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A NOTE ON TYPE 2 DEGENERATE DAEHEE POLYNOMIALS AND NUMBERS OF THE SECOND KIND

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Abstract: In this paper, we construct the type 2 degenerate Daehee numbers and polynomials of the second kind and their higher-order analogues, and investigate some properties of these numbers and polynomials. In addition, we give some new identities and relations between the type 2 degenerate Daehee polynomials of the second kind and degenerate Bernoulli polynomials of the second kind, an identity involving higher-order analogues of those polynomials and the degenerate Stirling numbers of the second kind, and an expression of higher-order analogues of those polynomials in terms of higher-order type 2 degenerate Bernoulli polynomials and the degenerate Stirling numbers of the first kind.

Keywords and Phrases: Type 2 degenerate Daehee polynomials and numbers, higher-order type 2 degenerate Daehee polynomials and numbers, Stirling numbers, degenerate central factorial numbers.

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1. Introduction

In [1, 2], Carlitz initiated the study of the degenerate Bernoulli and Euler polynomials and obtained some arithmetic and combinatorial results. In recent years, many mathematicians have drawn their attention to various degenerate versions of some old and new polynomials and numbers, namely some degenerate versions of Bernoulli numbers and polynomials of the second kind, Changhee numbers of the second kind, Daehee numbers of the second kind, Bernstein polynomials, central Bell numbers, and polynomials, central factorial numbers of the second kind, Cauchy numbers, Eulerian numbers and polynomials, Fubini polynomials, Stirling numbers of the first kind, Stirling polynomials of the second kind, central complete Bell polynomials, Bell numbers, and polynomials, type 2 Bernoulli numbers and polynomials, type 2 Bernoulli polynomials of the second kind, poly-Bernoulli numbers, and polynomials, poly-Cauchy polynomials, and of Frobenius-Euler polynomials, to name a few [7-10, 18-28] and the references therein.

The degenerate versions of some special numbers and polynomials have been studied by many researchers. The notion of degeneracy provides a powerful tool in defining special numbers and polynomials of their degenerate versions. The most important applications of these polynomials are in the theory of finite differences, analytic number theory, and applications in classical analysis and statistics. Despite the applicability of special functions in classical analysis and statistics, they also arise in communications systems, quantum mechanics, nonlinear wave propagation, electric circuit theory, electromagnetic theory, and so on.

The type 2 Bernoulli polynomials $B_n(x), (n \geq 0)$ and the type 2 Euler polynomials $E_n(x), (n \geq 0)$ are respectively defined by (see [4])

$$e^{xt}\frac{t}{2}\csc h\frac{t}{2} = \frac{t}{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} B_n(x)\frac{t^n}{n!}, \tag{1.1}$$

and

$$e^{xt}\sec h\frac{t}{2} = \frac{2}{e^{\frac{t}{2}} + e^{-\frac{t}{2}}}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}. \tag{1.2}$$

In the case when $x = 0$, $B_n = B_n(0)$ and $E_n = E_n(0)$ are called the type 2 Bernoulli and Euler numbers.

The generalized Bernoulli polynomials of order r are defined by

$$\left(\frac{t}{e^t - 1}\right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x)\frac{t^n}{n!} \quad \mid t \mid < 2\pi \quad \text{(see [1, 7])}. \tag{1.3}$$

When $x = 0, B_n^{(r)} = B_n^{(r)}(0)$ are called the generalized Bernoulli numbers of order r.

For any non-zero $\lambda \in \mathbb{R}$ (or $\mathbb{C}$), the degenerate exponential function is defined by

$$e_\lambda^x(t) = (1 + \lambda t)^{\frac{x}{\lambda}}, e_\lambda(t) = (1 + \lambda t)^{\frac{1}{\lambda}} \quad \text{(see [7-10])}. \tag{1.4}$$

By binomial expansion, we get

$$e_\lambda^x(t) = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!} \quad \text{(see [12, 15])}, \tag{1.5}$$

where $(x)_{0,\lambda} = 1$, $(x)_{n,\lambda} = (x - \lambda)(x - 2\lambda) \cdots (x - (n-1)\lambda) \quad (n \geq 1)$.

