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INTUITIONISTIC FUZZY CHARACTERISTIC IDEAL OF A Γ-RING

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Abstract: In this paper, we define the notion of intuitionistic fuzzy characteristic ideal (IFCI) of a Γ-ring which is analogue of a characteristic ideal in the ordinary ring theory and derive various new results. The correlation between the set of all automorphisms of Γ-ring and the corresponding automorphisms of its operator rings have been innovated. Then a one to one correlation between the set of all intuitionistic fuzzy characteristic ideals of Γ-ring and that of its operator ring has been constituted. This is used to obtain a similar bijection for characteristic ideals.

Keywords and Phrases: Γ-ring, Intuitionistic fuzzy characteristic ideal, Γ-Automorphism.

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1. Introduction

The concept of a Γ-ring was first introduced by Nobusawa [9]. Barnes [5] weakened slightly the conditions in the definition of the Γ-ring in the sense of Nobusawa. Since then, many researchers have investigated various properties of this Γ-ring. Any ring can be regarded as a Γ-ring by suitably choosing Γ. Many fundamental results in ring theory have been extended to Γ-rings. R. Paul [13] studied various types of ideals in Γ-ring and the corresponding operator rings. The idea of intuitionistic fuzzy sets was first published by Atanassaov [3, 4], as a generalization of the notion of fuzzy set given by Zadeh [18]. Kim et al. in [8] considered the intuitionistic fuzzification of ideal of Γ-ring which were further studied by Palaniappan et al. in [10, 11]. Cho et al. in [6] and Palaniappan et al. in [12], studied intuitionistic fuzzy ideal and intuitionistic fuzzy prime ideal in Γ-near-rings. The notion of intuitionistic fuzzy bi-ideals in Γ-near-rings was introduced by Ezhilmaran et al. in [7]. Alhaleem et al. in [2] studied intuitionistic fuzzy normed subrings and ideals. The characteristic ideals and characteristic ideals of Γ-semigroups was studied by Sardar et al. in [14]. Aggarwal et al. in [1] studied some theorems on fuzzy prime ideals of Γ-rings. Sharma et al. in [15, 16, 17] studied extension of intuitionistic fuzzy ideals, intuitionistic fuzzy prime radical and intuitionistic fuzzy primary ideal and translational subset (ideals) in Γ-rings.

The objective of this paper is to study the various properties of intuitionistic fuzzy characteristic ideal of a Γ-ring. We shall also investigate the relationship between the intuitionistic fuzzy characteristic ideal of a Γ-ring with its level cut sets. A connection between the set of all automorphisms of Γ-ring and the corresponding automorphisms of its operator rings will be characterized. Finally we will study the correspondence between the set of all intuitionistic fuzzy characteristic ideals of Γ-ring and the set of all intuitionistic fuzzy characteristic ideals of its operator ring. The structuring of the paper is as follows.

In part 2 we recollect some groundwork for their use in the continuation of the development of the subject matter. In part 3 we set in motion of the notion of intuitionistic fuzzy characteristic ideal (IFCI) of Γ-ring M. With the help of an example we show that an intuitionistic fuzzy ideal of Γ-ring M need not be an intuitionistic fuzzy characteristic ideal. We also characterized intuitionistic fuzzy characteristic ideal with the help of its level cut Γ-ideals. In part 4 we inaugurate the notion of automorphism of operator rings of a Γ-ring and also the notion of corresponding automorphism in Γ-rings. While proving some more related results we establish a linkage among the set of all intuitionistic fuzzy characteristic ideals of Γ-ring and that of its operator ring.

2. Preliminaries

Let us recall some definitions and results, which are necessary for the development of the paper.

Definition 2.1. ([5, 9]) *Let $(M, +)$ and $(\Gamma, +)$ be additive abelian groups. Then M is called a Γ-ring (in the sense of Barnes* [5]*) if there exist mapping $M \times \Gamma \times M \to M$ [image of (x, α, y) is denoted by $x\alpha y$, $x, y \in M, \alpha \in \Gamma$] satisfying the following con-*

ditions:

(1) $x\alpha y \in M$.

(2) $(x+y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha+\beta)y = x\alpha y + x\beta y$, $x\alpha(y+z) = x\alpha y + x\alpha z$.

(3) $(x\alpha y)\beta z = x\alpha(y\beta z)$ *for all* $x, y, z \in M$ *and* $\alpha, \beta \in \Gamma$.

The subset N of a Γ-ring M is a left ideal of M if N is an additive subgroup of M and $M\Gamma N = \{x\alpha y | x \in M, \alpha \in \Gamma, y \in N\}$ is contained in N. Similarly, right ideal $N\Gamma M$ of M can be defined. If N is both a left and a right ideal then N is a two-sided ideal, or simply an ideal of M. A mapping $f : M \to M^{'}$ of Γ-rings is called a Γ-homomorphism [5] if $f(x+y) = f(x) + f(y)$ and $f(x\alpha y) = f(x)\alpha f(y)$ for all $x, y \in M, \alpha \in \Gamma$. When $M^{'} = M$, then a Γ-homomorphism is called a Γ-endomorphism, further a one to one Γ-endomorphism is called a Γ-automorphism.

Throughout this study $Aut(M)$ will denote the set of all Γ-automorphisms of M. We now review some intuitionistic fuzzy logic concepts. We refer the reader to follow [3] and [4] for complete details.

Definition 2.2. ([18]) *A fuzzy set* μ *in* X *is a mapping* $\mu : X \to [0,1]$.

Definition 2.3. ([3, 4]) *An intuitionistic fuzzy set (IFS) A in* X *can be represented as an object of the form* $A = \{< x, \mu_A(x), \nu_A(x) >: x \in X\}$, *where the functions* $\mu_A : X \to [0,1]$ *and* $\nu_A : X \to [0,1]$ *denote the degree of membership (namely* $\mu_A(x)$*) and the degree of non-membership (namely* $\nu_A(x)$*) of each element* $x \in X$ *to* A *respectively and* $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ *for each* $x \in X$. *It is shortly denoted by* $A(x) = (\mu_A(x), \nu_A(x))$, *for all* $x \in X$.

Proposition 2.4. ([3, 4]) *If* A *and* B *are two intuitionistic fuzzy sets of* X, *then*

(i) $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x)$ *and* $\nu_A(x) \geq \nu_B(x), \forall x \in X$;

(ii) $A = B \Leftrightarrow A \subseteq B$ *and* $B \subseteq A$, *i.e.,* $A(x) = B(x)$, *for all* $x \in X$.

For any subset Y of X, the intuitionistic fuzzy characteristic function (IFCF) χ_Y is an intuitionistic fuzzy set of X, defined as $\chi_Y(x) = (1,0), \forall x \in Y$ and $\chi_Y(x) = (0,1), \forall x \in X \backslash Y$. Let $\alpha, \beta \in [0,1]$ with $\alpha + \beta \leq 1$. Then the set $A_{(\alpha,\beta)} = \{x \in X : \mu_A(x) \geq \alpha$ and $\nu_A(x) \leq \beta\}$ is called the (α,β)-level cut subset of X with respect to IFS A. Further if $f : X \to Y$ is a mapping and A, B be respectively IFS of X and Y. Then the image $f(A)$ is an IFS of Y and is defined as $\mu_{f(A)}(y) = Sup\{\mu_A(x) : f(x) = y\}$, $\nu_{f(A)}(y) = Inf\{\nu_A(x) : f(x) = y\}$, for all $y \in Y$ and the inverse image $f^{-1}(B)$ is an IFS of X and is defined as $\mu_{f^{-1}(B)}(x) = \mu_B(f(x))$, $\nu_{f^{-1}(B)}(x) = \nu_B(f(x))$, for all $x \in X$, i.e., $f^{-1}(B)(x) = B(f(x))$, for all $x \in X$. Also the IFS A of X is called f-invariant if $f(x) = f(y)$ implies $A(x) = A(y)$, where $x, y \in X$.

Definition 2.5. ([8]) *Let A be an IFS of a Γ-ring M. Then A is called an intuitionistic fuzzy ideal (IFI) of M if for all $m, n \in M, \alpha \in \Gamma$, the following are satisfied*

(i) $\mu_A(m-n) \geq \mu_A(m) \wedge \mu_A(n)$;

(ii) $\mu_A(m\alpha n) \geq \mu_A(m) \vee \mu_A(n)$;

(iii) $\nu_A(m-n) \leq \nu_A(m) \vee \nu_A(n)$;

(iv) $\nu_A(m\alpha n) \leq \nu_A(m) \wedge \nu_A(n)$.

