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ON NON-HOMOGENEOUS QUINARY QUINTIC EQUATION $(x^4 - y^4) = 125(z^2 - w^2)p^3$

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Abstract: The quinary quintic non-homogeneous diophantine equation represented by $(x^4 - y^4) = 125(z^2 - w^2)p^3$ is analyzed for its patterns of non-zero distinct integral solutions and some properties among the solutions are also illustrated.

Keywords and Phrases: Non-homogeneous quintic equation, quintic equation with five unknowns, integral solutions.

2020 Mathematics Subject Classification: 11D41.

1. Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems [1, 2, 8, 9]. Particularly,in [3, 4] quintic equations with three unknowns are studied for their integral solutions. In [5] quintic equations with four unknowns for their non-zero integer solutions. [6, 7] analyze quintic equations with five unknowns for their non-zero integer solutions. This communication concerns with yet another interesting non-homogeneous quintic equation with five unknowns given by $(x^4 - y^4) = 125(z^2 - w^2)p^3$ for finding its infinitely many non-zero distinct integer solutions and some properties among the solutions are also illustrated.

2. Notations

1. Polygonal number of rank n with size m :
   $t_{m,n} = n[1 + \frac{(n-1)(m-2)}{2}]$
2. Centered pyramidal number of rank n with size m :
   $CP_{m,n} = \frac{m(n-1)n(n+1)+6n}{6}$
3. Centered icosipentagonal pyramidal number of rank n with size 25 :
   $CP_n^{25} = \frac{n(25n^2-19)}{6}$
4. Pyramidal number of rank n with size m :
   $P_n^m = \frac{1}{6}[n(n+1)][(m-2)n + (5-m)]$
5. Pentagonal pyramidal number of rank n with size 5 : $P_n^5 = \frac{1}{2}n^2(n+1)$

3. Definition

Nasty number : A positive integer n is said to be a nasty number if it has atleast four different factors such that the difference between one pair of factors equals the sum of another pair of factors.

4. Method of Analysis

The non-homogeneous fifth degree equation with five unknowns is

$$(x^4 - y^4) = 125(z^2 - w^2)p^3 \tag{1}$$

Introduction of the linear transformations

$$x = u + v\ ,\ y = u - v\ ,\ z = 2u + v,\ w = 2u - v \tag{2}$$

in (1) leads to

$$u^2 + v^2 = 125p^3 \tag{3}$$

To solve the above equation, the method of factorization is employed as illustrated below:
Let

$$p = a^2 + b^2 \tag{4}$$

Write the number 125 as

$$125 = (10 + 5i)(10 - 5i) \tag{5}$$

Substituting (4), (5) in (3) and applying the method of factorization, consider

$$u + iv = (10 + 5i)(a + ib)^3 \tag{6}$$

Equating the real and imaginary parts, one obtains

$$u = 10(a^3 - 3ab^2) - 5(3a^2b - b^3)$$
$$v = 10(3a^2b - b^3) + 5(a^3 - 3ab^2)$$

In view of (2) the values of x, y, z, w satisfying (1) are given by

$$x = x(a, b) = 15(a^3 - 3ab^2) + 5(3a^2b - b^3)$$
$$y = y(a, b) = 5(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$z = z(a, b) = 25(a^3 - 3ab^2)$$
$$w = w(a, b) = 15(a^3 - 3ab^2) - 20(3a^2b - b^3)$$

Properties:

i. $6[50P_a^5 - 25(a - 1) - z(a, 1)]$ is a nasty number

ii. $50P_a^5 - t_{52,a} - z(a, 1) \equiv 0(mod 11)$

iii. $z(a, 1) + 56a = 6CP_a^{25}$

Note 1:
It is to be noted that in addition to (5), 125 may be represented as the product of complex conjugates as shown below:

- **Choice(i)** : 125 = (5+10i)(5-10i)
- **Choice(ii)** : 125 = (11+2i)(11-2i)
- **Choice(iii)**: 125 = (2+11i)(2-11i)

For each of the above choices, the corresponding solutions to (1) are presented below:
Solution for choice(i):

$$x = 15(a^3 - 3ab^2) - 5(3a^2b - b^3)$$
$$y = -5(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$z = 20(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$w = -25(3a^2b - b^3)$$

Solution for choice(ii):

$$\begin{aligned} x &= 13(a^3-3ab^2)+9(3a^2b-b^3) \\ y &= 9(a^3-3ab^2)-13(3a^2b-b^3) \\ z &= 24(a^3-3ab^2)+7(3a^2b-b^3) \\ w &= 20(a^3-3ab^2)-15(3a^2b-b^3) \end{aligned}$$

Solution for choice(iii):

$$\begin{aligned} x &= 13(a^3-3ab^2)-9(3a^2b-b^3) \\ y &= -9(a^3-3ab^2)-13(3a^2b-b^3) \\ z &= 15(a^3-3ab^2)-20(3a^2b-b^3) \\ w &= -7(a^3-3ab^2)-24(3a^2b-b^3) \end{aligned}$$

Note 2:
It is worth mentioning below, that, the two choices for the values of z and w in the linear transformations (2) may be taken as follows:

- **Choice(iv)** : x = u+v, y = u-v, z = uv+2 , w = uv-2
- **Choice(v)** : x = u+v, y = u-v, z = 2uv+1, w = 2uv+1
- **Choice(vi)** : x = u+v, y = u-v, z = u+2v , w = u-2v