Note that

$$\lim_{\lambda \longrightarrow 0} e_\lambda^x(t) = \sum_{n=0}^{\infty} x^n \frac{t^n}{n!} = e^{xt}.$$

In [1], Carlitz considered the degenerate Bernoulli polynomials given by

$$\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x) \frac{t^n}{n!} \quad (\lambda \in \mathbb{R}). \tag{1.6}$$

When $x = 0, \beta_{n,\lambda} = \beta_{n,\lambda}(0)$ are called the degenerate Bernoulli numbers.

As is well known, the higher-order degenerate Bernoulli polynomials are considered by Carlitz [2] as follows:

$$\left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1 + \lambda t)^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty} \beta_{n,\lambda}^{(r)}(x). \tag{1.7}$$

In the special case when $x = 0$, $\beta_{n,\lambda}^{(r)} = \beta_{n,\lambda}^{(r)}(0)$ are called the higher-order degenerate Bernoulli numbers.

From (1.3) and (1.7), we note that

$$\begin{aligned} \sum_{n=0}^{\infty} \lim_{\lambda \to 0} \beta_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} &= \lim_{\lambda \to 0} \left(\frac{t}{(1 + \lambda t)^{\frac{1}{\lambda}} - 1} \right)^r (1 + \lambda t)^{\frac{x}{\lambda}} \\ &= \left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}. \end{aligned} \tag{1.8}$$

Thus, by (1.8), we get

$$\lim_{\lambda \to 0} \beta_{n,\lambda}^{(r)}(x) = B_n^{(r)}(x) \quad (n \geq 0).$$

Jang and Kim introduced the type 2 degenerate Bernoulli polynomials defined by (see [7])

$$\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{t^n}{n!}. \tag{1.9}$$

When $x=0$, $B_{n,\lambda}=B_{n,\lambda}(0)$ are called the type 2 degenerate Bernoulli numbers.

The Bernoulli polynomials of the second kind of order r are given by (see [4])

$$\left(\frac{t}{\log(1+t)}\right)^r(1+t)^x=\sum_{n=0}^{\infty}b_n^{(r)}(x)\frac{t^n}{n!}. \tag{1.10}$$

From (1.3) and (1.10), we note that

$$b_n^{(r)}(x)=B_n^{(n-r+1)}(x+1)\quad (n\geq 0).$$

Kim *et al.* [16] introduced the degenerate Bernoulli polynomials of the second kind of order α defined by

$$\left(\frac{t}{\log_\lambda(1+t)}\right)^\alpha(1+t)^x=\sum_{n=0}^{\infty}b_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.11}$$

Note that $\lim_{\rightarrow 0}b_{n,\lambda}^{(\alpha)}(x)=b_n^{(\alpha)}(x)\quad (n\geq 0)$.

The Daehee polynomials are defined by

$$\frac{\log(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_n(x)\frac{t^n}{n!}\quad \text{(see [5, 12, 15])}. \tag{1.12}$$

When $x=0$, $D_n=D_n(0)$ are called the Daehee numbers.

Kim *et al.* [21] introduced the new type degenerate Daehee polynomials defined by

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x=\sum_{n=0}^{\infty}D_{n,\lambda}(x)\frac{t^n}{n!}\quad \text{(see [22])}. \tag{1.13}$$

When $x=0$, $D_{n,\lambda}=D_{n,\lambda}(0)$ are called the degenerate Daehee numbers.

