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MORE CONDITIONS ON A Γ− SEMIRING AND IDEALS OF A IZUKA AND BOURNE FACTOR Γ− SEMIRING

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Abstract: In this paper, we impose some more conditions on a Γ− semiring such as cancellative, centreless, semi subtractive, zero divisor, simple and characterize the results of a semiring by using cancellative semi- subtractive, zero divisor of cancellative semi- subtractive and cancellative rigid semi- subtractive Γ− semiring with a strong identity. Furthermore, we study maximal and minimal ideals of strongly multiplicative Γ− idempotent Γ− semiring and the results regarding division Γ− semiring, Γ− semi field and Γ− field for Izuka and Borne factor Γ− semiring.

Keywords and Phrases: Multiplicatively cancellative Γ− semiring, semi - subtractive, maximal and minimal ideals, division Γ− semiring, Bourne and Izuka Γ− congruence relation.

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1. Introduction

In 1995, Murali Krishna Rao [9, 14] introduced the notion of a Γ− semiring as a generalization of Γ− ring, ternary semiring and semiring. The set of all negative integers $\mathbf{Z}^-$ is not a semiring with respect to usual addition and multiplication but $\mathbf{Z}^-$ forms a Γ− semiring where $\Gamma = Z$. Historically semiring first appear implicitly in Dedekind and later in Macaulay, Neither and Lorenzen in connection with the study of a ring. However, semirings first appear explicitly in Vandiver, also in connection with the axiomatization of arithmetic of natural numbers. The concept of semiring was first introduced by Vandiver [20] in 1934. Indeed the first Mathematical structure we encounter the set of all non-negative integers is a semiring. The theory of semirings and ordered semiring have wide applications in linear and combinatorial optimization problems such as path problems, transformation and assignment problems, matching problems and eigenvalue value problems. The theory of ordered semiring is very popular since it has wide applications in the theory of computer science, optimization theory and theoretical physics. Semirings are useful in the areas of theoretical computer science as well as in the solutions of graph theory in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches. As a generalization of ring, the notion of a $\Gamma-$ ring was introduced by Nobusawa [7] in 1964. The important reason for the development of $\Gamma-$ semiring is a generalization of results of the rings, $\Gamma-$ rings, semirings, semi groups and ternary semirings. M. M. K Rao and Venkateswarlu [12] introduce the notion $\Gamma-$ incline, zero divisor free $\Gamma-$ semiring and field $\Gamma-$ semiring and study properties of regular $\Gamma-$ incline and field $\Gamma-$ semiring.

Since a semiring which is both centreless and entire is an information algebra. Such semirings are very important applications in graph theory and the theory of discrete event dynamical systems [2]. Also the family of all congruence relations on a semiring R is a complete lattice with meets and joins defined as follows: (1) If Y is a non empty family of congruence relations on R then $\wedge Y$ is the congruence relation on R defined by $r(\wedge y)r'$ if and only if $r\rho r'$ for all relation ρ in Y. (2) If Y is a non empty family of congruence relations on R then $\vee Y$ is the congruence relation on R defined by $r(\vee y)r'$ if and only if there exist elements $r = s_0, s_1, ...s_n = r'$ of R and elements $\rho_1, ...\rho_n$ of Y such that $s_{i-1}\rho_i s_i$ for all $1 \leqslant i \leqslant n$. Indeed, by an easy modification of a result of Funayama and Nakayama, The set of all congruences is in fact a frame [19], and hence a semiring [1]. Also a recent interesting applications of idempotent analysis has been in the study of amoebas as part of an emerging area known as "tropical algebraic geometry", which has important applications in string theory and other applications in physics [6]. So, it is worth looking at this idea in the more general context of $\Gamma-$ semiring.

As a continuation of previous papers "Some conditions on $\Gamma-$ semirings" [15] and "Ideals of a Bourne factor $\Gamma-$ semirings" [17], here, we study the consequences of imposing more condition like centreless, semi subtractive, zero divisor, division $\Gamma-$ semiring, additively and multiplicatively cancellative $\Gamma-$ semiring. Further, we study maximal and minimal ideals by imposing the cancellative of strongly multiplicative Γ− idempotent and strong identity of Γ− semiring. Finally, we characterize some results regarding Γ− field, Γ− semi field, plain and division Γ− semiring of Izuka and Bourne factor Γ− semiring.

2. Preliminaries

Here we give some basic definitions required for the development of this paper. Throughout this paper, R represents a Γ− semiring.

Definition 2.1. [7] *Let R and Γ be two additive abelian groups. Then R is called a $\Gamma-$ ring if there exists a mapping $R \times \Gamma \times R \to R$ written as $(x\alpha y) \to x\alpha y$ for all $x, y, z \in R$ and $\alpha, \beta \in \Gamma$ satisfying the following conditions:*

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)z = x\alpha z + x\beta z$

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$.

