial condition

$$u(x, y, 0) = \sqrt{xy}. \tag{33}$$

Taking the Sumudu transform on the both sides of equation (32), and making use of the result given by equation (33), we have

$$S[u(x, y, t)] = \sqrt{xy} + \omega^\alpha S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu\Big]. \tag{34}$$

Operating with the inverse Sumudu transform on both sides of equation (34) gives

$$u(x, y, t) = \sqrt{xy} + S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu\Big]\Big]. \tag{35}$$

Substituting the results from equations (16) to (18) in the equation (35) and applying the equation (20), we determine the components of the STIM solution as follows

$$u_0(x,y,t) = u(x,y,0) = \sqrt{xy} \quad , \tag{36}$$

$$\begin{aligned} u_1(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + hu_0\Big]\Big] \\ &= \sqrt{xy}\frac{ht^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{37}$$

$$\begin{aligned} u_2(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1)^2}{\partial y^2} + h(u_0+u_1)\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + hu_0\Big]\Big] \\ &= \sqrt{xy}\frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{38}$$

$$\begin{aligned} u_3(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1+u_2)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1+u_2)^2}{\partial y^2} + h(u_0+u_1+u_2)\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1)^2}{\partial y^2} + h(u_0+u_1)\Big]\Big] \\ &= \sqrt{xy}\frac{h^3t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{39}$$

and so on. The other components can be found accordingly.
Thus, the approximate analytical solution in the series form can be obtained as

$$u(x,y,t) \cong \lim_{\mathbb{N}\to\infty}\sum_{m=0}^{\mathbb{N}} u_m(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+,...,$$

$$\begin{aligned} u(x,y,t) &= \sqrt{xy} + \frac{ht^\alpha}{\Gamma(\alpha+1)}\sqrt{xy} + \frac{h^2t^{2\alpha}}{\Gamma(2\alpha+1)}\sqrt{xy} + \frac{h^3t^{3\alpha}}{\Gamma(3\alpha+1)}\sqrt{xy}+,..., \\ &= \sqrt{xy}\sum_{n=0}^{\infty}\frac{h^nt^{n\alpha}}{\Gamma(n\alpha+1)} \\ &= \sqrt{xy}\ E_\alpha(ht^\alpha). \end{aligned} \tag{40}$$

which is the same result was obtained by Sharma and Bairwa [25] using ILTM.

Remark 2. *For $\alpha = 1$ the result in equation (40) reduces to the following form*

$$u(x,y,t) = \sqrt{xy}\ e^{ht}. \tag{41}$$

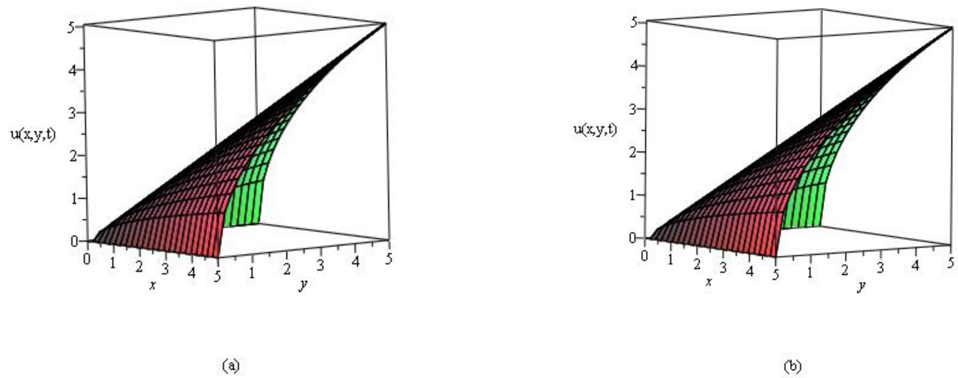


Figure 2. Solution graph for Example 2: (a) Exact Solution (b) approximate solution at $\alpha = 1,\ h = 0.01,\ t = 1$

This exact solution for the standard form of the biological population model was obtained earlier by Roul [24] using the HPM method.

Example 3. Consider the following generalized time-fractional biological population model [11]

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u, \quad t \geq 0, 0 < \alpha \leq 1, \tag{42}$$

with the initial condition

$$u(x, y, 0) = \sqrt{\sin x \sinh y}. \tag{43}$$

Taking the Sumudu transform on the both sides of equation (42), and making use of the result given by equation (43), we have

$$S[u(x, y, t)] = \sqrt{\sin x \sinh y} + \omega^\alpha S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u\Big]. \tag{44}$$

Operating with the inverse Sumudu transform on both sides of equation (44) gives

$$u(x, y, t) = \sqrt{\sin x \sinh y} + S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + u\Big]\Big]. \tag{45}$$