Recently, Jang and Kim [4, 5] introduced the type 2 degenerate Daehee polynomials of order α defined by

$$\left(\frac{\log(1+t)}{(1+\lambda\log(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log(1+t))^{-\frac{1}{2\lambda}}}\right)^\alpha(1+\lambda\log(1+t))^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}d_{n,\lambda}^{(\alpha)}(x)\frac{t^n}{n!}. \tag{1.14}$$

When $x = 0$, $d_{n,\lambda}^{(\alpha)} = d_{n,\lambda}^{(\alpha)}(0)$ are called the type 2 degenerate numbers of order α.

The degenerate Stirling numbers of the first kind are defined by (see [18, 24])

$$\frac{1}{k!}(\log_\lambda(1+t))^k = \sum_{n=k}^{\infty} S_{1,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0). \tag{1.15}$$

Note here that $\lim_{\lambda\to 0} S_{1,\lambda}(n,k) = S_1(n,k)$, where $S_1(n,k)$ are called the Stirling numbers of the first kind given by

$$\frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!} \quad (k \geq 0) \quad \text{(see [1-15])}.$$

The degenerate Stirling numbers of the second kind (see [22]) are given by

$$\frac{1}{k!}(e_\lambda(t)-1)^k = \sum_{n=k}^{\infty} S_{2,\lambda}(n,k)\frac{t^n}{n!} \quad (k \geq 0). \tag{1.16}$$

It is clear that $\lim_{\lambda\to 0} S_{2,\lambda}(n,k) = S_2(n,k)$, where $S_2(n,k)$ are called the Stirling numbers of the second kind given by

$$\frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad (k \geq 0) \quad \text{(see [16-28])}.$$

The degenerate central factorial numbers of the second kind (see [20]) are given by

$$\frac{1}{k!}\left(e_\lambda^{\frac{1}{2}}(t) - e_\lambda^{-\frac{1}{2}}(t)\right)^k = \sum_{n=k}^{\infty} T_\lambda(n,k)\frac{t^n}{n!}. \tag{1.17}$$

Note that $\lim_{\lambda\to 0} T_\lambda(n,k) = T(n,k)$ are called the central factorial numbers of the second kind given by

$$\frac{1}{k!}\left(e^{\frac{1}{2}}(t) - e^{-\frac{1}{2}}(t)\right)^k = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!} \quad \text{(see [11, 19])}.$$

Motivated by the works of Sharma *et al.* [27], we first define type 2 degenerate Daehee numbers and polynomials of the second kind. We investigate some new properties of these numbers and polynomials and derive some new identities and relations between the type 2 degenerate Daehee numbers and polynomials of the second kind and Carlitzs degenerate Bernoulli polynomials.

2. Type 2 degenerate Daehee polynomials and numbers

For $\lambda \in \mathbb{R}$, the degenerate logarithm function $\log_\lambda(1+t)$, which is the compositional inverse of the degenerate exponential function $e_\lambda(t)$ and the motivation for the definition of degenerate polylogarithm function by (see [21])

$$\log_\lambda(1+t) = \sum_{n=1}^{\infty} \lambda^{n-1}(1)_{n,1/\lambda}\frac{t^n}{n!} = \frac{1}{\lambda}\sum_{n=1}^{\infty}(\lambda)_n\frac{t^n}{n!} = \frac{1}{\lambda}\left((1+t)^\lambda - 1\right). \tag{2.1}$$

Note that

$$\lim_{\lambda\to 0}\log_\lambda(1+t) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{t^n}{n!} = \log(1+t).$$

Let $\lambda \in \mathbb{R}$. The new type of degenerate Daehee polynomials of the second kind are defined by (see [27])

$$\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{\lambda}} - 1}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}\widehat{D}_{n,\lambda}(x)\frac{t^n}{n!}. \tag{2.2}$$

When $x = 0$, $\widehat{D}_{n,\lambda} = \widehat{D}_{n,\lambda}(0)$ are called the new type of degenerate Daehee numbers of the second kind.