Definition 2.2. [9] *Let R and Γ be two additive commutative semi group. Then R is called a $\Gamma-$ semiring if there exists a mapping $R \times \Gamma \times R \to R$ denoted by $x\alpha y$ for all $x, y \in R$ and $\alpha \in \Gamma$ satisfying the following conditions:*

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$

(ii) $x(\alpha + \beta)z = x\alpha z + x\beta z$

(iii) $x\alpha(y + z) = x\alpha y + x\alpha z$

(iv) $(x\alpha y)\beta z = x\alpha(y\beta z)$ *for all* $x, y, z \in R$ *and* $\alpha, \beta \in \Gamma$.

Definition 2.3. [15] *A $\Gamma-$ semiring R is said to have a zero element if $0\gamma x = 0 = x\gamma 0$ and $x + 0 = x = 0 + x$ for all $x \in R$ and $\gamma \in \Gamma$.*

Definition 2.4. [15] *A $\Gamma-$ semiring R is said to have an identity element if for all $x \in R$ there exists $\alpha \in \Gamma$ such that $1\alpha x = x = x\alpha 1$.*

Definition 2.5. *A $\Gamma-$ semiring R is said to have a strong identity element if for all $x \in R$, $1\alpha x = x = x\alpha 1$, for all $\alpha \in \Gamma$.*

Definition 2.6. [9] *A $\Gamma-$ semiring R is said to be commutative if $x\gamma y = y\gamma x$ for all $x, y \in R$ and for all $\gamma \in \Gamma$.*

Definition 2.7. [17] *A $\Gamma-$ semiring R is centreless if and only if $x+y = 0$ implies* *that* $x = y = 0$.

Definition 2.8. [17] *Let* R *be a* $\Gamma-$ *semiring. Then* R *is said to be right additively cancellable if and only if for each* $x \in R$, $x + z = y + z$ *implies that* $x = y$. *Left additively cancellable elements are similarly defined. An element of* R *is additively cancellable if and only if it is both left and right additively cancellable. We will denote the set of all additively cancellable elements by* $AC(\Gamma R)$.

Definition 2.9. [12] *A* $\Gamma-$ *semiring* R *with zero element is said to satisfy cancellation law if for all* $a, b, c \in R$ *and* $\alpha \in \Gamma$ *we have that* $a \neq 0$, $a\alpha b = a\alpha c$ *and* $b\alpha a = c\alpha a$ *implies* $b = c$.

We will denote the set of all $\Gamma-$ *multiplicatively cancellable elements by* $MC(\Gamma R)$.

Definition 2.10. *If* R *is both additive and multiplicatively cancellative then* R *is cancellative. We denote the set of all cancellative elements by* $C(\Gamma R)$.

Definition 2.11. [17] *A non empty subset* I *of* R *is said to be left (right) ideal of* R *if* I *is sub semi group of* $(R, +)$ *and* $x\alpha y \in I(y\alpha x \in I)$ *for all* $y \in I, x \in R$ *and* $\alpha \in \Gamma$.

If R *is a* $\Gamma-$ *semiring with zero element then it is easy to verify that every ideal of* R *has zero element.*

Definition 2.12. [17] *If* I *is both left and right ideal of* R, *then* I *is known to be an ideal of* R.

Definition 2.13. [8] *Let* R *be a* $\Gamma-$ *ring with unity. An element* $x \in R$ *is called unity of* R *if it has a multiplicative inverse in* R. *If every non zero element of* R *is a unity then we say* R *is division* $\Gamma-$ *ring.*

Remark 2.14. *All through here,* R *will signify with "0" and "1" as zero and identity element except if in any case expressed.*

3. More Conditions on a $\Gamma-$ Semiring

"Let x be an element of a $\Gamma-$ semiring R. An element y of R is an additive inverse of x if and only if $x+y = 0 = y+x$. If x has an additive inverse, then such an inverse is unique. For if $x+y = 0 = x+y'$ then $y = y+0 = y+x+y' = 0+y' = y'$. We will denote the additive inverse of an element x, if it exists, by $-x$. Let us denote the set of all elements of R having additive inverse by $A(\Gamma R)$. This set is non empty since $0 \in A(\Gamma R)$, with $-0 = 0$. Moreover, if $x + y \in A(\Gamma R)$ then both x and y belong to $A(\Gamma R)$. Clearly R is a $\Gamma-$ ring if and only if $R = A(\Gamma R)$ and R is centreless if and only if $A(\Gamma R) = 0$.

We now turn from additive inverse to multiplicative inverse.