For simplicity and brevity, the integer solution to (1) for the corresponding choices of z, w and 125 are exhibited below:
Solution for choices(iv) and (5):

$$\begin{aligned} x &= 15(a^3-3ab^2)+5(3a^2b-b^3) \\ y &= 5(a^3-3ab^2)+15(-3a^2b+b^3) \\ z &= 50[(a^3-3ab^2)^2-(3a^2b-b^3)^2]+75[(a^3-3ab^2)(3a^2b-b^3)]+2 \\ w &= 50[(a^3-3ab^2)^2-(3a^2b-b^3)^2]+75[(a^3-3ab^2)(3a^2b-b^3)]-2 \end{aligned}$$

Solution for choices(iv) and (i):

$$\begin{aligned} x &= 15(a^3-3ab^2)-5(3a^2b-b^3) \\ y &= -5(a^3-3ab^2)-15(3a^2b-b^3) \\ z &= 50[(a^3-3ab^2)^2-(3a^2b-b^3)^2]-75[(a^3-3ab^2)(3a^2b-b^3)]+2 \\ w &= 50[(a^3-3ab^2)^2-(3a^2b-b^3)^2]-75[(a^3-3ab^2)(3a^2b-b^3)]-2 \end{aligned}$$

Solution for choices(iv) and (ii):

$$x = 13(a^3 - 3ab^2) + 9(3a^2b - b^3)$$
$$y = 9(a^3 - 3ab^2) - 13(3a^2b + b^3)$$
$$z = 22[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] + 117[(a^3 - 3ab^2)(3a^2b - b^3)] + 2$$
$$w = 22[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] + 117[(a^3 - 3ab^2)(3a^2b - b^3)] - 2$$

Solution for choices(iv) and (iii):

$$x = 13(a^3 - 3ab^2) - 9(3a^2b - b^3)$$
$$y = -9(a^3 - 3ab^2) - 13(3a^2b + b^3)$$
$$z = 22[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] - 117[(a^3 - 3ab^2)(3a^2b - b^3)] + 2$$
$$w = 22[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] - 117[(a^3 - 3ab^2)(3a^2b - b^3)] - 2$$

Solution for choices(v) and (5):

$$x = 15(a^3 - 3ab^2) + 5(3a^2b - b^3)$$
$$y = 5(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$z = 100[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] + 150[(a^3 - 3ab^2)(3a^2b - b^3)] + 1$$
$$w = 100[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] + 150[(a^3 - 3ab^2)(3a^2b - b^3)] - 1$$

Solution for choices(v) and (i):

$$x = 15(a^3 - 3ab^2) - 5(3a^2b - b^3)$$
$$y = -5(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$z = 100[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] - 150[(a^3 - 3ab^2)(3a^2b - b^3)] + 1$$
$$w = 100[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] - 150[(a^3 - 3ab^2)(3a^2b - b^3)] - 1$$

Solution for choices(v) and (ii):

$$x = 13(a^3 - 3ab^2) + 9(3a^2b - b^3)$$
$$y = 9(a^3 - 3ab^2) - 13(3a^2b + b^3)$$
$$z = 44[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] + 234[(a^3 - 3ab^2)(3a^2b - b^3)] + 1$$
$$w = 44[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] + 234[(a^3 - 3ab^2)(3a^2b - b^3)] - 1$$

Solution for choices(v) and (iii):

$$x = 13(a^3 - 3ab^2) - 9(3a^2b - b^3)$$
$$y = -9(a^3 - 3ab^2) - 13(3a^2b + b^3)$$
$$z = 44[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] - 234[(a^3 - 3ab^2)(3a^2b - b^3)] + 1$$
$$w = 44[(a^3 - 3ab^2)^2 - (3a^2b - b^3)^2] - 234[(a^3 - 3ab^2)(3a^2b - b^3)] - 1$$

Solution for choices(vi) and (5):

$$x = 15(a^3 - 3ab^2) + 5(3a^2b - b^3)$$
$$y = 5(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$z = 20(a^3 - 3ab^2) + 15(3a^2b - b^3)$$
$$w = -25(3a^2b - b^3)$$

Solution for choices(vi) and (i):

$$x = 15(a^3 - 3ab^2) - 5(3a^2b - b^3)$$
$$y = -5(a^3 - 3ab^2) - 15(3a^2b - b^3)$$
$$z = 25(a^3 - 3ab^2)$$
$$w = -15(a^3 - 3ab^2) - 20(3a^2b - b^3)$$

Solution for choices(vi) and (ii):

$$x = 13(a^3 - 3ab^2) + 9(3a^2b - b^3)$$
$$y = 9(a^3 - 3ab^2) - 13(3a^2b + b^3)$$
$$z = 15(a^3 - 3ab^2) + 20(3a^2b - b^3)$$
$$w = 7(a^3 - 3ab^2) - 24(3a^2b - b^3)$$

Solution for choices(vi) and (iii):

$$x = 13(a^3 - 3ab^2) - 9(3a^2b - b^3)$$
$$y = -9(a^3 - 3ab^2) - 13(3a^2b + b^3)$$
$$z = 24(a^3 - 3ab^2) - 7(3a^2b - b^3)$$
$$w = -20(a^3 - 3ab^2) - 15(3a^2b - b^3)$$

5. Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous quinary quintic equation given by $(x^4 - y^4) = 125(z^2 - w^2)p^3$. As the quintic equations are rich in variety, one may search for integer solutions to other forms of quintic equation with variables grater than or equal to five.

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