From (1.16) and (2.2), we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\widehat{D}_{n,\lambda}(x) &= \sum_{m=0}^{\infty}\widehat{D}_{n,\lambda}(x)\frac{1}{m!}(e_\lambda(t)-1)^m \\
&= \sum_{m=0}^{\infty}\widehat{D}_{n,\lambda}(x)\sum_{n=m}^{\infty}S_{2,\lambda}(n,m)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}S_{2,\lambda}(n,m)\widehat{D}_{n,\lambda}(x)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.3}$$

Comparing the coefficients on both sides of (2.3), we have

$$\widehat{D}_{n,\lambda}(x) = \sum_{m=0}^{n}S_{2,\lambda}(n,m)\widehat{D}_{n,\lambda}(x).$$

Motivated by (2.1) and (2.2), we define the type 2 degenerate Daehee polynomials of the second kind which are given by the following generating function to be

$$\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}} - (1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}} = \sum_{n=0}^{\infty}d^*_{n,\lambda,2}(x)\frac{t^n}{n!}. \tag{2.4}$$

When $x=0, d^*_{n,\lambda,2}=d^*_{n,\lambda,2}(0)$ are called the type 2 degenerate Daehee numbers of the second kind.

Theorem 2.1. *For $n \geq 0$, we have*

$$d^*_{n,\lambda,2}(x)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}(x)_{\lambda,m}S_{1,\lambda}(k,m)d^*_{n-k,\lambda,2}.$$

Proof. Using (2.4), we note that

$$\begin{aligned}\sum_{n=0}^{\infty}d^*_{n,\lambda,2}(x)\frac{t^n}{n!}&=\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}d^*_{n,\lambda,2}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}(x)_{\lambda,m}\frac{(\lambda\log_\lambda(1+t))^m}{m!}\\&=\sum_{n=0}^{\infty}d^*_{n,\lambda}\frac{t^n}{n!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(x)_{\lambda,m}\lambda^m S_{1,\lambda}(k,m)\frac{t^k}{k!}\\&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}(x)_{\lambda,m}S_{1,\lambda}(k,m)\binom{n}{k}d^*_{n-k,\lambda,2}\right)\frac{t^n}{n!}.\end{aligned}\tag{2.5}$$

Therefore, by (2.5), we obtain the result.

Theorem 2.2. *For $n \geq 0$, we have*

$$d^*_{n,\lambda,2}(x)=\sum_{m=0}^{n}B_{m,\lambda}(x)S_{1,\lambda}(n,m).$$

Proof. By changing t by $\log_\lambda(1+t)$ in (1.9), we have

$$\begin{aligned}\sum_{n=0}^{\infty}B_{n,\lambda}(x)\frac{(\log_\lambda(1+t))^n}{n!}&=\frac{\log_\lambda(1+t)(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\\&=\sum_{n=0}^{\infty}d^*_{n,\lambda,2}(x)\frac{t^n}{n!}.\end{aligned}\tag{2.6}$$

On the other hand,

$$\begin{aligned}\sum_{m=0}^{\infty}B_{m,\lambda}(x)\frac{(\log_\lambda(1+t))^m}{m!}&=\sum_{m=0}^{\infty}B_{m,\lambda}(x)\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^n}{n!}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{n}B_{m,\lambda}(x)S_{1,\lambda}(n,m)\frac{t^n}{n!}.\end{aligned}\tag{2.7}$$

Therefore, by (2.6) and (2.7), we get the result.

Theorem 2.3. *For $n \geq 0$, we have*

$$B_{n,\lambda}(x) = \sum_{m=0}^{n} d^*_{m,\lambda,2}(x) S_{2,\lambda}(n,m).$$

Proof. By replacing t by $e_\lambda(t) - 1$ in (2.4), we get

$$\begin{aligned} \sum_{m=0}^{\infty} d^*_{m,\lambda,2}(x) \frac{1}{m!}(e_\lambda(t)-1)^m &= \frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}} - (1+\lambda t)^{-\frac{1}{2\lambda}}}(1+\lambda t)^{\frac{x}{\lambda}} \\ &= \sum_{n=0}^{\infty} B_{n,\lambda}(x) \frac{t^n}{n!}. \end{aligned} \tag{2.8}$$