An element x of a $\Gamma-$ semiring R is unit if and only if there exists an element y of R satisfying $x\alpha y = 1 = y\alpha x$ for all $\alpha \in \Gamma$. The element y of R is called the inverse of x in R. We will denote the inverse of an element x, if it exist by x^{-1}. It is straightforward to see that if x and y are units of R then $(x\alpha y)^{-1} = y^{-1}\alpha^{-1}x^{-1}$. Thus in particular $(x^{-1})^{-1} = x$. Let us denote the set of all elements of R having units by $U(\Gamma R)$. This set is non empty since $1 \in U(\Gamma R)$ and is not all of R" [17].

Definition 3.1. [10] *A $\Gamma-$ semiring R is called a division $\Gamma-$ semiring if each non zero element of R has a multiplicative inverse.*

Example 3.2. Let $\mathbb{N}$ be the set of positive integers and $R = (\mathbb{N}, +)$ be the semigroup of positive integers and let $\Gamma = (2\mathbb{N}, +)$ be the semigroup of even positive integers. Then R is a multiplicatively cancellative $\Gamma-$ semiring, but is not division $\Gamma-$ semiring. For this $U(\Gamma R) = \{1\}$.

Theorem 3.3. *Let R be a division $\Gamma-$ semiring with a strong identity. Then R is cancellative if and only if $AC(\Gamma R) \neq \{0\}$.*

Proof. Let R be additively cancellative $\Gamma-$ semiring then $AC(\Gamma R) = R \neq \{0\}$. Conversely, let $0 \neq r \in AC(\Gamma R)$ and $x, y, z \in R$ be such that $x + y = x + z$. In case $x = 0$ then $y = z$. Otherwise, multiply $r\alpha x^{-1}$ both side of the equation for some $r, x \in R$ and $\alpha \in \Gamma$. Therefore $(r\alpha x^{-1})\beta(x + y) = (r\alpha x^{-1})\beta(x + z)$, $\alpha, \beta \in \Gamma$ so $r\alpha(x^{-1}\beta x) + r\alpha(x^{-1}\beta y) = r\alpha(x^{-1}\beta x) + r\alpha(x^{-1}\beta z)$ thus $r\alpha 1 + r\alpha(x^{-1}\beta y) = r\alpha 1 + r\alpha(x^{-1}\beta z)$ hence $r + r\alpha(x^{-1}\beta y) = r + r\alpha(x^{-1}\beta z)$ gives that $r\alpha(x^{-1}\beta y) = r\alpha(x^{-1}\beta z)$ (by left cancellation law). Now, multiply both side of the equation on the left by $x\gamma r^{-1}$, we get $(x\gamma r^{-1})\delta[r\alpha(x^{-1}\beta y)] = (x\gamma r^{-1})\delta[r\alpha(x^{-1}\beta z)]$, $\gamma, \delta \in \Gamma$ therefore $x\gamma(r^{-1}\delta r)\alpha(x^{-1}\beta y) = x\gamma(r^{-1}\delta r)\alpha(x^{-1}\beta z)$ so $(x\gamma 1)\alpha(x^{-1}\beta y) = (x\gamma 1)\alpha(x^{-1}\beta z)$ thus $(x\alpha x^{-1})\beta y = (x\alpha x^{-1})\beta z$ implies that $1\beta y = 1\beta z$. Hence $y = z$.

Definition 3.4. [13] *A $\Gamma-$ semiring R is said to be semi subtractive $\Gamma-$ semiring R if for every $x, y \in R$ there exists $r \in R$ such that $r + x = y$ or $r + y = x$.*

Definition 3.5. [13] *Let R be a $\Gamma-$ semiring. An element $x \in R$ is said to be zeroid if there exists $r \in R$ such that $x + r = r$ or $r + x = r$. Set of all zeroids is denoted by Z(R). If $Z(R) \neq R$, then it is non zeroid.*

Every additive idempotent element of $\Gamma-$ semiring R is zeroid of R.

A $\Gamma-$ semiring R is plain if and only if $Z(R) = \{0\}$. In case R is cancellative, then it is definitely plain. However, the following result provides partial converse of this fact.

Theorem 3.6. *Let R be a semi subtractive $\Gamma-$ semiring. Then R is cancellative if and only if it is plain.*

Theorem 3.7. *Let R is a division* $\Gamma-$ *semiring. Then R is either centreless or division* $\Gamma-$ *ring.*

Proof. Let R be not centreless. Then there exists $0 \neq x \in R$ having an additive inverse $(-x)$. Let $0 \neq z \in R$ and $\alpha, \beta \in \Gamma$ then $z + (z\alpha x^{-1})\beta(-x) = z\alpha 1 + (z\alpha x^{-1})\beta(-x) = z\alpha(x^{-1}\beta x) + (z\alpha x^{-1})\beta(-x) = (z\alpha x^{-1})\beta[x + (-x)] = (z\alpha x^{-1})\beta 0 = 0$. This implies that z has an additive inverse. Then $(R, +)$ is a group. So, R is a $\Gamma-$ ring, at that point R be a division $\Gamma-$ ring.

