*South East Asian J. of Mathematics and Mathematical Sciences*
*Vol. 18, No. 1 (2022), pp. 97-112*

**ISSN (Online): 2582-0850**

**ISSN (Print): 0972-7752**

# ON SOME REVERSES OF MINKOWSKI'S, HÖLDER'S AND HARDY'S TYPE INEQUALITIES USING $\psi$-FRACTIONAL INTEGRAL OPERATORS

**Bhagwat R. Yewale and Deepak B. Pachpatte**

Department of Mathematics,
Dr. B. A. M. University, Jaisingpura,
Aurangabad - 431004, Maharashtra, INDIA

E-mail : yewale.bhagwat@gmail.com, pachpatte@gmail.com

(**Received:** *Jul. 30, 2021* **Accepted:** *Mar. 10, 2022* **Published:** *Apr. 30, 2022)*

**Abstract:** In present paper, we establish new reverses of Minkowski, Hölder and Hardy type inequalities by using $\psi$-Riemann-Liouville fractional integral operator.

**Keywords and Phrases:** Minkowski inequality, Hölder inequality, Hardy inequality, $\psi$-Riemann-Liouville fractional integral, fractional integral operator.

**2020 Mathematics Subject Classification:** 26A33, 26D10.

## 1. Introduction

In 2006, Bougoffa [6] introduced the following reverse Minkowski integral inequality:

Let $\zeta$ and $\eta$ be positive functions defined on [c, d]. Then

$$\left(\int_c^d \zeta^p(t)dt\right)^{\frac{1}{p}} + \left(\int_c^d \eta^p(t)dt\right)^{\frac{1}{p}} \leq K\left(\int_c^d \left(\zeta(t)+\eta(t)\right)^p dt\right)^{\frac{1}{p}}, \tag{1.1}$$

where $0 < l \leq \dfrac{\zeta(t)}{\eta(t)} \leq Ł$, for all t $\in$ [c, d] and $K = \dfrac{Ł(l+2)+1}{(l+1)(Ł+1)}$.

Inequalities play an important role in mathematical analysis due to its wide applications in various branches of Mathematics. In recent years, many researchers have generalized and improved the above inequality (1.1) in a number of ways. For instance, in 2012, through his contribution to developing the theory of integral inequality, Sulaiman in [35] obtained some reverses of Minkowski's inequalities which provides a new bound for inequality (1.1), while Sroysang [33] and Benaissa [3] established some more generalized integral inequalities related to the reverse Minkowski's inequality. Similarly some researchers have generalized the Hardy and Hölder type inequalities. Pachpatte in [20] proved some variants of Hardy type inequalities involving the function of two variables. In [32, 34], the authors obtained some improvements in the Hardy-like inequalities. Pečarić [21] established an extension of Hölder type inequalities by using log convexity. Benaissa and Budak [4] obtained various results on integral inequalities of Hölder's type. Since inequalities have a great impact on substantial fields of research, many authors have paid a great deal of attention in obtaining various types of inequalities; for more details see [1, 2, 5, 8, 9, 15, 30, 37, 38].

Fractional calculus is important in mathematics due to its applications in diverse fields. It plays a very important role, especially in the study of various fractional differential equations and inequalities. Recently, many researchers have worked on the generalization of integral inequalities using different mathematical approaches. The use of fractional integral operators is one of the most popular and effective methods as it extends classical integral inequalities to fractional integral inequalities. In 2010, Dahmani [11] presented the fractional version of inequality (1.1) for Riemann-Liouville operator.

Sousa with Olivera [31] and Restrepo et al. [29] presented some reverse Minkowski's type inequalities by employing generalized Katugampola and weighted fractional operator respectively. Rahman et al. [22] established reverse Minkowski's type inequalities by considering generalized proportional fractional operator. In 2020, reverse Minkowski's type inequalities were studied by Rashid et al. [23] via generalized proportional fractional definition with respect to another function. In [12], authors established some Hardy type inequalities using generalized fractional integral operator while in [14], authors studied Hardy type inequalities for Hilfer and generalized fractional operators. Using different fractional operators, a wide range of inequalities have been obtained and analysed; see [7, 10, 13, 19, 24-28, 38] and the references therein. In [36], Wu et al. presented generalized Hardy-like inequalities and a fractional version of it was obtained by Khameli et al. [16] using Riemann-Liouville integral operators.

In 2020, Benaissa [3] established the following generalized Hardy type integral inequality:

(i) For $p \geq 1$, we have

$$p \int_c^d \frac{\left(\int_c^s \zeta(t)dt\right)^p}{\eta(s)} ds \leq (d-c)^p \int_c^d \frac{\zeta^p(s)}{\eta(s)} ds - \int_c^d \frac{(s-c)^p}{\eta(s)} \zeta^p(s) ds \tag{1.2}$$

(ii) For $0 < p < 1$,

$$p \int_c^d \frac{\left(\int_c^s \zeta(t)dt\right)^p}{\eta(s)} ds \geq \frac{(d-c)^p}{\eta(d)} \int_c^d \zeta^p(s) ds - \frac{1}{\eta(d)} \int_c^d (s-c)^p \zeta^p(s) ds, \tag{1.3}$$

where $\zeta, \eta$ are positive functions defined on [c, d] and $\eta$ is non decreasing.