On the other hand,

$$\begin{aligned} \sum_{m=0}^{\infty} d^*_{m,\lambda,2}(x) \frac{1}{m!}(e_\lambda(t)-1)^m &= \sum_{m=0}^{\infty} d^*_{m,\lambda}(x) \sum_{m=n}^{\infty} S_{2,\lambda}(n,m) \frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} d^*_{m,\lambda,2}(x) S_{2,\lambda}(n,m) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.9}$$

Therefore, by (2.8) and (2.9), we obtain the required result.

Theorem 2.4. *For $n \geq 0$, we have*

$$\sum_{m=0}^{n} (x)_{m,\lambda} \lambda^m S_{1,\lambda}(n,m) = \sum_{m=0}^{n} \binom{n}{m} b^*_{m,\lambda,2} d^*_{n-m,\lambda,2}(x).$$

Proof. From (2.4), we observe that

$$(1+\lambda \log_\lambda(1+t))^{\frac{x}{\lambda}} = \frac{(1+\lambda \log_\lambda(1+t))^{\frac{1}{2\lambda}} - (1+\lambda \log_\lambda(1+t))^{-\frac{1}{2\lambda}}}{\log_\lambda(1+t)} \sum_{n=0}^{\infty} d^*_{n,\lambda}(x) \frac{t^n}{n!}$$

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{m=0}^{n} (x)_{m,\lambda} \lambda^m S_{1,\lambda}(n,m) \frac{t^n}{n!} &= \left(\sum_{m=0}^{\infty} b^*_{m,\lambda,2} \frac{t^m}{m!} \right) \left(\sum_{n=0}^{\infty} d^*_{n,\lambda,2}(x) \frac{t^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} b^*_{m,\lambda,2} d^*_{n-m,\lambda,2}(x) \right) \frac{t^n}{n!}. \end{aligned} \tag{2.10}$$

Therefore, by (2.10), we obtain the result.

For $r \in \mathbb{N}$, we define type 2 higher-order degenerate Daehee polynomials of the second kind by

$$\left(\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^r(1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$

$$=\sum_{n=0}^{\infty}d^{*(r)}_{n,\lambda,2}(x)\frac{t^n}{n!}. \tag{2.11}$$

When $x=0$, $d^{*(r)}_{n,\lambda,2}=d^{*(r)}_{n,\lambda,2}(0)$ are called the higher-order type 2 degenerate Daehee numbers of the second kind.

Theorem 2.5. *For $n\geq 0$, we have*

$$d^{*(r)}_{n,\lambda,2}(x+y)=\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}d^{*(r)}_{n-k,\lambda,2}(x)(y)_{m,\lambda}\lambda^m S_{1,\lambda}(k,m).$$

Proof. From (2.9), we note that

$$\sum_{n=0}^{\infty}d^{*(r)}_{n,\lambda,2}(x+y)\frac{t^n}{n!}$$

$$\begin{aligned}
&=\left(\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^r(1+\lambda\log_\lambda(1+t))^{\frac{x+y}{\lambda}}\\
&=\sum_{n=0}^{\infty}d^{*(r)}_{n,\lambda,2}(x)\frac{t^n}{n!}\sum_{m=0}^{\infty}(y)_{m,\lambda}\frac{(\lambda\log_\lambda(1+t))^m}{m!}\\
&=\sum_{n=0}^{\infty}d^{*(r)}_{n,\lambda,2}(x)\frac{t^n}{n!}\sum_{k=0}^{\infty}\sum_{m=0}^{k}(y)_{m,\lambda}\lambda^m S_{1,\lambda}(k,m)\frac{t^k}{k!}\\
&=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\sum_{m=0}^{k}\binom{n}{k}d^{*(r)}_{n-k,\lambda,2}(x)(y)_{m,\lambda}\lambda^m S_{1,\lambda}(k,m)\right)\frac{t^n}{n!}.
\end{aligned} \tag{2.12}$$

Thus, by (2.12), we obtain the result.