Theorem 3.8. *Let R be a multiplicatively cancellative* $\Gamma-$ *semiring. In case, an element having finite multiplicative order other than 1, then R is* $\Gamma-$ *ring.*

Proof. Let $1 \neq x \in R$ satisfying $(x\alpha)^{n-1}x = 1, \alpha \in \Gamma$ where $(x\alpha)^{n-1} = (x\alpha)(x\alpha) \overset{(n-1)\, times}{\ldots} (x\alpha)$. Let $y = 1 + x + (x\alpha)x + (x\alpha)^2 x + \ldots + (x\alpha)^{n-2}x$. Then $x\alpha y = x\alpha[1 + x + (x\alpha)x + (x\alpha)^2 x + \ldots + (x\alpha)^{n-2}x] = x\alpha 1 + (x\alpha)x + (x\alpha)^2 x + \ldots + (x\alpha)^{n-2}x + (x\alpha)^{n-1}x = 1 + x + (x\alpha)x + (x\alpha)^2 x + \ldots + (x\alpha)^{n-2}x = y$. This suggests that $x\alpha y = 1\alpha y$. In any case, $x \neq 1$, hence $y = 0$ so $1 \in A(\Gamma R)$, showing that $A(\Gamma R) = R$. Hence R is $\Gamma-$ ring.

Definition 3.9. [17] *A non zero element x in a* $\Gamma-$ *semiring R is a left zero divisor if and only if there exists a non zero element* $y \in R$ *and* $\alpha \in \Gamma$ *satisfying* $x\alpha y = 0$. *It is a right zero divisor if and only if there exists non zero* $y \in R$ *and* $\alpha \in \Gamma$ *satisfying* $y\alpha x = 0$. *It is a zero divisor if and only if it is both left and right zero divisor. A* $\Gamma-$ *semiring R having no* $\Gamma-$ *zero divisor is* $\Gamma-$ *entire.*

Definition 3.10. *A* $\Gamma-$ *semiring which is both centreless and* $\Gamma-$ *entire is an information algebra.*

Theorem 3.11. *Let R be a cancellative semi subtractive* $\Gamma-$ *semiring and x be any element of R. Then x is not a zero divisor but multiplicatively cancellative.*

Proof. Let $x \in R$ such that x is not a zero divisor. Let $y, z \in R$ and $\alpha \in \Gamma$ be such that $y\alpha x = z\alpha x$. Since R is semi subtractive $\Gamma-$ semiring, therefore there exists an element $r \in R$ such that $y = z + r$ or $z = y + r$. Let $y = z + r$. Therefore, $z\alpha x + 0 = z\alpha x = y\alpha x = (z + r)\alpha x = z\alpha x + r\alpha x$. Since R is cancellative, this implies that $r\alpha x = 0$. Once more, since x is certainly not a zero divisor, therefore we should have $r = 0$. So $y = z$, showing that x is right multiplicatively cancellative. An overall affirmation shows that x is also left multiplicative cancellative.

Definition 3.12. [17] *An ideal I of a* $\Gamma-$ *semiring R is called k-ideal if for* $x, y \in R, x + y \in I$ *and* $y \in I$ *implies that* $x \in I$.

Definition 3.13. [17] *If R has no non-zero left k-ideal then R is called left rigid. Right rigid is similarly defined.*

The following Corollary is proved in [17].

Corollary 3.14. [17] *Let R be a $\Gamma-$ semiring. Then R is left rigid if and only if for each $0 \neq r \in R$ there exist elements $x, y \in R$ and $\alpha \in \Gamma$ satisfying $x\alpha r + 1 = y\alpha r$.*

Theorem 3.15. *Let R be a cancellative rigid semi subtractive $\Gamma-$ semiring with strong identity. Then R is division $\Gamma-$ semiring.*

Proof. Let $0 \neq r \in R$. By Corollary 3.14, there exist elements $x, y \in R$ and $\alpha, \beta \in \Gamma$ such that $x\alpha r + 1 = y\alpha r$. Since R is semi subtractive $\Gamma-$ semiring, so $z \in R$ such that $x = y + z$ or $y = x + z$. Now, in case $x = y + z$ then $x\alpha r = y\alpha r + z\alpha r = x\alpha r + 1 + z\alpha r$ implies that $0 = 1 + z\alpha r$, since R is cancellative. So $1 \in A(\Gamma R)$. Hence R is a $\Gamma-$ ring. Since each left ideal of a $\Gamma-$ ring is k-ideal, so we assume that R has a no non-zero remaining left ideal and it is satisfactory to exhibit that R is a division $\Gamma-$ ring. Let $y = x + z$ then $x\alpha r + z\alpha r = y\alpha r = x\alpha r + 1$ implies that $z\alpha r = 1$ implies that $z \neq 0$. Therefore in a similar way either R is a $\Gamma-$ ring or there exists an element $z' \in R$ and $\beta \in \Gamma$ satisfying $z'\beta z = 1$. But $z' = z'\beta 1 = z'\beta(z\alpha r) = (z'\beta z)\alpha r = 1\alpha r = r$, for all $\alpha \in \Gamma$ implies that $r \in U(\Gamma R)$ with $z = r^{-1}$. In this way, every non zero element of R is unit. Hence R is division $\Gamma-$ semiring.