Motivated by the above literature, in this paper we established the reverses of Minkowski's, Hölder's and Hardy's type integral inequalities proved in [3, 4] by employing $\psi$-fractional integral operator. The paper is organized as follows: In the next section 2, we will give some preliminaries, definitions and lemmas. In section 3, we established Hardy type integral inequalities by using $\psi$-fractional integral operator. Reverses of Minkowski and Hölder inequalities for $\psi$-fractional integral are given in section 4.

## 2. Preliminaries

In this section, we give some preliminaries, basic definitions of fractional integral operators and lemmas which are helpful in proving our main results.

**Definition 1.** [17] *The Riemann-Liouville fractional integral operator of the integrable function $\zeta$ on [c, d] of order $\beta > 0$ is defined as*

$$\mathfrak{J}_{c+}^{\beta} \zeta(t) = \frac{1}{\Gamma(\beta)} \int_c^t (t-s)^{\beta-1} \zeta(s) ds, \quad \text{for all } t > c,$$

*where $\Gamma$ is the Gamma function.*

**Definition 2.** [17] *The Hadamard fractional integral of the integrable function $\zeta$ on [c, d] of order $\beta > 0$ is defined as*

$${}^{H}\mathfrak{J}_{c+}^{\beta} \zeta(t) = \frac{1}{\Gamma(\beta)} \int_c^t \left(log \frac{t}{s}\right) \frac{\zeta(s)}{s} ds, \quad (c < t < d),$$

*where $\Gamma$ is the Gamma function.*

**Definition 3.** [17] *Let $\zeta$ be an integrable function defined on [c, d] and $\psi \in C^1[c,d]$ an increasing function such that $\psi'(t) \neq 0$, for all $t \in [c,d]$. Then $\psi$-Riemann-Liouville fractional integral of the function $\zeta$ with respect to the function $\psi$ of order $\beta > 0$ is defined by*

$$\mathfrak{J}_{c+}^{\beta,\psi} \zeta(t) = \frac{1}{\Gamma(\beta)} \int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1} \zeta(s) ds, \quad \text{for all } t > c,$$

*where* $\Gamma$ *is the Gamma function.*

**Lemma 2.1.** (Hölder inequality) [18, 36]. *Let* $\zeta$ *and* $\eta$ *be non-negative integrable functions defined on* $[c, d]$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *with* $p>1$, *then*

$$\int_c^d \zeta(s)\eta(s)ds \leq \Big(\int_c^d \zeta^p(s)ds\Big)^{\frac{1}{p}}\Big(\int_c^d \eta^q(s)ds\Big)^{\frac{1}{q}}. \tag{2.1}$$

**Lemma 2.2.** (Reverse Hölder inequality) [36]. *Let* $\zeta$ *and* $\eta$ *be positive integrable functions defined on* $[c, d]$ *and* $\frac{1}{p}+\frac{1}{q}=1$ *with* $0<p<1$, *then*

$$\int_c^d \zeta(s)\eta(s)ds \geq \Big(\int_c^d \zeta^p(s)ds\Big)^{\frac{1}{p}}\Big(\int_c^d \eta^q(s)ds\Big)^{\frac{1}{q}}. \tag{2.2}$$

## 3. Hardy Type Inequalities Using $\psi$-fractional Integral

In this section, we prove Hardy type inequalities using $\psi$-fractional integral. Our proofs based on the applications of the well known Fubini's theorem.

**Theorem 3.1.** *Let* $\beta>0$, $p>1$ *and* $\zeta$, $\eta$ *be two positive functions defined on* $[c,d]\subseteq[0,\infty)$ *such that* $\eta$ *is non-decreasing and* $\psi$ *is defined as in Definition 3, then following inequalities hold:*

$$\begin{aligned}\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \leq{}& \frac{1}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}\Big[(\psi(d)-\psi(c))^{\beta p-\beta+1}\mathfrak{I}_{c+}^{\beta,\psi}\frac{\zeta^p(d)}{\eta(d)}\\ &-\mathfrak{I}_{c+}^{\beta,\psi}\Big(\frac{\zeta^p(d)}{\eta(d)}(\psi(d)-\psi(c))^{\beta p-\beta+1}\Big)\Big].\end{aligned} \tag{3.1}$$

**Proof.** For $p>1$, we have

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt = \int_c^d \eta^{-1}(t)\Big(\frac{1}{\Gamma(\beta)}\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta(s)ds\Big)^p dt. \tag{3.2}$$

By using Hölder inequality (2.1), we get

$$\begin{aligned}\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \leq{}& \int_c^d \eta^{-1}(t)\Big[\Big(\frac{1}{\Gamma(\beta)}\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta^p(s)ds\Big)^{\frac{1}{p}}\\ &\Big(\frac{1}{\Gamma(\beta)}\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\Big)^{\frac{p-1}{p}}\Big]^p dt\\ ={}& \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta)}\int_c^d \eta^{-1}(t)\Big(\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta^p(s)ds\Big)\\ &\Big(\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\Big)^{p-1}dt\end{aligned}$$

$$=\frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_c^d \eta^{-1}(t)(\psi(t)-\psi(c))^{\beta(p-1)}$$
$$\Big(\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta^p(s)ds\Big)dt.$$

Since $\eta$ is non decreasing and by changing the order of integration, we obtain

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \leq \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_c^d \eta^{-1}(s)\psi'(s)(\psi(d)-\psi(s))^{\beta-1}\zeta^p(s)$$
$$\Big(\int_s^d (\psi(t)-\psi(c))^{\beta(p-1)}dt\Big)ds.$$

It follows that

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \leq \frac{1}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)\Gamma(\beta)}\Big[\int_c^d \eta^{-1}(s)\psi'(s)(\psi(d)-\psi(s))^{\beta-1}$$
$$(\psi(d)-\psi(c))^{\beta p-\beta+1}\zeta^p(s)ds - \int_c^d \eta^{-1}(t)\psi'(s)(\psi(d)-\psi(s))^{\beta-1}$$
$$(\psi(s)-\psi(c))^{\beta p-\beta+1}\zeta^p(s)ds\Big].$$

From above we get the required inequality (3.1).