Substituting the results from equations (16) to (18) in the equation (45) and applying the equation (20), we determine the components of the STIM solution as follows

$$u_0(x,y,t) = u(x,y,0) = \sqrt{\sin x \sinh y}\ , \tag{46}$$

$$\begin{aligned} u_1(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + u_0\Big]\Big] \\ &= \sqrt{\sin x \sinh y}\ \frac{t^\alpha}{\Gamma(\alpha+1)}, \end{aligned} \tag{47}$$

$$\begin{aligned} u_2(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1)^2}{\partial y^2} + (u_0+u_1)\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + u_0\Big]\Big] \\ &= \sqrt{\sin x \sinh y}\ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \end{aligned} \tag{48}$$

$$\begin{aligned} u_3(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1+u_2)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1+u_2)^2}{\partial y^2} + (u_0+u_1+u_2)\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1)^2}{\partial y^2} + (u_0+u_1)\Big]\Big] \\ &= \sqrt{\sin x \sinh y}\ \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \end{aligned} \tag{49}$$

and so on. The other components can be found accordingly.
Thus, the approximate analytical solution in the series form can be obtained as

$$u(x,y,t) \cong \lim_{\mathbb{N}\to\infty} \sum_{m=0}^{\mathbb{N}} u_m(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+,...,$$

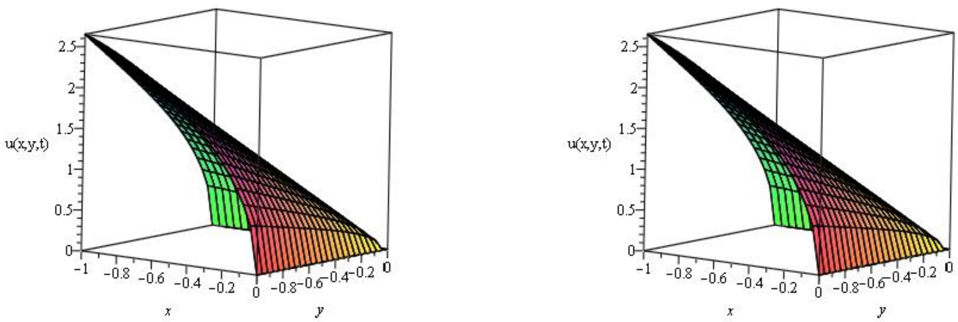
$$\begin{aligned} u(x,y,t) &= \sqrt{\sin x \sinh y} + \frac{t^\alpha}{\Gamma(\alpha+1)}\sqrt{\sin x \sinh y} \\ &+ \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\sqrt{\sin x \sinh y} + \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\sqrt{\sin x \sinh y}+,..., \\ &= \sqrt{\sin x \sinh y}\sum_{n=0}^{\infty}\frac{t^{n\alpha}}{\Gamma(n\alpha+1)} \\ &= \sqrt{\sin x \sinh y}\ E_\alpha(t^\alpha). \end{aligned} \tag{50}$$

which is the same result was obtained by Sharma and Bairwa [25] using ILTM.

Remark 3. *For $\alpha = 1$ the result in equation (50) reduces to the following form*

$$u(x, y, t) = \sqrt{\sin x \sinh y}\; e^{t}. \tag{51}$$

This exact solution for the standard form of the biological population model was obtained earlier by Roul [24] using the method of HPM.



(b) (b)

Figure 3. Solution graph for Example 3: (a) Exact Solution (b) approximate solution at $\alpha = 1,\ t = 1$

Example 4. Consider the following generalized time-fractional biological population model [11]

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{-1}(1 - ru), \quad t \geq 0, h, r \in \mathbb{R}, 0 < \alpha \leq 1, \tag{52}$$

with the initial condition

$$u(x, y, 0) = \Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{1/2}. \tag{53}$$

Taking the Sumudu transform on the both sides of equation (52), and making use of the result given by equation (53), we have

$$S[u(x, y, t)] = \Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{1/2} + \omega^{\alpha} S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{-1}(1 - ru)\Big]. \tag{54}$$

Operating with the inverse Sumudu transform on both sides of equation (54) gives

$$u(x, y, t) = \Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{1/2} + S^{-1}\Big[\omega^{\alpha} S\Big[\frac{\partial^2 u^2}{\partial x^2} + \frac{\partial^2 u^2}{\partial y^2} + hu^{-1}(1 - ru)\Big]\Big]. \tag{55}$$