Theorem 3.16. *Let R be a division $\Gamma-$ semiring. Then $\{0\}$ and R are the only ideals of R.*

Definition 3.17. [15] *A non empty subset S of a $\Gamma-$ semiring R is said to be a sub $\Gamma-$ semiring of R if $(S, +)$ is a sub semi group of $(R, +)$ and $x\gamma y \in S$ for all $x, y \in S$ and $\gamma \in \Gamma$.*

Theorem 3.18. *Let R be a semi subtractive $\Gamma-$ semiring. If I and J are sub $\Gamma-$ semiring of R such that $I\Gamma I \subseteq A(\Gamma R)$ or $J\Gamma J \subseteq A(\Gamma R)$.*

Proof. Let $I\Gamma I \nsubseteq A(\Gamma R)$. Then $x, x' \in I$ and $\alpha \in \Gamma$ such that $x\alpha x' \notin A(\Gamma R)$. Let $y, y' \in J$. If $r \in R$ such that $x + r = y$. Then $x'\alpha x + x'\alpha r = x'\alpha y \in I\Gamma J \subseteq A(\Gamma R)$. Therefore $x'\alpha x \in A(\Gamma R)$. Which is inconsistency. Hence $r \in R$ such that $x = y + r$, since R is semi subtractive. But $y\alpha y' + r\alpha y' = x\alpha y' \in I\Gamma J \subseteq A(\Gamma R)$. Thus $y\alpha y' \in A(\Gamma R)$. Hence $J\Gamma J \subseteq A(\Gamma R)$.

Definition 3.19. [15] *A $\Gamma-$ semiring R with zero element is simple if and only if $x + 1 = 1 = 1 + x$ for all $x \in R$.*

Clearly simple $\Gamma-$ semiring are additive idempotent but converse is not true.

Theorem 3.20. *Let R be a simple $\Gamma-$ semiring. Then $U(\Gamma R) = \{1\}$.*

Proof. Let $x \in U(\Gamma R)$ then $y \in R, \alpha \in \Gamma$ such that $x\alpha y = 1$. Hence by Lemma 3.2 (i) [15], we have $x = x + x\alpha y = x + 1 = 1$.

4. Maximal and Minimal Ideals of a $\Gamma-$ Semiring

In this section, we study the consequences of imposing some other conditions on maximal and minimal ideals.

Definition 4.1. [16] *A proper ideal M of a $\Gamma-$ semiring R is said to be maximal ideal if there does not exist any other proper ideal of R containing M properly.*

Theorem 4.2. *Let R be a $\Gamma-$ semiring and I be a maximal ideal of R. Then each ideal of R is contained in I.*
Proof. It is straightforward.

Theorem 4.3. *Let R be a $\Gamma-$ semiring with strong identity and I be one sided maximal ideal of R. Then for any element $x \in R$, $x \in U(\Gamma R)$ if and only if $x \notin I$.*
Proof. Let $x \in U(\Gamma R)$ and I be the maximal left ideal of R. If $x \in I$ then for all $\alpha \in \Gamma$, $x^{-1}\alpha x \in I$. This implies that $1 \in I$, which is a contradiction. Consequently $x \notin I$. Likewise $x \notin I$ for any maximal right ideal I. Conversely, let $x \notin I \subseteq R$. By Theorem 4.2, $R\Gamma x$ is not a left ideal of R. Therefore $R\Gamma x = R$. Consequently $x\Gamma R = R$. Thus there exist elements $y, z \in R$ and $\beta, \gamma \in \Gamma$ such that $y\beta x = 1 = x\gamma z$. In any case $y = y\beta 1 = y\beta(x\gamma z) = (y\beta x)\gamma z = 1\gamma z = z$ thus $x \in U(\Gamma R)$ and $y = x^{-1}$.