**Remark 3.1.** *For $\psi(s)=s$ and $\beta=1$, the inequality (3.1), reduces to the inequality (2.5) of Theorem(2.2) in* [3].

**Theorem 3.2.** *Let $\beta>0$, $0<p<1$ and $\zeta$, $\eta$ be two positive functions defined on $[c,d]\subseteq[0,\infty)$ such that $\eta$ is non-decreasing and $\psi$ is defined as in Definition 3, then following inequalities hold:*

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \frac{\eta^{-1}(d)}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}\Big[\frac{(-1)^{\beta p-\beta+1}\Gamma(\beta p+1)}{\Gamma(\beta)}\mathfrak{I}_d^{\beta p+1,\psi}\zeta^p(c)$$
$$-(\psi(d)-\psi(c))^{\beta p-\beta+1}\mathfrak{I}_d^{\beta,\psi}\zeta^p(c)\Big]. \quad (3.3)$$

**Proof.** For $0<p<1$, we have

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt = \int_c^d \eta^{-1}(t)\Big(\frac{1}{\Gamma(\beta)}\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta(s)ds\Big)^p dt. \quad (3.4)$$

By using reverse Hölder inequality (2.2), we get

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \int_c^d \eta^{-1}(t)\Big[\Big(\frac{1}{\Gamma(\beta)}\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta^p(s)ds\Big)^{\frac{1}{p}}$$
$$\Big(\frac{1}{\Gamma(\beta)}\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\Big)^{\frac{p-1}{p}}\Big]^p dt$$
$$= \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta)}\int_c^d \eta^{-1}(t)\Big(\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta^p(s)ds\Big)$$
$$\Big(\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}ds\Big)^{p-1}dt$$
$$=\frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_c^d \eta^{-1}(t)(\psi(t)-\psi(c))^{\beta(p-1)}$$
$$\Big(\int_c^t \psi'(s)(\psi(t)-\psi(s))^{\beta-1}\zeta^p(s)ds\Big)dt.$$

Since $\eta$ is non decreasing and by changing the order of integration, we have

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_c^d \eta^{-1}(d)\psi'(s)(\psi(c)-\psi(s))^{\beta-1}\zeta^p(s)$$
$$\Big(\int_s^d (\psi(t)-\psi(c))^{\beta(p-1)}dt\Big)ds.$$

Therefore

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \frac{1}{\Gamma(\beta)\Gamma^{p-1}(\beta+1)}\int_d^c \eta^{-1}(d)\psi'(s)(\psi(c)-\psi(s))^{\beta-1}\zeta^p(s)$$
$$\Big(\int_d^s (\psi(t)-\psi(c))^{\beta(p-1)}dt\Big)ds.$$

From above we get

$$\int_c^d \frac{[\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \frac{1}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)\Gamma(\beta)}\Big[\int_d^c \eta^{-1}(d)\psi'(s)(\psi(c)-\psi(s))^{\beta-1}$$
$$(\psi(s)-\psi(c))^{\beta p-\beta+1}\zeta^p(s)ds - \int_d^c \eta^{-1}(d)\psi'(s)(\psi(c)-\psi(s))^{\beta-1}$$
$$(\psi(d)-\psi(c))^{\beta p-\beta+1}\zeta^p(s)ds\Big].$$

It follows that

$$\int_c^d \frac{[\mathfrak{J}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \frac{\eta^{-1}(d)}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}\Bigg[\frac{1}{\Gamma(\beta)}\int_d^c \psi'(s)(\psi(c)-\psi(s))^{\beta-1}$$
$$(\psi(s)-\psi(c))^{\beta p-\beta+1}\zeta^p(s)ds - \frac{1}{\Gamma(\beta)}(\psi(d)-\psi(c))^{\beta p-\beta+1}$$
$$\int_d^c \psi'(s)(\psi(c)-\psi(s))^{\beta-1}\zeta^p(s)ds\Bigg].$$

Therefore

$$\int_c^d \frac{[\mathfrak{J}_{c+}^{\beta,\psi}\zeta(t)]^p}{\eta(t)}dt \geq \frac{\eta^{-1}(d)}{(\beta p-\beta+1)\Gamma^{p-1}(\beta+1)}$$
$$\Bigg[\frac{(-1)^{\beta p-\beta+1}}{\Gamma(\beta)}\int_d^c \psi'(s)(\psi(c)-\psi(s))^{\beta p}\zeta^p(s)ds$$
$$-\frac{1}{\Gamma(\beta)}(\psi(d)-\psi(c))^{\beta p-\beta+1}\int_d^c \psi'(s)(\psi(c)-\psi(s))^{\beta-1}\zeta^p(s)ds\Bigg].$$

From above we get the required inequality (3.3).