Example 2.6. Let D be a division ring with unity 1 and M be a set of (2×2) matrices of the type

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

where, $a, b \in D$. Take $\Gamma =$ set of matrices of M with translation of interchanging of row 1 and row 2, then M is a Γ-ring. It is easy to see that the set J of all (2×2) matrices of the type

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$$

where, $a \in D$, is a Γ-ideal of M.
Let $A = (\mu_A, \nu_A)$ be an IFS of M defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in J \\ 0.5, & \text{if } x \notin J \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in J \\ 0.3, & \text{if } x \notin J. \end{cases}$$

Then it is easy to verify that A is an IFI of Γ-ring M.

Theorem 2.7. ([8]) *Let K be a non-void subset of a Γ-ring M. Then K is Γ-ideal of M iff intuitionistic fuzzy characteristic function χ_K is an intuitionistic fuzzy ideal of M.*

3. Intuitionistic Fuzzy Characteristic Ideal of Γ-Ring

Definition 3.1. *Let A be an IFS in a Γ-ring M and $f: M \to M$ be a Γ-endomorphism, then A^f is an IFS on M defined as $A^f(x) = A(f(x))$, for all $x \in M$, i.e., $\mu_{A^f}(x) = \mu_A(f(x))$ and $\nu_{A^f}(x) = \nu_A(f(x))$, for all $x \in M$.*

Theorem 3.2. *Let A be an IFI of Γ-ring M and f be a Γ-endomorphism, then*

A^f is also an IFI of M.

Proof. Let A be an IFI of Γ-ring M. Let $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned} \mu_{A^f}(x-y) &= \mu_A(f(x-y)) \\ &= \mu_A(f(x)-f(y)) \\ &\geq \mu_A(f(x)) \wedge \mu_A(f(y)) \\ &= \mu_{A^f(x)} \wedge \mu_{A^f}(y). \end{aligned}$$

Thus $\mu_{A^f}(x-y) \geq \mu_{A^f(x)} \wedge \mu_{A^f}(y)$. Similarly, we can prove $\nu_{A^f}(x-y) \leq \nu_{A^f(x)} \vee \nu_{A^f}(y)$.

Also,

$$\begin{aligned} \mu_{A^f}(x\alpha y) &= \mu_A(f(x\alpha y)) \\ &= \mu_A(f(x)\alpha f(y)) \\ &\geq \mu_A(f(x)) \vee \mu_A(f(y)) \\ &= \mu_{A^f(x)} \vee \mu_{A^f}(y). \end{aligned}$$

i.e., $\mu_{A^f}(x\alpha y) \geq \mu_{A^f(x)} \vee \mu_{A^f}(y)$. Similarly, we can prove $\nu_{A^f}(x\alpha y) \leq \nu_{A^f(x)} \wedge \mu_{A^f}(y)$. Hence A^f is an IFI of Γ-ring M.

Definition 3.3. *A Γ-ideal K of M is said to be characteristic ideal if $f(K) = K$ for all $f \in Aut(M)$.*

Definition 3.4. *An IFI A of Γ-ring M is said to be an IFCI if $A^f(x) = A(x), \forall x \in M$ and for all $f \in Aut(M)$, i.e., $\mu_{A^f}(x) = \mu_A(x)$ and $\nu_{A^f}(x) = \nu_A(x)$ for all $x \in M$ and for all $f \in Aut(M)$.*

Example 3.5. Consider the Γ-ring M, where $M = \mathbb{Z}$, the ring of integers and $\Gamma = 2\mathbb{Z}$, the ring of even integers and $x\gamma y$ denote the usual product of integers $x, y \in M, \gamma \in \Gamma$.

Let $A = (\mu_A, \nu_A)$ be an intuitionistic fuzzy subset of M defined by

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \text{ is even integer} \\ 0.5, & \text{if } x \text{ is odd integer} \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \text{ is even integer} \\ 0.3, & \text{if } x \text{ is odd integer.} \end{cases}$$

Then it is easy to verify that A is an IFCI of Γ-ring M.

Example 3.6. Consider the Γ-ring M, where $M = \{[a_{ij}] : a_{ij} \in \mathbb{Z}, i = 1, 2, j = 1, 2, 3\}$, the set of (2×3) matrices and $\Gamma = \{[a_{ij}] : a_{ij} \in \mathbb{Z}, i = 1, 2, 3, j = 1, 2\}$, the set of (3×2) matrices whose entries are from the ring of integers $\mathbb{Z}$. Let

$A = (\mu_A, \nu_A)$ be an IFS of M defined by

$$A([a_{ij}]) = \begin{cases} (0.7, 0.2), & \text{if } a_{ij} = 0, \forall i, j \\ (0.3, 0.5), & \text{if } a_{ij} \neq 0 \text{ for atleast one } i \text{ and } j \end{cases}$$

Then it is easy to verify that A is an IFCI of Γ-ring M.

Example 3.7. Consider $M = \mathbf{Z}_2 \times \mathbf{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$, $\Gamma = \{(0,0), (1,1)\}$ and $K = \mathbf{Z}_2 \times \{0\} = \{(1,0), (0,0)\}$, where $\mathbf{Z}_2$ be the ring of integers modulo 2. Clearly, M and Γ are additive abelian groups and that M is Γ-ring. Also, here K is Γ-ideal of M. Consider the IFS A defined on M as

$$\mu_A(x) = \begin{cases} 1, & \text{if } x \in K \\ 0.5, & \text{if } x \notin K \end{cases}; \quad \nu_A(x) = \begin{cases} 0, & \text{if } x \in K \\ 0.3, & \text{if } x \notin K. \end{cases}$$

Then it is easy to check that A is an IFI of Γ-ring M, but it is not an IFCI, as there exists a Γ-automorphism $f : M \to M$ defined by $f(x, y) = (y, x)$, for all $(x, y) \in M$ such that $A^f((x, y)) \neq A((x, y))$, for all $(x, y) \in M$.

For example: $A^f((1, 0)) = (0.5, 0.3) \neq (1, 0) = A((1, 0))$.

Theorem 3.8. *Let A be an IFCI of Γ-ring M. Then for each $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$ the level cut set $A_{(\alpha,\beta)}$ is a characteristic ideal of Γ-ring M.*

Proof. Assume that A be an IFCI of Γ-ring M. It is sufficient to show that $f(A_{(\alpha,\beta)}) = A_{(\alpha,\beta)}$ for all $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$.

Let $x \in A_{(\alpha,\beta)}$. Since A be an IFCI of Γ-ring M, we have $\mu_{A^f}(x) = \mu_A(x) \geq \alpha$ and $\nu_{A^f}(x) = \nu_A(x) \leq \beta$ implies $\mu_A(f(x)) \geq \alpha$ and $\nu_A(f(x)) \leq \beta$, i.e., $f(x) \in A_{(\alpha,\beta)}$. Thus $f(A_{(\alpha,\beta)}) \subseteq A_{(\alpha,\beta)}$.

For the reverse inclusion, let $y \in A_{(\alpha,\beta)}$ and let $x \in M$ be such that $f(x) = y$. Then
$\mu_A(x) = \mu_{A^f}(x) = \mu_A(f(x)) = \mu_A(y) \geq \alpha$. Similarly, we can prove $\nu_A(x) \leq \beta$ implies $x \in A_{(\alpha,\beta)}$ and so $y = f(x) \in f(A_{(\alpha,\beta)})$ gives that $A_{(\alpha,\beta)} \subseteq f(A_{(\alpha,\beta)})$. Thus $f(A_{(\alpha,\beta)}) = A_{(\alpha,\beta)}$. Hence $A_{(\alpha,\beta)}$ is a characteristic ideal of Γ-ring M.

The following lemma is obvious, and we omit the proof.

Lemma 3.9. *Let A be an IFI of Γ-ring M and let $x \in M$. Then $A(x) = (\alpha, \beta)$ if and only if $x \in A_{(\alpha,\beta)}$ and $x \notin A_{(p,q)}$ for all $p > \alpha$ and $q < \beta$.*

Now we give the converse of the Theorem (3.8).