Theorem 2.6. *For $n\geq 0$, we have*

$$B^{(r)}_{m,\lambda}(x)=\sum_{m=0}^{n}d^{*(r)}_{m,\lambda,2}S_{2,\lambda}(n,m).$$

Proof. By replacing t by $e_\lambda(t)-1$ in (2.11), we get

$$\sum_{m=0}^{\infty} d_{m,\lambda,2}^{*(r)}(x)\frac{1}{m!}(e_\lambda(t)-1)^m = \left(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}} - 1 + \lambda t)^{-\frac{1}{2\lambda}}}\right)^r (1+\lambda t)^{\frac{x}{\lambda}}$$
$$= \sum_{n=0}^{\infty} B_{n,\lambda}^{(r)}(x)\frac{t^n}{n!}. \quad (2.13)$$

On the other hand,

$$\sum_{m=0}^{\infty} d_{m,\lambda,2}^{*(r)}(x)\frac{1}{m!}(e_\lambda(t)-1)^m = \sum_{m=0}^{\infty} d_{m,\lambda,2}^{*(r)}(x)\sum_{n=m}^{\infty} S_{2,\lambda}(n,m)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} d_{m,\lambda,2}^{*(r)}(x)S_{2,\lambda}(n,m)\right)\frac{t^n}{n!}. \quad (2.14)$$

Therefore, by (2.13) and (2.14), we get the result.

Theorem 2.7. *For $r,k\in\mathbb{N}$, with $r>k$, we have*

$$d_{n,\lambda,2}^{*(r)}(x) = \sum_{l=0}^{n}\binom{n}{l} d_{l,\lambda,2}^{*(r-k)} d_{n-l,\lambda,2}^{*(k)}(x), (n\geq 0).$$

Proof. By (2.11), we see that

$$\left(\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}} - (1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^r (1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$
$$= \left(\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}} - (1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{r-k}$$
$$\times \left(\frac{\log_\lambda(1+t)}{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}} - (1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}\right)^{k} (1+\lambda\log_\lambda(1+t))^{\frac{x}{\lambda}}$$
$$= \left(\sum_{l=0}^{\infty} d_{l,\lambda,2}^{*(r-k)}\frac{t^l}{l!}\right)\left(\sum_{m=0}^{\infty} d_{m,\lambda,2}^{*(k)}(x)\frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\binom{n}{l} d_{l,\lambda,2}^{*(r-k)} D_{n-l,\lambda,2}^{*(k)}(x)\right)\frac{t^n}{n!}. \quad (2.15)$$

Therefore, by (2.11) and (2.15), we obtain the result.

It is well known that the type 2 degenerate Bernoulli polynomials of order r are defined by (see [4])

$$\left(\frac{t}{(1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}}\right)^{r}(1+\lambda t)^{\frac{x}{\lambda}}=\sum_{n=0}^{\infty}B_{n,\lambda}^{(r)}(x)\frac{t^{n}}{n!}. \tag{2.16}$$

When $x=0$, $B_{n,\lambda}^{(r)}=B_{n,\lambda}^{(r)}(0)$ are called the type degenerate Bernoulli numbers.