**Remark 3.2.** *For $\psi(s)=s$ and $\beta=1$, the inequality (3.3), reduces to the inequality (2.6) of Theorem(2.2) in* [3].

## 4. Reverse Minkowski and Hölder type Inequality using $\psi$-fractional Integral

In this section, we obtain some reverse Minkowski type and Hölder type inequalities for $\psi$-Riemann-Liouville fractional integral.

Our next result deals with reverses of Minkowski type inequality by using $\psi$-fractional integral operator.

**Theorem 4.1.** *Let $\beta>0$, $p\geq 1$. Let $\zeta$ and $\eta$ be two positive functions defined on $[c,t]$, for all $t>c\geq 0$ such that $\mathfrak{J}_{c+}^{\beta,\psi}\zeta^p(t)<\infty, \mathfrak{J}_{c+}^{\beta,\psi}\eta^p(t)<\infty$ and $\psi$ is defined as in Definition 3. If $0<k<l\leq \frac{\theta\zeta(s)}{\eta(s)}\leq Ł$, for $\theta>0$, $s\in[c,t]$, then following inequalities hold:*

$$\frac{Ł+\theta}{\theta(Ł-k)}\left[\mathfrak{J}_{c+}^{\beta,\psi}(\theta\zeta-k\eta)^p(t)\right]^{\frac{1}{p}} \leq \left[\mathfrak{J}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}}+\left[\mathfrak{J}_{c+}^{\beta}\eta^p(t)\right]^{\frac{1}{p}}$$
$$\leq \frac{l+\theta}{\theta(l-k)}\left[\mathfrak{J}_{c+}^{\beta,\psi}(\theta\zeta-k\eta)^p(t)\right]^{\frac{1}{p}}. \tag{4.1}$$

**Proof.** Since for $\theta>0, s\in[c,\ t], t>0$, we have

$$0<k<l\leq\frac{\theta\zeta(s)}{\eta(s)}\leq Ł, \tag{4.2}$$

then

$$-\frac{1}{l} \leq -\frac{\eta(s)}{\theta\zeta(s)} \leq -\frac{1}{Ł},$$

which implies

$$\frac{1}{k} - \frac{1}{l} \leq \frac{1}{k} - \frac{\eta(s)}{\theta\zeta(s)} \leq \frac{1}{k} - \frac{1}{Ł}.$$

Therefore

$$\frac{l-k}{kl} \leq \frac{\theta\zeta(s) - k\eta(s)}{k\theta\zeta(s)} \leq \frac{Ł-k}{kŁ}.$$

From above we get

$$\frac{Ł}{Ł-k} \leq \frac{\theta\zeta(s)}{\theta\zeta(s) - k\eta(s)} \leq \frac{l}{l-k}.$$

It follows that

$$\frac{Ł}{\theta(Ł-k)}\big(\theta\zeta(s) - k\eta(s)\big) \leq \zeta(s) \leq \frac{l}{\theta(l-k)}\big(\theta\zeta(s) - k\eta(s)\big). \tag{4.3}$$

Taking $p^{th}$ power of (4.3), we get

$$\left[\frac{Ł}{\theta(Ł-k)}\right]^p \big(\theta\zeta(s) - k\eta(s)\big)^p \leq \zeta^p(s) \leq \left[\frac{l}{\theta(l-k)}\right]^p \big(\theta\zeta(s) - k\eta(s)\big)^p. \tag{4.4}$$

Multiplying (4.4) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t) - \psi(s))^{\beta-1}$, $s \in (c,t)$ and integrating with respect to $s$ from $c$ to $t$, we get

$$\begin{aligned}\left[\frac{Ł}{\theta(Ł-k)}\right]\left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta - k\eta)^p(t)\right]^{\frac{1}{p}} \leq& \left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}} \\ \leq& \left[\frac{l}{\theta(l-k)}\right]\left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta - k\eta)^p(t)\right]^{\frac{1}{p}}.\end{aligned} \tag{4.5}$$

Now from the condition (4.2), we have

$$l - k \leq \frac{\theta\zeta(s) - k\eta(s)}{\eta(s)} \leq Ł - k.$$

Therefore

$$\frac{1}{Ł-k} \leq \frac{\eta(s)}{\theta\zeta(s) - k\eta(s)} \leq \frac{1}{l-k}.$$

It follows that

$$\frac{(\theta\zeta(s) - k\eta(s))^p}{(Ł-k)^p} \leq \eta^p(s) \leq \frac{(\theta\zeta(s) - k\eta(s))^p}{(l-k)^p}. \tag{4.6}$$

Multiplying by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s\in(c,t)$ to (4.6) and integrating with respect to $s$ from $c$ to $t$, we get

$$\begin{aligned}\frac{1}{Ł-k}\left[\mathfrak{J}_{c+}^{\beta,\psi}(\theta\zeta-k\eta)^p(t)\right]^{\frac{1}{p}} \leq & \left[\mathfrak{J}_{c+}^{\beta,\psi}\eta^p(t)\right]^{\frac{1}{p}} \\ \leq & \frac{1}{l-k}\left[\mathfrak{J}_{c+}^{\beta,\psi}(\theta\zeta-k\eta)^p(t)\right]^{\frac{1}{p}}.\end{aligned} \tag{4.7}$$

Adding inequalities (4.5) and (4.7), we get required inequality (4.1).

**Remark 4.1.** *For $\psi(s)=s$ and $\beta=1$, the inequality (4.1), reduces to the inequality (2.2) of Theorem(2.1) in* [3].