Theorem 3.10. *Let A be an IFI of Γ-ring M. If for each $\alpha, \beta \in [0, 1]$ such that $\alpha + \beta \leq 1$ the level cut set $A_{(\alpha,\beta)}$ is a characteristic ideal of M, then A is an IFCI of Γ-ring M.*

Proof. Let A be an IFI of Γ-ring M. Let $x \in M$, $f \in Aut(M)$ and $A(x) = (\alpha, \beta)$. By Lemma (3.9), $x \in A_{(\alpha,\beta)}$ and $x \notin A_{(p,q)}$ for all $p > \alpha$ and $q < \beta$.
From hypothesis it follows that $f(A_{(\alpha,\beta)}) = A_{(\alpha,\beta)}$. Thus $f(x) \in f(A_{(\alpha,\beta)}) = A_{(\alpha,\beta)}$, and so $\mu_A(f(x)) \geq \alpha$, $\nu_A(f(x)) \leq \beta$.

Let $\mu_A(f(x)) = p$ and $\nu_A(f(x)) = q$ and we assume that $p > \alpha$ and $q < \beta$. Then $f(x) \in A_{(p,q)} = f(A_{(p,q)})$. Since f is one to one implies $x \in A_{(p,q)}$. This is a contradiction.

Hence $\mu_{A^f}(x) = \mu_A(f(x)) = \alpha = \mu_A(x)$ and $\nu_{A^f}(x) = \nu_A(f(x)) = \beta = \nu_A(x)$, showing that A is an IFCI of Γ-ring M.

Theorem 3.11. *A non-empty subset K of a Γ-ring M is a characteristic ideal of M iff its IFCF χ_K is an IFCI of Γ-ring M.*
Proof. Let K be a characteristic ideal of Γ-ring M. Then by definition $f(K) = K, \forall f \in Aut(M)$. Let χ_K be the IFCF with respect to K. Then by Theorem (2.7) χ_K be an IFI of Γ-ring M. Also,
If $x \in K$ then $\forall f \in Aut(M)$, we have $f(x) \in f(K) = K$ and so $\chi_K(f(x)) = (1, 0) = \chi_K(x)$.
If $x \notin K$ then $\forall f \in Aut(M)$, we have $f(x) \notin f(K) = K$ and so $\chi_K(f(x)) = (0, 1) = \chi_K(x)$.

Thus we see that $\chi_K(f(x)) = \chi_K(x), \forall x \in M, \forall f \in Aut(M)$, i.e., $\mu_{\chi_K^f}(x) = \mu_{\chi_K}(x)$ and $\nu_{\chi_K^f}(x) = \nu_{\chi_K}(x)$, $\forall x \in M, \forall f \in Aut(M)$. Hence χ_K is an IFCI of Γ-ring M.

Conversely, let us suppose that χ_K be an IFCI of Γ-ring M. Then by Theorem (2.7) K is an Γ-ideal of M. So we need only to show that $f(K) = K.\forall f \in Aut(M)$. Let $f \in Aut(M)$ and $x \in K$, then $\mu_{\chi_K^f}(x) = \mu_{\chi_K}(x) = 1$ and $\nu_{\chi_K^f}(x) = \nu_{\chi_K}(x) = 0$ implies $\mu_{\chi_K}(f(x)) = 1$ and $\nu_{\chi_K}(f(x)) = 0$ implies $f(x) \in K$. Thus we obtain $f(K) \subseteq K$, for all $f \in Aut(M)$. Since $f \in Aut(M)$ implies $f^{-1} \in Aut(M)$ and so $f^{-1}(K) \subseteq K$. Hence $K \subseteq f(K)$ and so $f(K) = K$, i.e., K is characteristic ideal of M.

4. Operator rings and corresponding intuitionistic fuzzy ideals of Γ-ring

Definition 4.1. ([6, 9]) *Let M be a Γ-ring. Let us signify a relation σ on $M \times \Gamma$ as follows:*

$$(x, \alpha)\sigma(y, \beta) \text{ if and only if } x\alpha m = y\beta m, \forall m \in M \text{ and } \gamma x\alpha = \gamma y\beta, \forall \gamma \in \Gamma.$$

Thus σ is an equivalence relation on $M \times \Gamma$. Set $[x, \alpha]$ be the equivalence class containing (x, α). Let $L = \{[x, \alpha] : x \in M, \alpha \in \Gamma\}$. Then L is a ring with respect to the compositions

$$[x,\alpha]+[y,\alpha]=[x+y,\alpha]\ ;\ [x,\alpha]+[x,\beta]=[x,\alpha+\beta]\ ;$$
$$\sum_i[x_i,\alpha_i]\sum_j[y_j,\beta_j]=\sum_{i,j}[x_i\alpha_i y_j,\beta_j].$$

This ring L is called the left operator ring of Γ-ring M. Dually the right operator ring R of Γ-ring M is formed where the compositions on R are defined as:

$$[\alpha,x]+[\beta,x]=[\alpha+\beta,x];\ [\alpha,x]+[\alpha,y]=[\alpha,x+y];$$
$$\sum_i[\alpha_i,x_i]\sum_j[\beta_j,y_j]=\sum_{i,j}[\alpha_i,x_i\beta_j y_j].$$

Remark 4.2.

(1) *If there exists an element $1_L=\sum_i[e_i,\delta_i]\in L$ (or $1_R=\sum_i[\gamma_i,a_i]\in R$) such that $\sum_i e_i\delta_i x=x$ (resp. $\sum_i x\gamma_i a_i=x$) for all $x\in M$ then $\sum_i[e_i,\delta_i]$ (resp. $\sum_i[\gamma_i,a_i]$) is called the left (resp. right) unity of M. Also $1_L=\sum_i[e_i,\delta_i]$ (resp. $1_R=\sum_i[\gamma_i,a_i]$) is the unity of L (resp. R).*

(2) *If we define a mapping $L\times M\to M$ by $(\sum_i[x_i,\alpha_i],y)\to\sum_i x_i\alpha_i y$, then we can show that the above mapping is well defined and M is a left L-module, and we call L the left operator ring of the Γ-ring M. Similarly, we can construct a right operator ring R of M so that M is a right R-module.*

Let M be a Γ-ring with left operator ring L. For $P\subseteq L$ and $Q\subseteq M$, we define $P^+=\{x\in M:[x,\alpha]\in P,\forall\alpha\in\Gamma\}$ and $Q^{+'}=\{[x,\alpha]\in L: x\alpha y\in Q,\forall y\in M\}$. Similarly, if M is a Γ-ring with and right operator ring R. For $P\subseteq R$ and $Q\subseteq M$, we define
$P^*=\{x\in M:[\alpha,x]\in P,\forall\alpha\in\Gamma\}$ and $Q^{*'}=\{[\alpha,x]\in R: y\alpha x\in Q,\forall y\in M\}$

Then in [6], it was shown that if P (resp. Q) is a right ideal of L (resp. M), then P^+ (resp. $Q^{+'}$) is a right ideal of M (resp. L) and there exists an inclusion preserving mapping $Q\to Q^{+'}$. Also if P (resp. Q) is a left ideal of R (resp. M), then P^* (resp. $Q^{*'}$) is a left ideal of M (resp. R) and there exists an inclusion preserving mapping $Q\to Q^{*'}$.

Definition 4.3. *Let M be a Γ-ring and L be the left operator ring of M. Then the bijection $f:L\to L$ is said to be automorphism if*

1. *$f([x,\alpha]+[y,\alpha])=f([x,\alpha])+f([y,\alpha])$ and $f([x,\alpha]+[x,\beta])=f([x,\alpha])+f([x,\beta])$,*

2. *$f(\sum_i[x_i,\alpha_i]\sum_j[y_j,\beta_j])=f(\sum_i[x_i,\alpha_i])f(\sum_j[y_j,\beta_j])$,*

3. *$f(\sum_i[e_i,\delta_i])=\sum_i[e_i,\delta_i]$, if $\sum_i[e_i,\delta_i]$ is the left unity of M,*

4. $f(\sum_i[a_i,\gamma_i]) = \sum_i[a_i,\gamma_i]$, *if* $\sum_i[a_i,\gamma_i]$ *is the right unity of* M.

Similarly we can define the automorphism on the right operator ring R *of the* Γ*-ring* M.