Definition 4.4. [11] *Let R be a $\Gamma-$ semiring. An element x of a $\Gamma-$ semiring R is said to be strongly multiplicative $\Gamma-$ idempotent if $x = x\gamma x$ for all $\gamma \in \Gamma$. If every element of R is strongly multiplicative $\Gamma-$ idempotent then R is called strongly multiplicative $\Gamma-$ idempotent.*

Definition 4.5. [5] *An ideal M of a $\Gamma-$ semiring R is said to be minimal if and only if it does not contain any ideal of R other than itself and 0.*

Theorem 4.6. *Let R be a $\Gamma-$ semiring. Let K be a minimal left ideal of R and $x \in R, \alpha \in \Gamma$. Then $K\Gamma x$ is a left ideal of R which is either $\{0\}$ or minimal.*
Proof. It is obvious that, $K\Gamma x$ is a left ideal of R. Now, let $K\Gamma x \neq \{0\}$ be such that $Q \subset K\Gamma x$, where $0 \neq Q$ is a left ideal of R. Let us define $P = \{r \in K | r\alpha x \in Q$ for all $x \in Q, \alpha \in \Gamma\}$ then P is a left ideal of R and $P \subset K$ and $K \neq \{0\}$, which is inconsistency to the minimality of K. Hence $K\Gamma x$ must be minimal.

Theorem 4.7. *Let R be a $\Gamma-$ semiring. If K be a minimal left ideal of R such that $J \supset K$, J is an ideal of R and J' be the sum of all minimal left ideals of R contained in J. Then J' is an ideals of R.*
Proof. Since J' be sum of all minimal left ideals of R contained in J. Then J' is left ideal of R. Let $x \in R, \alpha \in \Gamma$ and K be a minimal left ideal of R contained in J, then $K\Gamma x \subseteq J$ so by Theorem 4.6, $K\Gamma x \subseteq J'$. Consequently $J'\Gamma x \subseteq J'$ for each

$x \in R$. Thus J' is an ideal of R.

Theorem 4.8. *Let R be a strongly multiplicative $\Gamma-$ idempotent $\Gamma-$ semiring and K be a minimal left ideal of R. In case $0 \neq e \in K$ and e be strongly multiplicative $\Gamma-$ idempotent. Then $e\Gamma K$ is a division $\Gamma-$ semiring with multiplicative identity e.*

Proof. Clearly $e\Gamma K$ is additive commutative semi group. Let Γ be a additive commutative semi group. Define a mapping $((e\Gamma K) \times \Gamma \times (e\Gamma K) \to (e\Gamma K)$ defined by $[(e\alpha x), \beta, (e\gamma y)] \to [(e\alpha x)\beta(e\gamma y)]$, for all $(e\alpha x), (e\gamma y) \in e\Gamma K$, $\alpha, \beta, \gamma \in \Gamma$. Clearly $e\Gamma K$ is a $\Gamma-$ semiring. Since $K\Gamma e$ is a non zero left ideal of R such that $K\Gamma e \subset K$. Therefore $K\Gamma e = K$. Thus $x = y\alpha e$ for all $x, y \in K$ and $\alpha \in \Gamma$. Now, since e is strongly multiplicative $\Gamma-$ idempotent so, $(e\gamma x)\beta e = (e\gamma(y\alpha e))\beta e = (e\gamma y)\alpha(e\beta e) = (e\gamma y)\alpha e = e\gamma(y\alpha e) = e\gamma x$, for all $\alpha, \beta, \gamma \in \Gamma$. Similarly $e\beta(e\gamma x) = (e\beta e)\gamma x = e\gamma x$ for all $\beta, \gamma \in \Gamma$. This implies that $(e\gamma x)\beta e = e\gamma x = e\beta(e\gamma x)$. Then e is multiplicative identity of $e\Gamma K$. Now, let $0 \neq e\alpha x \in e\Gamma K$ then $e\alpha x = (e\beta e)\alpha x = e\beta(e\alpha x) \in K\Gamma(e\alpha x)$ and so $K\Gamma(e\alpha x)$ is a non zero left ideal of R contained in K. Thus $K = K\Gamma(e\alpha x)$ and so $e\Gamma K = e\Gamma K\Gamma(e\alpha x)$. In particular, there exists an element k of K and $\gamma, \alpha, \beta \in \Gamma$ satisfying $(e\gamma k)\beta(e\alpha x) = e$. Since $e \in K$ is strongly $\Gamma-$ idempotent so $e\beta e \in e\Gamma K$ that is $e \in e\Gamma K$. Again, there exists an element h of K and $\gamma, \delta, \rho \in \Gamma$ satisfying $(e\delta h)\rho(e\gamma k) = e$ therefore $e\delta h = (e\delta h)\rho e = (e\delta h)\rho((e\gamma k)\beta(e\alpha x)) = ((e\delta h)\rho(e\gamma k))\beta(e\alpha x) = e\beta(e\alpha x) = (e\beta e)\alpha x = e\alpha x$. Hence $e\Gamma K$ is a division $\Gamma-$ semiring.