In next theorem we prove reverse Hölder type inequalities in the sense of $\psi$-fractional integral operator.

**Theorem 4.2.** *Let $\beta>0$, $p>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $m>0$, $n>0$, $t>c\geq 0$. Let $\zeta$ and $\eta$ be two positive integrable functions defined on $[c,\ t]$ and $\psi$ is defined as in Definition 3. Let $w$ be a weight function defined on $[c,\ t]$ . If $0<l\leq\frac{\zeta^m(s)}{\eta^n(s)}\leq Ł$, for $s\in[c,t]$, then following inequalities hold:*

$$\left(\mathfrak{J}_{c+}^{\beta,\psi}\zeta^m(t)w(t)\right)^{\frac{1}{p}}\left(\mathfrak{J}_{c+}^{\beta,\psi}\eta^n(t)w(t)\right)^{\frac{1}{q}} \leq \left(\frac{Ł}{l}\right)^{\frac{1}{pq}}\left(\mathfrak{J}_{c+}^{\beta,\psi}\zeta^{\frac{m}{p}}(t)\eta^{\frac{n}{q}}(t)w(t)\right). \tag{4.8}$$

**Proof.** Since for $m,n>0$ and $s\in[c,t]$, we have

$$0<l\leq\frac{\zeta^m(s)}{\eta^n(s)}\leq Ł. \tag{4.9}$$

Then

$$\left(\frac{1}{l}\right)^{\frac{1}{q}}\geq\frac{\eta^{\frac{n}{q}}(s)}{\zeta^{\frac{m}{q}}(s)}\geq\left(\frac{1}{Ł}\right)^{\frac{1}{q}}. \tag{4.10}$$

Multiplying (4.10) by $\zeta^m(s)$, we get

$$\left(\frac{1}{l}\right)^{\frac{1}{q}}\zeta^m(s)\geq\frac{\eta^{\frac{n}{q}}(s)\zeta^m(s)}{\zeta^{\frac{m}{q}}(s)}\geq\left(\frac{1}{Ł}\right)^{\frac{1}{q}}\zeta^m(s). \tag{4.11}$$

From (4.11), we have

$$\left(\frac{1}{l}\right)^{\frac{1}{q}}\zeta^m(s)\geq\eta^{\frac{n}{q}}(s)\zeta^{\frac{m}{p}}(s)\geq\left(\frac{1}{Ł}\right)^{\frac{1}{q}}\zeta^m(s).$$

Which gives

$$l^{\frac{1}{q}}\zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s) \leq \zeta^m(s) \leq Ł^{\frac{1}{q}}\zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s). \tag{4.12}$$

Multiplying right hand side of (4.12) by *w(s)*, we obtain

$$\zeta^m(s)w(s) \leq Ł^{\frac{1}{q}}\zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s)w(s). \tag{4.13}$$

Multiplying (4.13) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (c,t)$ and integrating with respect to $s$ from $c$ to $t$, we get

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^m(t)w(t)\right]^{\frac{1}{p}} \leq Ł^{\frac{1}{pq}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^{\frac{m}{p}}(t)\eta^{\frac{n}{q}}(t)w(t)\right]^{\frac{1}{p}}. \tag{4.14}$$

Now from (4.9), we have

$$l^{\frac{1}{p}} \leq \frac{\zeta^{\frac{m}{p}}(s)}{\eta^{\frac{n}{p}}(s)} \leq Ł^{\frac{1}{p}}. \tag{4.15}$$

Multiplying (4.15) by $\eta^n(s)$, we get

$$l^{\frac{1}{p}}\eta^n(s) \leq \frac{\zeta^{\frac{m}{p}}(s)\eta^n(s)}{\eta^{\frac{n}{p}}(s)} \leq Ł^{\frac{1}{p}}\eta^n(s).$$

Which gives

$$l^{\frac{1}{p}}\eta^n \leq \zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s) \leq Ł^{\frac{1}{p}}\eta^n(s). \tag{4.16}$$

It follows that

$$\left(\frac{1}{Ł}\right)^{\frac{1}{p}}\zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s) \leq \eta^n(s) \leq \left(\frac{1}{l}\right)^{\frac{1}{p}}\zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s). \tag{4.17}$$

Multiplying right hand side inequality of (4.17) by *w(s)*, we obtain

$$\eta^n(s)w(s) \leq \left(\frac{1}{l}\right)^{\frac{1}{p}}\zeta^{\frac{m}{p}}(s)\eta^{\frac{n}{q}}(s)w(s). \tag{4.18}$$

Multiplying (4.18) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (c,t)$ and integrating with respect to $s$ from $c$ to $t$, we get

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^n(t)w(t)\right]^{\frac{1}{q}} \leq \left(\frac{1}{l}\right)^{\frac{1}{pq}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^{\frac{m}{p}}(t)\eta^{\frac{n}{q}}(t)w(t)\right]^{\frac{1}{q}}. \tag{4.19}$$

Multiplying (4.14) and (4.19), we get required inequality (4.8).

**Remark 4.2.** *For $\psi(s) = s$ and $\beta = 1$, the inequality (4.8), reduces to the inequality (6) of Theorem(2.1) in* [4].