Proposition 4.4. ([10]) *Every left (or right) ideal of* Γ*-ring* M *defines a left (or right) ideal of the right operator ring* R *and conversely.*

Definition 4.5. *Let* L *and* R *be respectively be the left and right operator ring of* Γ*-ring* M. *Then for any fixed IFS* A *of* L *(or* R*) and for any fixed IFS* B *of* M *we define intuitionistic fuzzy sets* A^+, A^* *of* M *and* $B^{+'}$ *of* L, $B^{*'}$ *of* R *by*

$\mu_{A^+}(x) = Inf_{\alpha\in\Gamma}(\mu_A([x,\alpha]))$ *and* $\nu_{A^+}(x) = Sup_{\alpha\in\Gamma}(\mu_A([x,\alpha]))$, *where* $x \in M$.

$\mu_{A^*}(x) = Inf_{\alpha\in\Gamma}(\mu_A([\alpha,x]))$ *and* $\nu_{A^*}(x) = Sup_{\alpha\in\Gamma}(\mu_A([\alpha,x]))$, *where* $x \in M$.

$\mu_{B^{+'}}(\sum_i[x_i,\alpha_i]) = Inf_{m\in M}(\mu_B(\sum_i x_i\alpha_i m))$ *and* $\nu_{B^{+'}}(\sum_i[x_i,\alpha_i])$
$= Sup_{m\in M}(\mu_B(\sum_i x_i\alpha_i m))$, *where* $[x_i,\alpha_i] \in L$.

$\mu_{B^{*'}}(\sum_i[\alpha_i,x_i]) = Inf_{m\in M}(\mu_B(\sum_i m\alpha_i x_i))$ *and* $\nu_{B^{*'}}(\sum_i[\alpha_i,x_i])$
$= Sup_{m\in M}(\mu_B(\sum_i m\alpha_i x_i))$, *where* $[\alpha_i,x_i] \in R$.

Proposition 4.6. *Let* M *be a* Γ*-ring and* L *be the left operator ring of* M *and* A *is an IFI of* L. *Then* A^+ *is an IFI of* M.

Proof. Let A is an IFI of L. Then $\mu_A(0_L) = 1, \nu_A(0_L) = 0$.
Now $\mu_{A^+}(0_M) = Inf_{\alpha\in\Gamma}(\mu_A([0_M,\alpha])) = Inf_{\alpha\in\Gamma}(\mu_A(0_L)) = 1$. Similarly, we can show that $\nu_{A^+}(0_M) = 0$. So A^+ is non-empty.
Let $x, y, m \in M, \alpha, \beta \in \Gamma$ be any elements, then we have

$$\begin{aligned}
\mu_{A^+}(x-y) &= Inf_{\alpha\in\Gamma}(\mu_A([x-y,\alpha])) \\
&= Inf_{\alpha\in\Gamma}(\mu_A([x,\alpha]-[y,\alpha])) \\
&\geq Inf_{\alpha\in\Gamma}\{\mu_A([x,\alpha]) \wedge \mu_A([y,\alpha])\} \\
&= Inf_{\alpha\in\Gamma}(\mu_A([x,\alpha])) \wedge Inf_{\alpha\in\Gamma}(\mu_A([y,\alpha])) \\
&= \mu_{A^+}(x) \wedge \mu_{A^+}(y).
\end{aligned}$$

Thus $\mu_{A^+}(x-y) \geq \mu_{A^+}(x) \wedge \mu_{A^+}(y)$. Similarly, we can prove $\nu_{A^+}(x-y) \leq \nu_{A^+}(x) \vee \nu_{A^+}(y)$. Also,

$$\begin{aligned}
\mu_{A^+}(x\beta y) &= Inf_{\alpha\in\Gamma}(\mu_A([x\beta y,\alpha])) \\
&= Inf_{\alpha\in\Gamma}(\mu_A([x,\beta][y,\alpha])) \\
&\geq Inf_{\alpha\in\Gamma}(\mu_A([x,\beta]))[\text{ and } \geq Inf_{\alpha\in\Gamma}(\mu_A([y,\alpha]))] \\
&= Inf_{\beta\in\Gamma}(\mu_A([x,\beta])) \vee Inf_{\alpha\in\Gamma}(\mu_A([y,\alpha])) \\
&= \mu_{A^+}(x) \vee \mu_{A^+}(y).
\end{aligned}$$

Thus $\mu_{A^+}(x\alpha y) \geq \mu_{A^+}(x) \vee \mu_{A^+}(y)$. Similarly, we can prove $\nu_{A^+}(x\alpha y) \leq \nu_{A^+}(x) \wedge \nu_{A^+}(y)$. Hence A^+ is an IFI of M.

Proposition 4.7. *Let M be a Γ-ring and L be the left operator ring of M and B is an IFI of M. Then $B^{+'}$ is an IFI of L.*

Proof. Let B is an IFI of M. Then $\mu_B(0_M) = 1, \nu_B(0_M) = 0$.
Now $\mu_{B^{+'}}([0_M, \alpha]) = Inf_{m\in M}(\mu_B(0_M\alpha m)) = \mu_B(0_M) = 1$. Similarly, we can show that $\nu_{B^{+'}}([0_M, \alpha]) = 0$. So $B^{+'}$ is non-empty.
Let $\sum_i[x_i, \alpha_i], \sum_j[y_j, \beta_j] \in L$, $m \in M, \alpha_i, \beta_j \in \Gamma$ be any elements, then we have

$$\begin{aligned}
\mu_{B^{+'}}(\sum_i[x_i,\alpha_i] - \sum_j[y_j,\beta_j]) &= Inf_{m\in M}(\mu_B(\sum_i x_i\alpha_i m - \sum_j y_j\beta_j m)) \\
&\geq Inf_{m\in M}\{\mu_B(\sum_i x_i\alpha_i m) \wedge \mu_B(\sum_j y_j\beta_j m)\} \\
&= (Inf_{m\in M}(\mu_B(\sum_i x_i\alpha_i m))) \wedge (Inf_{m\in M}(\mu_B(\sum_j y_j\beta_j m))) \\
&= \mu_{B^{+'}}(\sum_i[x_i,\alpha_i]) \wedge \mu_{B^{+'}}(\sum_j[y_j,\beta_j]).
\end{aligned}$$

Thus $\mu_{B^{+'}}(\sum_i[x_i, \alpha_i] - \sum_j[y_j, \beta_j]) \geq \mu_{B^{+'}}(\sum_i[x_i, \alpha_i]) \wedge \mu_{B^{+'}}(\sum_j[y_j, \beta_j])$. Similarly, we can show $\nu_{B^{+'}}(\sum_i[x_i, \alpha_i] - \sum_j[y_j, \beta_j]) \leq \nu_{B^{+'}}(\sum_i[x_i, \alpha_i]) \vee \mu_{B^{+'}}(\sum_j[y_j, \beta_j])$ Also.

$$\begin{aligned}
\mu_{B^{+'}}(\sum_i[x_i,\alpha_i]\sum_j[y_j,\beta_j]) &= \mu_{B^{+'}}(\sum_{i,j}[x_i\alpha_i y_j,\beta_j]) \\
&= Inf_{m\in M}(\mu_B(\sum_{i,j} x_i\alpha_i y_j\beta_j m)) \\
&= Inf_{m\in M}(\mu_B(\sum_{i,j}(x_i\alpha_i)(y_j\beta_j m))) \\
&= Inf_{m'_j\in M}(\mu_B(\sum_{i,j} x_i\alpha_i m'_j))[\text{ where } m'_j = y_j\beta_j m \in M] \\
&= Inf_{m'_j\in M}[\mu_B(\sum_i x_i\alpha_i m'_1 + \sum_i x_i\alpha_i m'_2 + ...)] \\
&\geq Inf_{m'_j\in M}[\vee_j \mu_B(\sum_i x_i\alpha_i m'_j)] \\
&= \vee_j[Inf_{m'_j\in M}(\sum_i x_i\alpha_i m'_j)] \\
&= \vee_j[\mu_{B^{+'}}(\sum_i[x_i,\alpha_i])]
\end{aligned}$$

$$= \mu_{B^{+'}}(\sum_i [x_i, \alpha_i]).$$

Also, we can prove that $\mu_{B^{+'}}(\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j]) \geq \mu_{B^{+'}}(\sum_j [y_j, \beta_j])$. Thus we have
$\mu_{B^{+'}}(\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j]) \geq \mu_{B^{+'}}(\sum_i [x_i, \alpha_i]) \vee \mu_{B^{+'}}(\sum_j [y_j, \beta_j])$. Similarly, we can prove
$\nu_{B^{+'}}(\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j]) \leq \nu_{B^{+'}}(\sum_i [x_i, \alpha_i]) \wedge \nu_{B^{+'}}(\sum_j [y_j, \beta_j])$. Hence $B^{+'}$ is an IFI of L.