5. Ideals of a Izuka and Bourne Factor $\Gamma-$ Semirings

"An equivalence relation ρ defined on a $\Gamma-$ semiring R satisfying the condition that if $r\rho_A r'$ in R then $(r + s)\rho(r' + s')$ and $(r\alpha s)\rho(r'\alpha s')$ for all $r, s, r', s' \in R$ and $\alpha \in \Gamma$ is called a $\Gamma-$ congruence on the $\Gamma-$ semiring R. For a proper ideal A of $\Gamma-$ semiring R the $\Gamma-$ congruence on R, denoted by ρ_A defined as $s\rho_A s'$ if and only if $s + a_1 = s' + a_2$ for some $a_1, a_2 \in A$ is called Bourne $\Gamma-$ congruence on R defined by the ideal A.

We denote the Bourne $\Gamma-$ congruence (ρ_A) class of an element r of R by R/ρ_A or simply R/A and denote the set of all such $\Gamma-$ congruence classes of the $\Gamma-$ semiring R by R/ρ_A or simply R/A. It should be noted here that for any proper ideal A of R and for any $s \in R$, s/A is not necessarily equal to $s+A = \{s+a : a \in A\}$ but surely contain it. For any proper ideal A of $\Gamma-$ semiring R, if the Bourne $\Gamma-$ congruence (ρ_A), defined by A, is proper that is $0/A \neq R$ then R/A is a $\Gamma-$ semiring with the following operation: $s/A + s'/A = (s + s')/A$ and $(s/A)\alpha(s'/A) = (s\alpha s')/A$ for all $\alpha \in \Gamma$. We call this $\Gamma-$ semiring the Bourne factor $\Gamma-$ semiring of R" [17].

Remark 5.1. [14] *For a proper ideal X of a $\Gamma-$ semiring R, the $\Gamma-$ congruence* *on R denoted by σ_x and defined as $r\sigma_x r'$ if and only if $r + x_1 + z = r' + x_2 + z$ for some $x_1, x_2 \in X$ and for some $z \in R$ is called Izuka $\Gamma-$ congruence on R defined by the ideal X. Now, we denote Izuka $\Gamma-$ congruence class of an element r of R by $r[/]X$ and denote the set of all such $\Gamma-$ congruence class of an element of the $\Gamma-$ semiring R by $R[/]X$. If the Izuka $\Gamma-$ congruence σ_X, defined by X, is proper i.e. $0[/]X \neq R$ then $R[/]X$ is a $\Gamma-$ semiring with the following operations: $r[/]X + r'[/]X = (r + r')[/]X$ and $(r[/]X)\alpha(r'[/]X) = (r\alpha r')[/]X$ for all $\alpha \in \Gamma$. We call this $\Gamma-$ semiring, the Izuka $\Gamma-$ semiring of R by X.*

We now state the following theorems regarding Izuka $\Gamma-$ semiring and Bourne factor $\Gamma-$ semiring which are analogous to the corresponding Theorems in semirings, proofs of which are easy and straightforward and so we omit the proofs.

Theorem 5.2. *Let R be a $\Gamma-$ semiring. In case J is an ideal of R such that $R \neq 0[/]J = K$ then the $\Gamma-$ semirings $R[/]K$ and R/K are plain.*

Corollary 5.3. *Let R be a $\Gamma-$ semiring. In case R is non zeroid. Then $R/Z(R)$ is plain.*

Theorem 5.4. *Let R be a semi subtractive $\Gamma-$ semiring and J be a k-ideal of R. Then $J \supseteq Z(R)$ if and only if R/J is cancellative.*

Theorem 5.5. *Let R be a cancellative $\Gamma-$ semiring. Then R/J is cancellative for each ideal J of R.*

Definition 5.6. *A commutative division $\Gamma-$ semiring R is said to be a $\Gamma-$ semi field.*

Definition 5.7. [11] *A $\Gamma-$ semigroup R is said to be $\Gamma-$ group if it satisfy the following*
(i) If there exists $1 \in R$ and for each $x \in R$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.
(ii) If for each element $0 \neq x \in R$ there exists $y \in R, \alpha \in \Gamma$ such that $x\alpha y = y\alpha x = 1$.

Definition 5.8. [11] *A commutative $\Gamma-$ ring R is said to be $\Gamma-$ field if R is a $\Gamma-$ group.*

Example 5.9. Let M be the set of all rational numbers and $\Gamma = M$ be a commutative semi group with respect to usual addition. Define a mapping $M \times \Gamma \times M \to M$ by $x\alpha y$ as usual multiplication for all $x, y \in M$ and $\alpha \in \Gamma$ then M is field $\Gamma-$ semiring.

Theorem 5.10. *Let R be a multiplicatively cancellative, centreless and commu-*

tative $\Gamma-$ semiring with strong identity having no non-trivial proper congruence relation then either $R = \mathbb{B} = \{0, 1\}$ or R is a $\Gamma-$ field.