**Corollary 4.1.** *Let $\beta > 0$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\zeta$ and $\eta$ be two positive integrable functions on $[c, t]$ and $\psi$ is defined as in Definition 3. If $0 < l \leq \frac{\zeta^{p-1}(s)}{\eta(s)} \leq Ł$, for $s \in [c, t]$, where $t > c \geq 0$, then following inequalities hold:*

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{Ł}{l}\right)^{\frac{1}{p}}\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)\eta(t).$$

**Corollary 4.2.** *Let $\beta > 0$, $p > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Let $\zeta$ and $\eta$ be two positive integrable functions on [c, t], for all $t > c \geq 0$, and $\psi$ is defined as in Definition 3. If $0 < l \leq \frac{\zeta(s)}{\eta^{q-1}(s)} \leq Ł$, for $s \in [c, t]$, then following inequalities hold:*

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{Ł}{l}\right)^{\frac{1}{q}}\mathfrak{I}_{c+}^{\beta,\psi}\zeta(t)\eta(t).$$

**Remark 4.3.** *For $\psi(s) = s$ and $\beta = 1$, the inequalities in corollary (4.1) and (4.2) reduces to the inequalities in corollary (2.4) and (2.5) in* [4] *respectively.*

**Theorem 4.3.** *Let $\beta > 0$, $\theta > 0, k > 0, p > 0, q > 0, \mu > 0, \nu > 0$. Let $\zeta$, $\eta$ be non-negative integrable functions defined on [c, t], for all $t > c \geq 0$ and $\psi$ is defined as in Definition 3. If $0 < k < l \leq \frac{\theta\zeta(s)}{\eta(s)} \leq Ł$, for all $s \in [c, t]$, where $t \geq c > 0$, then following inequalities hold:*

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{Ł}{\theta}\right)\left(\frac{\theta}{l}\right)^{\frac{2\mu}{\mu+\nu}}(l+k)^{\frac{\mu-\nu}{\mu+\nu}}(Ł+k)^{\frac{\nu-\mu}{\mu+\nu}}$$
$$\left[\mathfrak{I}_{c+}^{\beta,\psi}\left(\zeta^\mu(t)\eta^\nu(t)\right)^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\left(\zeta^\mu(t)\eta^\nu(t)\right)^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \quad (4.20)$$

**Proof.** Since for $\theta > 0$, $s \in [c, t]$, $t > c \geq 0$ we have

$$0 < k < l \leq \frac{\theta\zeta(s)}{\eta(s)} \leq Ł, \quad (4.21)$$

then

$$l + k \leq \frac{\theta\zeta(s) + k\eta(s)}{\eta(s)} \leq Ł + k. \quad (4.22)$$

From above we have

$$(l+k)^q \leq \Big(\frac{\theta\zeta(s) + k\eta(s)}{\eta(s)}\Big)^q \leq (Ł+k)^q. \quad (4.23)$$

Multiplying left hand side inequality of (4.23) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (c,t)$ and integrating with respect to $s$ from $c$ to $t$, we get

$$(l+k)\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^q(t)\right]^{\frac{1}{q}} \leq \left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^q(t)\right]^{\frac{1}{q}}. \tag{4.24}$$

Also, from (4.21) we have

$$\frac{Ł+k}{Ł} \leq \frac{\theta\zeta(s)+k\eta(s)}{\theta\zeta(s)} \leq \frac{l+k}{l}. \tag{4.25}$$

Multiplying the left hand side inequality of (4.25) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (c,t)$ and integrating with respect to $s$ from $c$ to $t$, we get

$$\theta\left(\frac{Ł+k}{Ł}\right)\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}} \leq \left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^p(t)\right]^{\frac{1}{p}}. \tag{4.26}$$

Multiplying inequalities (4.24) and (4.26), we obtain

$$\begin{aligned}&\left(\frac{\theta}{Ł}\right)(Ł+k)(l+k)\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^p(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^q(t)\right]^{\frac{1}{q}}\\ \leq\;& \left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^p(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{a+}^{\beta,\psi}(\theta\zeta+k\eta)^q(t)\right]^{\frac{1}{q}}.\end{aligned} \tag{4.27}$$

From the right hand side inequalities of (4.22) and (4.25), we get

$$(\theta\zeta(s)+k\eta(s))^{\nu} \leq (Ł+k)^{\nu}\eta^{\nu}(s) \tag{4.28}$$

and

$$(\theta\zeta(s)+k\eta(s))^{\mu} \leq \left(\frac{\theta}{l}(l+k)\right)^{\mu}\zeta^{\mu}(s). \tag{4.29}$$

Adding inequalities (4.28) and (4.29), we obtain

$$\theta\zeta(s)+k\eta(s) \leq \left(\frac{\theta}{l}\right)^{\frac{\mu}{\mu+\nu}}(l+k)^{\frac{\mu}{\mu+\nu}}(Ł+k)^{\frac{\nu}{\mu+\nu}}\left(\zeta^{\mu}(s)\eta^{\nu}(s)\right)^{\frac{1}{\mu+\nu}}. \tag{4.30}$$

Multiplying (4.30) by $\frac{1}{\Gamma(\beta)}\psi'(s)(\psi(t)-\psi(s))^{\beta-1}$, $s \in (c,t)$ and integrating with respect to $s$ from $c$ to $t$, we get

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^p(t)\right]^{\frac{1}{p}} \leq \left(\frac{\theta}{l}\right)^{\frac{\mu}{\mu+\nu}}(l+k)^{\frac{\mu}{\mu+\nu}}(Ł+k)^{\frac{\nu}{\mu+\nu}}\left[\mathfrak{I}_{c+}^{\beta,\psi}(\zeta^{\mu}(t)\eta^{\nu}(t))^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}}. \tag{4.31}$$