Similarly, we can prove the following propositions.

Proposition 4.8. *Let M be a Γ-ring and R be the right operator ring of M and A an IFI of R. Then A^* an IFI of M.*

Proposition 4.9. *Let M be a Γ-ring and R be the right operator ring of M and B an IFI of M. Then $B^{*'}$ an IFI of R.*

Theorem 4.10. *Let M be a Γ-ring with unities and L be its left operator ring. Then ∃ an inclusion preserving one to one map $A \to A^{+'}$ between the set of all intuitionistic fuzzy ideals of M and the set of intuitionistic fuzzy ideals of L.*
Proof. First we show that $((A^+)')^+ = A$, where A is an IFI of M. Let $x \in M$. Then

$$\begin{aligned}
\mu_{((A^+)')^+}(x) &= Inf_{\alpha\in\Gamma}(\mu_{(A^+)'}([x,\alpha])) \\
&= Inf_{\alpha\in\Gamma}[Inf_{m\in M}(\mu_A(x\alpha m))] \\
&\geq Inf_{\alpha\in\Gamma}[Inf_{m\in M}(\mu_A(x))] \\
&= \mu_A(x).
\end{aligned}$$

Thus $\mu_{((A^+)')^+}(x) \geq \mu_A(x)$. Similarly, we can prove $\nu_{((A^+)')^+}(x) \leq \nu_A(x)$. Thus $A \subseteq ((A^+)')^+$.
Let $\sum_i [\gamma_i, a_i]$ be the right unity of M. Then $\sum_i x\gamma_i a_i = x$, for all $x \in M$. Now,

$$\begin{aligned}
\mu_A(x) &= \mu_A(\sum_i x\gamma_i a_i) \\
&\geq Inf_i[\mu_A(x\gamma_i a_i)] \\
&\geq Inf_{\gamma\in\Gamma}[Inf_{m\in M}(\mu_A(x\gamma m))] \\
&= Inf_{\gamma\in\Gamma}(\mu_{(A^+)'}([x,\gamma])) \\
&= \mu_{((A^+)')^+}(x).
\end{aligned}$$

Similarly, we can prove $\nu_A(x) \leq \nu_{((A^+)')^+}(x)$. So $((A^+)')^+ \subseteq A$. Hence $A = ((A^+)')^+$.
Again, let A be an IFI of L. Now,

$$\begin{aligned}
\mu_{((A^+)^+)'}(\sum_i [x_i, \alpha_i]) &= Inf_{m\in M}(\mu_{A^+}(\sum_i x_i\alpha_i m)) \\
&= Inf_{m\in M}[Inf_{\beta\in\Gamma}(\mu_A([\sum_i x_i\alpha_i m, \beta]))] \\
&= Inf_{m\in M}[Inf_{\beta\in\Gamma}(\mu_A(\sum_i [x_i, \alpha_i][m, \beta]))] \\
&\geq \mu_A(\sum_i [x_i, \alpha_i]).
\end{aligned}$$

Thus $\mu_{((A^+)^+)'}(\sum_i [x_i, \alpha_i] \geq \mu_A(\sum_i [x_i, \alpha_i])$. Similarly, we can prove $\nu_{((A^+)^+)'}(\sum_i [x_i, \alpha_i] \leq \nu_A(\sum_i [x_i, \alpha_i])$. So $A \subseteq ((A^+)^+)'$.
Let $\sum_j [a_j, \gamma_j]$ be the right unity of M, then

$$\begin{aligned}
\mu_A(\sum_i [x_i, \alpha_i]) &= \mu_A(\sum_i [x_i, \alpha_i] \sum_j [a_j, \gamma_j]) \\
&\geq \wedge_j[\mu_A(\sum_i [x_i, \alpha_i][a_j, \gamma_j])]] \\
&\geq Inf_{m\in M}[Inf_{\gamma\in\Gamma}(\mu_A([x_i, \alpha_i][a_j, \gamma_j]))] \\
&= \mu_{((A^+)^+)'}(\sum_i [x_i, \alpha_i]).
\end{aligned}$$

Thus $\mu_A(\sum_i [x_i, \alpha_i]) \geq \mu_{((A^+)^+)'}(\sum_i [x_i, \alpha_i])$. Similarly, we can prove $\nu_A(\sum_i [x_i, \alpha_i]) \leq \nu_{((A^+)^+)'}(\sum_i [x_i, \alpha_i])$ and so $((A^+)^+)' \subseteq A$ and hence $A = ((A^+)^+)'$.
Thus the correspondence $A \to A^{+'}$ is a bijection. Now let A_1, A_2 be intuitionistic fuzzy ideals of M such that $A_1 \subseteq A_2$. Then for all $\sum_i [x_i, \alpha_i] \in L$, we have

$$\begin{aligned}
\mu_{A_1^{+'}}(\sum_i [x_i, \alpha_i]) &= Inf_{m\in M}(\mu_{A_1}(\sum_i x_i\alpha_i m)) \\
&\leq Inf_{m\in M}(\mu_{A_2}(\sum_i x_i\alpha_i m)) \\
&= \mu_{A_2^{+'}}(\sum_i [x_i, \alpha_i]).
\end{aligned}$$

Thus $\mu_{A_1^{+'}}(\sum_i [x_i, \alpha_i]) \leq \mu_{A_2^{+'}}(\sum_i [x_i, \alpha_i])$. Similarly, we can show $\nu_{A_1^{+'}}(\sum_i [x_i, \alpha_i]) \geq \nu_{A_2^{+'}}(\sum_i [x_i, \alpha_i])$. Thus $A_1^{+'} \subseteq A_2^{+'}$. Similarly we can show that if A_1, A_2 are intuitionistic fuzzy ideals of L such that $A_1 \subseteq A_2$, then $A_1^+ \subseteq A_2^+$. Hence $A \to A^{+'}$ is an inclusion preserving one to one map.

Similarly, we can prove the following theorem.

Theorem 4.11. *Let M be a Γ-ring with unities and R be its right operator ring. Then $\exists$ an inclusion preserving one to one map $B \to B^{*'}$ between the set of all IFIs of M and the set of IFIs of R.*

Proposition 4.12. *Let K be an ideal of the left operator ring L of a Γ-ring M. Then $(\chi_K)^+ = \chi_{K^+}$, where χ_K denote the IFCF of K.*

Proof. Let $x \in K^+$. Then $[x, \alpha] \in K$ for all $\alpha \in \Gamma$. This mean $Inf_{\alpha\in\Gamma}(\mu_{\chi_K}([x,\alpha])) = 1$ and $Sup_{\alpha\in\Gamma}(\nu_{\chi_K}([x,\alpha])) = 0$. Also $\mu_{\chi_{K^+}}(x) = 1$ and $\nu_{\chi_{K^+}}(x) = 0$. Thus $Inf_{\alpha\in\Gamma}(\mu_{\chi_K}([x,\alpha])) = \mu_{\chi_{K^+}}(x)$ and $Sup_{\alpha\in\Gamma}(\nu_{\chi_K}([x,\alpha])) = \nu_{\chi_{K^+}}(x), \forall x \in K^+$, i.e., $(\chi_K)^+(x) = \chi_{K^+}(x), \forall x \in K^+$.

Now suppose $x \notin K^+$. Then $\exists \beta \in \Gamma$ such that $[x, \beta] \notin K$. Hence $\mu_{\chi_K}([x,\beta]) = 0, \nu_{\chi_K}([x,\beta]) = 1$ and so $Inf_{\alpha\in\Gamma}(\mu_{\chi_K}([x,\alpha])) = 0$ and $Sup_{\alpha\in\Gamma}(\nu_{\chi_K}([x,\alpha])) = 1$. Thus $Inf_{\alpha\in\Gamma}(\mu_{\chi_K}([x,\alpha])) = \mu_{\chi_{K^+}}(x)$ and $Sup_{\alpha\in\Gamma}(\nu_{\chi_K}([x,\alpha])) = \nu_{\chi_{K^+}}(x), \forall x \notin K^+$, i.e., $(\chi_K)^+(x) = \chi_{K^+}(x), \forall x \notin K^+$. Hence $(\chi_K)^+ = \chi_{K^+}$.