Substituting the results from equations (16) to (18) in the equation (55) and applying the equation (20), we determine the components of the STIM solution as follows

$$u_0(x,y,t) = u(x,y,0) = \Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{1/2}, \tag{56}$$

$$\begin{aligned} u_1(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + hu_0^{-1}(1-ru_0)\Big]\Big] \\ &= h\frac{t^\alpha}{\Gamma(\alpha+1)}\Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{1/2}, \end{aligned} \tag{57}$$

$$\begin{aligned} u_2(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1)^2}{\partial y^2} + h(u_0+u_1)^{-1}(1-r(u_0+u_1))\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 u_0^2}{\partial x^2} + \frac{\partial^2 u_0^2}{\partial y^2} + hu_0^{-1}(1-ru_0)\Big]\Big] \\ &= -2h^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{-3/2}, \end{aligned} \tag{58}$$

$$\begin{aligned} u_3(x,y,t) &= S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1+u_2)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1+u_2)^2}{\partial y^2} \\ &+ h(u_0+u_1+u_2)^{-1}(1-r(u_0+u_1+u_2))\Big]\Big] \\ &- S^{-1}\Big[\omega^\alpha S\Big[\frac{\partial^2 (u_0+u_1)^2}{\partial x^2} + \frac{\partial^2 (u_0+u_1)^2}{\partial y^2} + h(u_0+u_1)^{-1}(1-r(u_0+u_1))\Big]\Big] \\ &= 3h^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{-5/2}, \end{aligned} \tag{59}$$

and so on. The other components can be found accordingly.
Thus, the approximate analytical solution in the series form can be obtained as

$$u(x,y,t) \cong \lim_{\mathbb{N}\to\infty}\sum_{m=0}^{\mathbb{N}} u_m(x,y,t) = u_0(x,y,t)+u_1(x,y,t)+u_2(x,y,t)+u_3(x,y,t)+,...,$$

$$\begin{aligned} u(x,y,t) &= \Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{1/2} + h\frac{t^\alpha}{\Gamma(\alpha+1)}\Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{-1/2} \\ &- 2h^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}\Big(\frac{hr}{4}x^2 + \frac{hr}{4}y^2 + y + 5\Big)^{-3/2} \end{aligned}$$

$$+\,3h^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\Big(\frac{hr}{4}x^2+\frac{hr}{4}y^2+y+5\Big)^{-5/2}+,...,$$
$$=u_0+\frac{ht^\alpha}{u_0}\sum_{n=0}^{\infty}\frac{(n+1)}{\Gamma\Big((n+1)\alpha+1\Big)}\Big[\frac{-ht^\alpha}{u_0^2}\Big]^n. \tag{60}$$

which is the same result was obtained by El-Sayed *et al.* [11] using ADM.

Remark 4. *For $\alpha = 1$ the result in equation (60) reduces to the following form*

$$u(x,y,t)=u_0+\frac{ht}{u_0}exp\Big(\frac{-ht}{u_0^2}\Big). \tag{61}$$

This exact solution for the standard form of the biological population model was obtained earlier by Liu *et al.* [19] using the HPM method.

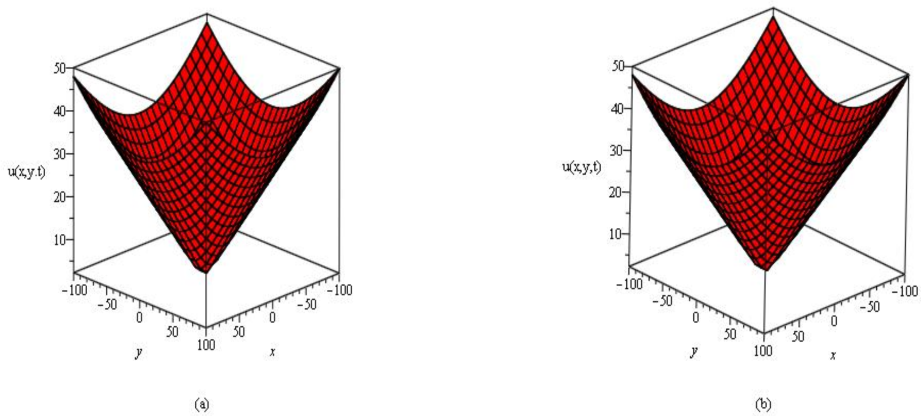


Figure 4. Solution graph for Example 4: (a) Exact Solution (b) approximate solution at $\alpha = 1,\ r = 48,\ h = 0.01,\ t = 1$

6. Conclusion

The Sumudu transform iterative method (STIM) is successfully used in this paper to solve generalized time-fractional biological population models. The fractional derivative is described in the Caputo sense and the solutions are obtained in closed form, in terms of Mittag-Leffler functions. The STIM solutions are highly compatible with the ADM, ILTM, and HPM solutions. The graphical representation of the obtained solutions was completed successfully by the Maple software. Furthermore, when compared to existing methods, the proposed approach is much easier to implement and requires fewer calculations. It can also be used to solve a variety of problems involving fractional-order derivatives.

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