Similarly, from (4.30), we have

$$\left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^q(t)\right]^{\frac{1}{q}} \leq \left(\frac{\theta}{l}\right)^{\frac{\mu}{\mu+\nu}}(l+k)^{\frac{\mu}{\mu+\nu}}(Ł+k)^{\frac{\nu}{\mu+\nu}}\left[\mathfrak{I}_{c+}^{\beta,\psi}(\zeta^{\mu}(t)\eta^{\nu}(t))^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \tag{4.32}$$

Multiplying inequalities (4.31) and (4.32), we get

$$\begin{aligned}&\left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^{p}(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}(\theta\zeta+k\eta)^{q}(t)\right]^{\frac{1}{q}}\\ \leq\ &\Big(\frac{\theta}{l}\Big)^{\frac{2\mu}{\mu+\nu}}(l+k)^{\frac{2\mu}{\mu+\nu}}(Ł+k)^{\frac{2\nu}{\mu+\nu}}\left[\mathfrak{I}_{c+}^{\beta,\psi}(\zeta^{\mu}(t)\eta^{\nu}(t))^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}}\\ &\left[\mathfrak{I}_{c+}^{\beta,\psi}(\zeta^{\mu}(t)\eta^{\nu}(t))^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \qquad (4.33)\end{aligned}$$

From the inequalities (4.27) and (4.33), we obtain

$$\begin{aligned}&\frac{\theta}{Ł}(Ł+k)(l+k)\left[\mathfrak{I}_{c+}^{\beta,\psi}\zeta^{p}(t)\right]^{\frac{1}{p}}\left[\mathfrak{I}_{c+}^{\beta,\psi}\eta^{q}(t)\right]^{\frac{1}{q}}\\ \leq\ &\Big(\frac{\theta}{l}\Big)^{\frac{2\mu}{\mu+\nu}}(l+k)^{\frac{2\mu}{\mu+\nu}}(Ł+k)^{\frac{2\nu}{\mu+\nu}}\left[\mathfrak{I}_{c+}^{\beta,\psi}(\zeta^{\mu}(t)\eta^{\nu}(t))^{\frac{p}{\mu+\nu}}\right]^{\frac{1}{p}}\\ &\left[\mathfrak{I}_{c+}^{\beta,\psi}(\zeta^{\mu}(t)\eta^{\nu}(t))^{\frac{q}{\mu+\nu}}\right]^{\frac{1}{q}}. \qquad (4.34)\end{aligned}$$

From inequality (4.34), we get required inequality (4.20).

**Remark 4.4.** *For $\psi(s) = s$ and $\beta = 1$, the inequality (4.20), reduces to the inequality (13) of Theorem(2.6) in* [4].

## 5. Conclusion

In this paper we obtained reverses of Hardy's, Minkowski's and Hölder's type inequalities using the $\psi$-Riemann-Liouville fractional integral. The obtained results are more generalized in nature. If we put different values of $\psi$, the $\psi$-Riemann-Liouville fractional integral operator are reduced to the many classical results as Riemann-Liouville, Hadamard and Erdélyi–Kober fractional integral operator for $\psi(s) = s$, $\psi(s) = ln\ s$ and $\psi(s) = s^{\sigma}$, respectively.

## References

[1] Agarwal, P., Cortez M. V., Oliveros, Y. R. and Ali, M. A., New Ostrowski type inequalities for generalized s-convex functions with applications to some special means of real numbers and to midpoint formula, AIMS Math., 7(1) (2021), 1429-1444.

[2] Ali, M. A., Abbas, M., Budak H., Agarwal, P., Murtaza, G. and Chu YM., New quantum boundaries for quantum Simpson's and quantum Newton's type inequalities for preinvex functions, Adv. Differ. Equ., 64 (2021).

[3] Benaissa, B., More on reverses of Minkowski's inequalities and Hardy's integral inequalities, Asian Eur. J. Math., 13(3) (2020), 1-7.

[4] Benaissa, B. and Budak, H., More on reverses of Hölder integral inequality, Korean J. Math., 28(1) (2020), 9-15.

[5] Benaissa, B., On the reverse of Minkowski's integral inequality, Kragujevac J. Math., 46(3) (2020), 407-416.

[6] Bougoffa, L., On Minkowski and Hardy integral inequalities, J. Inequal. Pure and Appl. Math., 7(2) (2006).

[7] Butt, S. I., Umar, M., Rashid, S., Akdemir A. O. and Chu, Y.-M., New Hermite–Jensen–Mercer-type inequalities via k-fractional integrals, Adv. Differ. Equ., 635 (2020).

[8] Butt, S. I., Agarwal, P., Yousaf, S., Akdemir A. O. and Guirao, J. L. G., Generalized fractal Jensen and Jensen–Mercer inequalities for harmonic convex function with applications, J. Inequal. Appl., 1 (2022).

[9] Chu, Y.-M., Xu, Q. and Zhang, X.-M., A note on Hardy's inequality, J. Inequal. Appl., 271 (2014).

[10] Chu, Y.-M., Rashid, S., Jarad, F., Noor, M. A., and Kalsoom, H., More new results on integral inequalities for generalized $\mathcal{K}$-fractional conformable integral operators, Discrete Contin. Dyn. Syst. Ser. S, 14(7) (2021), 2119-2135.