By applying similar argument as above we deduce the following Lemma.

Lemma 4.13. *Let K be an ideal of a Γ-ring M and L be the left operator ring of M. Then $(\chi_K)^{+'} = \chi_{K^{+'}}$.*

Proof. Let $\sum_i [x_i, \alpha_i] \in K^{+'}$. Then $\sum_i x_i\alpha_i m \in K, \forall m \in M$.
This means $Inf_{m\in M}\mu_{\chi_K}(\sum_i x_i\alpha_i m) = 1$ and $Sup_{m\in M}\nu_{\chi_K}(\sum_i x_i\alpha_i m) = 0$,
i.e., $\mu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i]) = 1$ and $\nu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i]) = 0$.

Also $\mu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = 1$ and $\nu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = 0$. Thus we have $\mu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = \mu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i])$ and $\nu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = \nu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i])$.
So $(\chi_K)^{+'}(\sum_i[x_i,\alpha_i]) = (\chi_{K^{+'}})(\sum_i[x_i,\alpha_i])$.

Let $\sum_i[x_i,\alpha_i] \notin K^{+'}$. Then $\sum_i x_i\alpha_i m \notin K, \forall m \in M$.
This means $Inf_{m\in M}\mu_{\chi_K}(\sum_i x_i\alpha_i m) = 0$ and $Sup_{m\in M}\nu_{\chi_K}(\sum_i x_i\alpha_i m) = 1$,
i.e., $\mu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i]) = 0$ and $\nu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i]) = 1$.

Also $\mu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = 0$ and $\nu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = 1$. Thus we have $\mu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = \mu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i])$ and $\nu_{(\chi_{K^{+'}})}(\sum_i[x_i,\alpha_i]) = \nu_{(\chi_K)^{+'}}(\sum_i[x_i,\alpha_i])$.
So $(\chi_K)^{+'}(\sum_i[x_i,\alpha_i]) = (\chi_{K^{+'}})(\sum_i[x_i,\alpha_i])$.

Thus from both the cases we get $(\chi_K)^{+'} = \chi_{K^{+'}}$.

Remark 4.14. *By drawing an analogy we can deduce results similar to the above*

Lemmas for right operator ring R of the Γ-ring M.

Theorem 4.15. *Let M be a Γ-ring with unities. Then $\exists$ an inclusion preserving one to one between the set of all ideals of M and that of its left operator ring L via the mapping $K \to K^{+'}$.*

Proof. Let $\phi : K \to K^{+'}$ be the mapping. This is actually a mapping follows from Proposition (4.9). Now let $\phi(K_1) = \phi(K_2)$. Then $K_1^{+'} = K_2^{+'}$. This implies $\chi_{K_1^{+'}} = \chi_{K_2^{+'}}$ (where χ_K is the IFCF of K). Hence by Lemma (4.13), $(\chi_{K_1})^{+'} = (\chi_{K_2})^{+'}$. This together with Theorem (4.10) gives $\chi_{K_1} = \chi_{K_2}$, hence $K_1 = K_2$. Consequently ϕ is one to one.

Let K be an ideal of L. Then its IFCF χ_K is an IFI of L. Hence by Theorem (4.10), $((\chi_K)^+)^{+'} = \chi_K$. This implies that $\chi_{(K^+)^{+'}} = \chi_K$ [by Lemma (4.12) and (4.13)]. Hence $(K^+)^{+'} = K$, i.e., $\phi(K^+) = K$. Now since K^+ is an ideal of M, it follows that ϕ is onto. Let K_1, K_2 be two ideals of M with $K_1 \subseteq K_2$. Then $\chi_{K_1} \subseteq \chi_{K_2}$. Hence by Theorem (4.10) we see that $(\chi_{K_1})^{+'} \subseteq (\chi_{K_2})^{+'}$, i.e., $\chi_{K_1^{+'}} \subseteq \chi_{K_2^{+'}}$ [by Lemma (4.13)] which gives $K_1^{+'} \subseteq K_2^{+'}$.

Remark 4.16. *Now by using a similar argument as above with the help of Lemma dual to Lemmas (4.12) and (4.13), Remark (4.14) and Theorem (4.12) we can deduce that $()^{*'}$ is an inclusion preserving one-to-one map (with $()^*$ as above) between the set of all ideals of M and that of its right operator ring R.*

Definition 4.17. *Let M be a Γ-ring and L be its left operator ring. Then for $f \in Aut(M)$, we define $f^{+'} : L \to L$ by $f^{+'}(\sum_i [x_i, \alpha_i]) = \sum_i [f(x_i), \alpha_i]$.*

We first show that the map $f^{+'}$ is well-defined.
Suppose $\sum_i [x_i, \alpha_i] = \sum_j [y_j, \beta_j]$, then $[x_i, \alpha_i] = [y_j, \beta_j]$, so, $x_i \alpha_i m = y_j \beta_j m, \forall m \in M$. Thus $\sum_i x_i \alpha_i m = \sum_j y_j \beta_j m$. This implies $f(\sum_i x_i \alpha_i m) = f(\sum_j y_j \beta m), \forall m \in M$.
Now for $a \in M$, we have $f(x_i)\alpha_i a = f(x_i)\alpha_i f(a')$ [As f is onto so there exists $a' \in M$ such that $f(a') = a$] $= f(x_i \alpha_i a') = f(y_j \beta_j a') = f(y_j)\beta_j f(a') = f(y_j)\beta_j a$. This implies $f(x_i)\alpha_i a = f(y_j)\beta_j a$. So $[f(x_i), \alpha_i] = [f(y_j), \beta_j] \Rightarrow \sum_i [f(x_i), \alpha_i] = \sum_j [f(y_j), \beta_j]$. Hence $f^{+'}(\sum_i [x_i, \alpha_i]) = f^{+'}(\sum_j [y_j, \beta_j])$. Therefore the map $f^{+'}$ is well-defined.

Proposition 4.18. *Let M be a Γ-ring and L be its left operator ring. Let $f \in Aut(M)$. Then $f^{+'} \in Aut(L)$.*
Proof. Let $f \in Aut(M)$ and $[x, \alpha], [y, \alpha], [x, \beta] \in L$. Then

$f^{+'}([x,\alpha]+[y,\alpha]) = f^{+'}([x+y,\alpha]) = [f(x+y),\alpha] = [f(x)+f(y),\alpha] = [f(x),\alpha] + [f(y),\beta]$
$f^{+'}([x,\alpha]+[x,\beta]) = f^{+'}([x,\alpha+\beta]) = [f(x),\alpha+\beta] = [f(x),\alpha]+[f(x),\beta]$.

$$\begin{aligned} f^{+'}(\sum_i [x_i,\alpha_i]\sum_j [y_j,\beta_j]) &= f^{+'}(\sum_{i,j}[x_i\alpha_i y_j,\beta_j]) \\ &= \sum_{i,j}[f(x_i\alpha_i y_j),\beta_j] \\ &= \sum_{i,j}[f(x_i)\alpha_i f(y_j),\beta_j] \\ &= \sum_i [f(x_i),\alpha_i]\sum_j [f(y_j),\beta_j] \\ &= f^{+'}(\sum_i [x_i,\alpha_i]) f^{+'}(\sum_j [y_j,\beta_j] \end{aligned}$$

Hence $f^{+'}$ is an endomorphism of L. As $f^{+'}$ is well-defined implies $f^{+'}$ is one to one map.
Further, let $\sum_i [x_i,\alpha_i] \in L$. Then $\exists$, $x_i' \in M$ such that $f(x_i') = x_i$. So $\sum_i [x_i',\alpha_i] \in L$ such that $f^{+'}(\sum_i [x_i',\alpha_i]) = \sum_i [f(x_i'),\alpha_i] = \sum_i [x_i,\alpha_i]$. Consequently, $f^{+'}$ is onto.