Proof. In case R has only two elements, then either $R = \mathbb{B}$ or R is a $\Gamma-$ field $Z/(2)$. Therefore, the result is obvious. So, let R have more than two elements. Let R be multiplicatively cancellative and each element $x \in R$ and $\alpha \in \Gamma$ define a congruence relation ρ_x on R by $r\rho_x r'$ if and only if $x\alpha r = x\alpha r'$. This is trivial congruence relation, if x is multiplicatively cancellable and proper if $x \neq 0$. Since R has no non-trivial proper congruence relation, so it should be multiplicatively cancellative. Therefore $R \setminus \{0\}$ is a submonoid of R. Eventually, except that R is centreless. Then $R \setminus \{0\}$ is closed under addition and multiplication. So, we have a non-trivial proper congruence relation ρ on R described by the condition that $x\rho y$ if and only if $x = y$ or $x \neq 0$ and $y \neq 0$. This is an inconsistency, thus R cannot be centreless. At that point $0 \subseteq A(\Gamma R)$ and atleast one non zero element contained in $A(\Gamma R)$. In addition, $A(\Gamma R)$ is an ideal of R. The congruence relation $\equiv_{A(\Gamma R)}$ define on R is not trivial and hence by assumption, it must be improper. Specifically, $1 \equiv_{A(\Gamma R)} 0$ thus there exists an element $y \in A(\Gamma R)$ satisfying $1 + y = 0$. Therefore, for any $r \in R, \alpha \in \Gamma$ we have $r + y\alpha r = 1\alpha r + y\alpha r = (1 + y)\alpha r = 0$ so each element of R has an additive inverse. Thus R is a $\Gamma-$ ring. Again, if $0 \neq x \in R$ and if $J = (x)$ is principal ideal of R generated by x then the congruence relation $\equiv_J$ is non trivial, thus it should be improper. Specifically, $1 \equiv_J 0$ and so $1 \in J$. Thus x is a unit. Hence R is a $\Gamma-$ field.

Corollary 5.11. [18] *Let R be a $\Gamma-$ semiring and P be any maximal ideal of R. Then P is prime.*

It is clear from Theorem 5.9 that a division $\Gamma-$ semiring or $\Gamma-$ semi field may have proper non-trivial congruence relation. If ρ is a proper congruence relation on a division $\Gamma-$ semiring, then surely R/ρ is a $\Gamma-$ semiring.

We now turn to considering the Bourne factor $\Gamma-$ semirings.

Theorem 5.12. *Let R be a commutative $\Gamma-$ semiring. If I is maximal k-ideal of R. Then R/I is $\Gamma-$ semi field.*

Proof. Let $0/I \neq x/I \in R/I$. In case $x\alpha x \in I$ then by commutative property, $(x)\Gamma(x) \subseteq I$ so by Corollary 5.10, we have $x \in I$, which is contradiction to the choice of x. Since $x\alpha x \in (x)$ for all $\alpha \in \Gamma$. So $I \subset I + (x)$. Hence by maximality of I, we have $R = I + (x)$. Consequently there exists an element $y \in I, r \in R$ and $\beta \in \Gamma$ such that $1 = y + r\beta x$. This implies that $1/I = (r\beta x)/I = (r/I)\beta(x/I)$. Thus $x/I \in U(\Gamma(R/I))$, hence R/I is a $\Gamma-$ semi field.

Theorem 5.13. *Let R be a $\Gamma-$ semiring with a strong identity. Then $\mathbb{B}$ is the only finite $\Gamma-$ semi field which is not a $\Gamma-$ field.*

Proof. Let $R \neq \mathbb{B}$ be finite $\Gamma-$ semi field. Let $x \in R \setminus \{0,1\}$. Since R is finite, then there exist positive integer $h < k$ such that $(x\alpha)^{h-1}x = (x\alpha)^{k-1}x$, for all $\alpha \in \Gamma$. Since R is $\Gamma-$ semi field, we have $1 = (x\alpha)^{n-1}x$ where $n = k - h$. Let $y = 1 + (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-2}x$. Then $x\alpha y = x\alpha(1 + (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-2}x) = (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-1}x = (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-2}x + 1 = 1 + (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-2}x = y$. If $y \neq 0$ then $x = x\alpha(y\beta y^{-1}) = (x\alpha y)\beta y^{-1} = y\beta y^{-1} = 1$, for all $\beta \in \Gamma$, which is a contradiction as $x \in R \setminus \{0,1\}$. Therefore, we must have $y = 0$. So $1 + (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-2}x = 0$. Which shows that 1 has an additive inverse $z = (x\alpha)1 + (x\alpha)x + (x\alpha)^2x + ... + (x\alpha)^{n-2}x$. Therefore for any $r \in R, \alpha \in \Gamma$, we have $r + r\alpha z = r\alpha 1 + r\alpha z = r\alpha(1 + z) = r\alpha 0 = 0$ for all $\alpha \in \Gamma$. So each element of R has an additive inverse, proving this R is $\Gamma-$ field.

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