[11] Dahmani, Z., On Minkowski and Hermite-Hadmard integral inequalities via fractional integration, Ann. Funct. Anal., 1(1) (2010), 51-58.

[12] Hernandez, J. E. H. and Cortez, M. J. V., On a Hardy's inequality for a fractional integral operator, An. Univ. Craiova Ser. Inform., 45(2) (2018), 232-242.

[13] Hussain, S., Khalid, J., and Chu, Y.-M., Some generalized fractional integral Simpson's type inequalities with applications, AIMS Math., 5(6) (2020), 5859–5883.

[14] Iqbal, S., Pečarić, J., Samraiz, M. and Tomovski, Ž., Hardy-type inequalities for generalized fractional integral operators, Tbil. Math. J., 10(1) (2017), 75-90.

[15] Kashuri, A. and Liko, R., Some inequalities similar to Hardy's inequality, PJMMS, 17(1) (2016), 1-6.

[16] Khameli, A., Dahmani, Z., Freha, K. and Sarikaya, M. Z., New Riemann-Liouville generalizations for some inequalities of Hardy type, Malaya J. Mat., 4(2) (2016), 277-283.

[17] Kilbas, A. A., Srivastava, H. M. and Trujillo, J. J., Theory and applications of fractional differential equations, Elsevier, Amsterdam, 2006.

[18] Mitrinovic, D. S., Pecaric, J. and Fink, A. M., Classical and New inequalities in analysis, Kluwer Acad. Publish., Dordrecht, 1993.

[19] Neang, P., Nonlaopon K., Tariboon, J., Ntouyas, S. K. and Agarwal, P., Some trapezoid and midpoint type inequalities via fractional (p, q)-calculus, Adv. Differ. Equ., 333 (2021).

[20] Pachpatte, B. G., On Hardy type integral inequalities for a functions of two variables, Demonstratio Math., XXVIII(2), (1995).

[21] Pečarić, J. and Smoljak, K., Improvement of an extension of a Hölder-type inequality, Anal. Math., 38 (2012), 135-146.

[22] Rahman, G., Khan, A., Abdeljawad, T. and Nisar, K. S., The Minkowski inequalities via generalized proportional fractional integral operators, Adv. Differ. Equ., 287 (2019).

[23] Rashid, S., Jarad, F. and Chu, Y.-M., A Note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function, Math. Probl. Eng., Vol. 2020, Art. ID 7630260.

[24] Rashid, S., Ahmad, H., Khalid, A. and Chu, Y.-M., On discrete fractional integral inequalities for a class of functions, Complexity, Vol. 2020, Art. ID 8845867.

[25] Rashid, S., Hammouch, Z., Jarad, F. and Chu, Y.-M., New estimates of integral inequalities via generalized proportional fractional integral operator with respect to another function, Fractals, 28(8), Art. ID 2040027.

[26] Rashid, S., Jarad, F., Kalsoom, H. and Chu, Y.-M., On Pólya–Szegö and Čebyšev type inequalities via generalized k-fractional integrals, Adv. Differ. Equ., 125 (2020).

[27] Rashid, S., Jarad, F., Noor M. A., Kalsoom, H. and Chu, Y.-M., Inequalities by means of generalized proportional fractional integral operators with respect to another function, Mathematics, 7(12), (2019).

[28] Rashid, S., Noor M. A., Noor, K. I. and Chu, Y.-M., Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions, AIMS Math., 5(3) (2020), 2629-2645.

[29] Restrepo, J. E., Chinchane, V. L. and Agarwal, P., Weighted reverse fractional inequalities of Minkowski's and Hölder's type, TWMS J. Pure Appl. Math., 10(2) (2019), 188-198.

[30] Sitthiwirattham, T., Murtaza, G., Ali, M. A., Promsakon, C., Sial, I. B. and Agarwal, P., Post-Quantum Midpoint-Type inequalities associated with twice-differentiable functions, Axioms, 11(2) (2022).

[31] Sousa, J. and Olivera, E., The Minkowski's inequality by means of a generalized fractional integral, AIMS Mathematics, 3(1) (2018), 131-147.

[32] Sroysang, B., A generalization of some integral inequalities similar to Hardy's Inequality, Math. Aeterna, 3(7) (2013), 593-596.

[33] Sroysang, B., More on reverses of Minkowski's integral inequality, Math. Aeterna, 3(7) (2013), 597-600.

[34] Sroysang, B., More on some Hardy type intgral inequalities, J. Math. Inequal., 8(3) (2014), 497-501.

[35] Sulaiman, W. T., Reverses of Minkowski's, Hölder's and Hardy's integral inequalities, Int. J. Mod. Math. Sci., 1(1) (2012), 14-24.

[36] Wu, S., Sroysang, B. and Li, S., A further generaliztion of certain integral inequalities similar to Hardy's inequality, J. Nonlinear Sci. Appl., 9 (2016), 1093-1102.

[37] Yilmaz, Y., Ozdemir, M. K. and Solak, I., A generaliztion of Hölder and Minkowski Inequalities, J. Inequal. Pure and Appl. Math., 7(5) (2006), art.193.

[38] Zhou, SS., Rashid, S., Jarad, F., Kalsoom, H. and Chu, Y.-M., New estimates considering the generalized proportional Hadamard fractional integral operators, Adv. Differ. Equ., 275 (2020).