Suppose L has the left unity $\sum_i [e_i,\delta_i]$. Then for any $\alpha_i \in \Gamma$, we have $f^{+'}\sum_i [e_i,\alpha_i] = \sum_i [f(e_i),\alpha_i] = \sum_i [e_i,\alpha_i]$. Again if M has the right unity $\sum_i [\gamma_i,a_i]$. Then for any $\alpha_i \in \Gamma$, we have $f^{+'}\sum_i [\gamma_i,\alpha_i] = \sum_i [f(\gamma_i),\alpha_i] = \sum_i [\gamma_i,\alpha_i]$. Hence $f^{+'} \in Aut(L)$.

We use the Remark (4.2)(ii) to frame the following precision and also to demonstrate the subsequent Propositions.

Definition 4.19. *Let M be a Γ-ring with right unity $\sum_i [\gamma_i,a_i]$ and L be its left operator ring. Then for $f \in Aut(L)$, we set $f^+ : M \to M$ by $f^+(x) = \sum_i f([x,\gamma_i])a_i$.*

We first show that the map f^+ is well-defined
Let $x, y \in M, \gamma_i, \beta_i \in \Gamma$ be such that $f^+(x) = f^+(y)$, then $\sum_i f([x,\gamma_i])a_i = \sum_i f([y,\gamma_i])a_i$

$\Rightarrow \sum_i [f([x,\gamma_i])a_i,\gamma_i] = \sum_i [f([y,\gamma_i])a_i,\gamma_i]$
$\Rightarrow \sum_i f([x,\gamma_i])\sum_i [a_i,\gamma_i] = \sum_i f([y,\gamma_i])\sum_i [a_i,\gamma_i]$
$\Rightarrow \sum_i f([x,\gamma_i])f(\sum_i [a_i,\gamma_i]) = \sum_i f([y,\gamma_i])f(\sum_i [a_i,\gamma_i])$ [Using Definition (4.3)]
$\Rightarrow f(\sum_i [x,\gamma_i]\sum_i [a_i,\gamma_i]) = f(\sum_i [y,\gamma_i]\sum_i [a_i,\gamma_i])$
$\Rightarrow f(\sum_i [x\gamma_i a_i,\gamma_i]) = f(\sum_i [y\gamma_i a_i,\gamma_i])$
$\Rightarrow \sum_i [x\gamma_i a_i,\gamma_i] = \sum_i [y\gamma_i a_i,\gamma_i]$ [Since f is one to one]
$\Rightarrow \sum_i [x,\gamma_i]\sum_i [a_i,\gamma_i] = \sum_i [y,\gamma_i]\sum_i [a_i,\gamma_i]$

$\Rightarrow \sum_i [x, \gamma_i] = \sum_i [y, \gamma_i] \Rightarrow [x, \gamma_i] = [y, \gamma_i] \Rightarrow x\gamma_i m = y\gamma_i m, \forall m \in M.$

In particular, take $m = a_i$, we get $\sum_i x\gamma_i a_i = \sum_i y\gamma_i a_i \Rightarrow x = y$. Hence f^+ is well-defined.

Proposition 4.20. *Let M be a Γ-ring with right unity $\sum_i [\gamma_i, a_i]$ and L be its left operator ring. Assume $f \in Aut(L)$, then $f^+ \in Aut(M)$.*

Proof. Let $x, y \in M, \alpha \in \Gamma$. Then

$$\begin{aligned}
f^+(x+y) &= \sum_i f([x+y, \gamma_i])a_i \\
&= \sum_i f([x, \gamma_i] + [y, \gamma_i])a_i \\
&= \sum_i (f([x, \gamma_i])a_i + f([y, \gamma_i])a_i) \\
&= \sum_i f([x, \gamma_i])a_i + \sum_i f([y, \gamma_i])a_i \\
&= f^+(x) + f^+(y). \\
f^+(x\alpha y) &= \sum_i f([x\alpha y, \gamma_i])a_i = \sum_i f([x, \alpha][y, \gamma_i])a_i \\
&= \sum_i f([x, \alpha]) \sum_i f([y, \gamma_i])a_i = \sum_i f([x\gamma_i a_i, \alpha]) \sum_i f([y, \gamma_i])a_i \\
&= \sum_i f([x, \gamma_i][a_i, \alpha]) \sum_i f([y, \gamma_i])a_i = \sum_i f([x, \gamma_i]) \sum_i f([a_i, \alpha]) \sum_i f([y, \gamma_i])a_i \\
&= \sum_i f([x, \gamma_i]) \sum_i [a_i, \alpha] \sum_i f([y, \gamma_i])a_i = \sum_i f([x, \gamma_i])a_i\alpha \sum_i f([y, \gamma_i])a_i \\
&= (\sum_i f([x, \gamma_i])a_i)\alpha(\sum_i f([y, \gamma_i])a_i) \\
&= f^+(x)\alpha f^+(y).
\end{aligned}$$

Hence f^+ is an endomorphism of M. As f^+ is well-defined implies that f^+ is one to one map.

Further, let $y \in M$. Since $f : L \to L$ is onto, $\exists \sum_i [x, \gamma_i] \in L$ such that $f(\sum_i [x, \gamma_i]) = \sum_i [y, \gamma_i]$.

$$\begin{aligned}
f^+(x) &= \sum_i f([x, \gamma_i])a_i = \sum_i f([x\gamma_i a_i, \gamma_i])a_i \\
&= \sum_i f([x, \gamma_i][a_i, \gamma_i])a_i = \sum_i f([x, \gamma_i]) \sum_i f([a_i, \gamma_i])a_i
\end{aligned}$$

$$\begin{aligned} &= \sum_i [y,\gamma_i]\sum_i [a_i,\gamma_i]a_i = \sum_i [y,\gamma_i][a_i,\gamma_i]a_i \\ &= \sum_i [y\gamma_i a_i,\gamma_i]a_i = \sum_i [y,\gamma_i]a_i \\ &= \sum_i y\gamma_i a_i = y. \end{aligned}$$

Hence f^+ is onto. Again if $\sum_i [e_i,\delta_i]$ is the left unity of M then $f^+(e) = \sum_i f([e,\delta_i])a_i = \sum_i [e,\delta_i]a_i = \sum_i e\delta_i a_i = e$. Consequently, $f^+ \in Aut(M)$.

Proposition 4.21. *Let M be a Γ-ring with left unity $\sum_i [e_i,\delta_i]$ and right unity $\sum_i [\gamma_i,a_i]$ and L be its left operator ring. Assume $f \in Aut(L)$, then $(f^{+'})^+ = f$.*

Proof. By Proposition (4.18), $f^{+'} \in Aut(L)$ hence by Proposition (4.20), $(f^{+'})^+ \in Aut(M)$. Let $x \in M$. Then $(f^{+'})^+(x) = f^{+'}(\sum_i [x,\gamma_i])a_i = \sum_i [f(x),\ \gamma_i]a_i = \sum_i f(x)\gamma_i a_i = f(x)$.
Hence $(f^{+'})^+ = f$.

Proposition 4.22. *Let M be a Γ-ring with left unity $\sum_i [e_i,\delta_i]$ and right unity $\sum_i [\gamma_i,a_i]$ and L be its left operator ring. Let $f \in Aut(M)$. Then $(f^+)^{+'} = f$.*

Proof. By Proposition (4.20), $f^+ \in Aut(M)$ whence by Proposition (4.18), $(f^+)^{+'} \in Aut(L)$. Let $\sum_i [x_i,\alpha_i] \in L$. Then

$$\begin{aligned} (f^+)^{+'}(\sum_i [x_i,\alpha_i]) &= \sum_i [f^+(x_i),\alpha_i] = \sum_i [f([x_i,\gamma_i])a_i,\alpha_i] \\ &= \sum_i f([x_i,\gamma_i])\sum_i [a_i,\alpha_i] = \sum_i f([x_i,\gamma_i])f(\sum_i [a_i,\alpha_i]) \\ &= \sum_i f([x_i,\gamma_i][a_i,\alpha_i]) = \sum_i f([x_i\gamma_i a_i,\alpha_i]) = \sum_i f([x_i,\alpha_i]) \\ &= f(\sum_i [x_i,\alpha_i]). \end{aligned}$$

Hence $(f^+)^{+'} = f$.

Theorem 4.23. *Let M be a Γ-ring and L be its left operator ring. Then there exists a bijection between the set of all automorphisms of M and the set of all automorphisms of L.*

Proof. Let us define the map $\phi : Aut(M) \to Aut(L)$ by $\phi(f) = f^{+'}, \forall f \in Aut(M)$. Consider $f,g \in Aut(M)$ such that $\phi(f) = \phi(g)$. Then $f^{+'} = g^{+'}$
$\Rightarrow f^{+'}(\sum_i [x_i,\alpha_i]) = g^{+'}(\sum_i [x_i,\alpha_i]), \forall \sum_i [x_i,\alpha_i] \in L \Rightarrow \sum_i [f(x_i),\alpha_i] = \sum_i [g(x_i),\alpha_i]$

$\Rightarrow f(x_i)\alpha_i m = g(x_i)\alpha_i m, \forall m \in M, \alpha_i \in \Gamma$. In particular, $f(x_i)\gamma_i a_i = g(x_i)\gamma_i a_i \Rightarrow f(x_i) = g(x_i)$. So $f = g$. Hence ϕ is one to one.

Suppose $f \in Aut(M)$. Then by Proposition (4.20), $f^{+'} \in Aut(M)$. Now $\phi(f^+) = f^{+'} = f$ (by Proposition (4.22)). Consequently, ϕ is onto. Hence ϕ is a bijection.

Proposition 4.24. *Let M be a Γ-ring with unities and L be its left operator ring and A be an IFCI of L. Then A^+ is an IFCI of M, where A^+ is explained in Definition (4.5).*

Proof. By Proposition (4.6), A^+ is an IFI of Γ-ring M. Let $x \in M$ and $f \in Aut(M)$. Then by Proposition (4.18), $f^{+'} \in Aut(L)$. Hence by using Definition (4.5) and (4.17) we obtain

$$\begin{aligned}\mu_{(A^+)^f}(x) &= \mu_{A^+}(f(x)) = Inf_{\alpha\in\Gamma}(\mu_A([f(x),\alpha])) \\ &= Inf_{\alpha\in\Gamma}(\mu_A(f^+([x,\alpha]))) = Inf_{\alpha\in\Gamma}(\mu_A([x,\alpha])) \\ &= \mu_{A^+}(x).\end{aligned}$$

Similarly, we can prove $\nu_{(A^+)^f}(x) = \nu_{A^+}(x)$, i.e., $(A^+)^f(x) = A^+(x), \forall f \in Aut(M)$. Hence A^+ is an IFCI of M.

Proposition 4.25. *Let M be a Γ-ring with unities and L be its left operator ring and B be an IFCI of M. Then $B^{+'}$ is an IFCI of L, where $B^{+'}$ is explained in Definition (4.5).*

Proof. By Proposition (4.7), $B^{+'}$ is an IFI of L. Let $\sum_i[x_i,\alpha_i] \in L$ and $g \in Aut(L)$. Then by Theorem (4.23) $\exists, f \in Aut(M)$ such that $f^{+'} = g$. Now

$$\begin{aligned}\mu_{(B^{+'})^g}(\sum_i[x_i,\alpha_i]) &= \mu_{B^{+'}}(g(\sum_i[x_i,\alpha_i])) = \mu_{B^{+'}}(f^{+'}(\sum_i[x_i,\alpha_i])) \\ &= \mu_{B^{+'}}(\sum_i[f(x_i),\alpha_i]) = Inf_{m\in M}(\mu_B(\sum_i f(x_i)\alpha_i m)) \\ &= Inf_{n\in M}(\mu_B(\sum_i f(x_i)\alpha_i f(n)))[\text{ As } f \text{ is a bijection so } f(n) = m\ ] \\ &= Inf_{n\in M}(\mu_B(\sum_i f(x_i\alpha_i n))) = Inf_{n\in M}(\mu_B(\sum_i x_i\alpha_i n)) \\ &\qquad\qquad\qquad\qquad [\text{ As } B \text{ is IFCI of M }] \\ &= \mu_{B^{+'}}(\sum_i[x_i,\alpha_i]).\end{aligned}$$

Similarly, we can prove $\nu_{(B^{+'})^g}(\sum_i[x_i,\alpha_i]) = \nu_{B^{+'}}(\sum_i[x_i,\alpha_i])$, i.e., $(B^{+'})^g(\sum_i[x_i,\alpha_i]) = B^{+'}(\sum_i[x_i,\alpha_i]), \forall g \in Aut(L)$. Hence $B^{+'}$ is an IFCI of L.

Theorem 4.26. *Let M be a Γ-ring with unities and L be its left operator ring. Then $\exists$ a one to one map between the set of all IFCIs of M and the set of all IFCIs of L.*

Proof. Let ϕ be a mapping from the set of all IFCIs of M to that of L. let D be an IFCI of M. Let us define $\phi(D) = D^{+'}$. Then by Proposition (4.25), $\phi(D)$ is an IFCI of L. Let A be an IFCI of L. Then by Proposition (4.24), A^+ is an IFCI of M. Then by Theorem (4.10), $(A^+)^{+'} = A$, i.e., $\phi(A^+) = A$. Thus ϕ is onto. Again if for D_1, D_2 of M such that $\phi(D_1) = \phi(D_2)$ then $D_1^{+'} = D_2^{+'} \Rightarrow (D_1^{+'})^+ = (D_2^{+'})^+ \Rightarrow D_1 = D_2$ (by Theorem (4,10)). Therefore ϕ is one to one, hence the proof.

Proposition 4.27. *Let M be a Γ-ring with left unity $\sum_i [e_i, \delta_i]$, right unity $\sum_i [\gamma_i, a_i]$ and L be its left operator ring. Let K be a characteristic ideal of L. Then K^+ is a characteristic ideal of M.*

Proof. Let $f \in Aut(M)$. Then by Proposition (4.18), $f^{+'} \in Aut(L)$. Hence $f^{+'}(K) = K$. Let $f(x) \in f(K^{+)})$, where $x \in K^+$. Then $[x, \alpha] \in K, \forall \alpha \in \Gamma$. Hence $f^{+'}([x, \alpha]) \in f^{+'}(K), \forall \alpha \in \Gamma \Rightarrow [f(x), \alpha] \in K, \forall \alpha \in \Gamma \Rightarrow f(x) \in K^+$. Thus $f(K^+) \subseteq K^+$. Hence $f^{-1}(K^+) \subseteq K^+$ (since $f \in Aut(M) \Rightarrow f^{-1} \in Aut(M) \Rightarrow K^+ \subseteq f(K^+)$. Hence $f(K^+) = K^+$. Consequently, K^+ is a characteristic ideal of M.

Theorem 4.28. *Let M be a Γ-ring with unities and L be its left operator ring. Then $\exists$ an inclusion preserving one to one between the set of all characteristic ideals of M and the set of all characteristic ideals of L via the mapping $K \to K^{+'}$.*

Proof. Let us denote the mapping $\psi : K \to K^{+'}$. Let K, I be two characteristic ideals of M such that $\psi(K) = \psi(I)$. Then $K^{+'} = I^{+'} \Rightarrow (K^{+'})^+ = (I^{+'})^+ \Rightarrow K = I$. (by Theorem (4.15). So ψ is one-one.

Let K be a characteristic ideal of L, then by proposition (4.27) K^+ is a characteristic ideal of M. Also $(K^{+'})^{+'} = K$. Thus $\psi(K^+) = (K^+)^{+'} = K$. Hence ψ is onto. From Theorem (4.15), it follows that ψ is inclusion preserving.

5. Conclusion

In this paper, we studied the notion of intuitionistic fuzzy characteristic ideal of a Γ-ring. We have constructed an example of an intuitionistic fuzzy ideal which is not an intuitionistic fuzzy characteristic ideal. A connection between the intuitionistic fuzzy characteristic ideal with its level cut sets has been studied. The relationships between the set of all automorphisms of Γ-ring and the corresponding automorphisms of its operator rings have been investigated. We proved that there exists a one to one map between the set of all intuitionistic fuzzy characteristic ideals of Γ-ring and the set of all intuitionistic fuzzy characteristic ideals of its operator ring. We see that these structures are useful in developing the concepts like intuitionistic fuzzy prime ideals, intuitionistic fuzzy primary ideals and intuitionistic fuzzy semiprime ideals of a Γ-ring.

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