Theorem 2.8. *For $n\geq 0$, we have*

$$d_{n,\lambda,2}^{(r)}(x)=\sum_{m=0}^{n}B_{m,\lambda}^{(r)}(x)S_{1,\lambda}(n,m).$$

Proof. Replacing t by $\log_{\lambda}(1+t)$ in (2.16), we get

$$\begin{aligned}&\left(\frac{\log_{\lambda}(1+t)}{(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_{\lambda}(1+t))^{-\frac{1}{2\lambda}}}\right)^{r}(1+\lambda\log_{\lambda}(1+t))^{\frac{x}{\lambda}}\\&=\sum_{m=0}^{\infty}B_{m,\lambda}^{(r)}(x)\frac{(\log_{\lambda}(1+t))^{m}}{m!}\\&=\sum_{m=0}^{\infty}B_{m,\lambda}^{(r)}(x)\sum_{n=m}^{\infty}S_{1,\lambda}(n,m)\frac{t^{n}}{n!}\\&=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}B_{m,\lambda}^{(r)}(x)S_{1,\lambda}(n,m)\right)\frac{t^{n}}{n!}.\end{aligned} \tag{2.17}$$

On the other hand,

$$\begin{aligned}&\left(\frac{\log_{\lambda}(1+t)}{(1+\lambda\log_{\lambda}(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_{\lambda}(1+t))^{-\frac{1}{2\lambda}}}\right)^{r}(1+\lambda\log_{\lambda}(1+t))^{\frac{x}{\lambda}}\\&=\sum_{n=0}^{\infty}d_{n,\lambda,2}^{*(r)}(x)\frac{t^{n}}{n!}.\end{aligned} \tag{2.18}$$

Therefore, by (2.17) and (2.18), we get the result.

Theorem 2.9. *For $n\geq 0$, we have*

$$\sum_{m=0}^{n}d_{m,\lambda,2}^{*(-k)}S_{2,\lambda}(n,m)=T_{\lambda}(n+k,k)\frac{1}{\binom{n+k}{n}}.$$

Proof. Let us take $r=-k$ and replacing t by $e_\lambda(t)-1$ in (2.11), we note that

$$\left(\frac{1}{t}\right)^k\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k=\sum_{m=0}^{\infty}d^{*(-k)}_{m,\lambda,2}\frac{(e_\lambda(t)-1)^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}d^{*(-k)}_{m,\lambda,2}S_{2,\lambda}(n,m)\frac{t^n}{n!}. \tag{2.19}$$

On the other hand,

$$\frac{1}{t^k}\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k=\frac{k!}{t^k}\frac{\left((1+\lambda t)^{\frac{1}{2\lambda}}-(1+\lambda t)^{-\frac{1}{2\lambda}}\right)^k}{k!}$$
$$=\frac{k!}{t^k}\sum_{n=k}^{\infty}T_\lambda(n,k)\frac{t^n}{n!}$$
$$=\sum_{n=0}^{\infty}T_\lambda(n+k,k)\frac{1}{\binom{n+k}{n}}\frac{t^n}{n!}. \tag{2.20}$$

Comparing the coefficients on equations (2.19) and (2.20), we get the result.

Theorem 2.10. *For $n\geq 0$, we have*

$$d^{*(-k)}_{n,\lambda,2}=\sum_{i=0}^{n}\sum_{m=k}^{i+k}\frac{\binom{n}{i}}{\binom{i+k}{i}}T_{2,\lambda}(m,k)S_{1,\lambda}(i+k,m)b^{(k)}_{n-i,\lambda}(1).$$

Proof. For $k\in\mathbb{N}$, let $r=-k$, $x=0$ in (2.11). Then we have

$$\sum_{n=0}^{\infty}d^{*(-k)}_{n,\lambda,2}\frac{t^n}{n!}=\left(\frac{(1+\lambda\log_\lambda(1+t))^{\frac{1}{2\lambda}}-(1+\lambda\log_\lambda(1+t))^{-\frac{1}{2\lambda}}}{\log_\lambda(1+t)}\right)^k$$
$$=\left(\frac{t}{\log_\lambda(1+t)}\right)^k\frac{k!}{t^k}\sum_{m=k}^{\infty}T_{2,\lambda}(m,k)\frac{1}{m!}(\log_\lambda(1+t))^m$$
$$=\left(\sum_{n=0}^{\infty}b^{(k)}_{n,\lambda}\frac{t^n}{n!}\right)\left(\frac{k!}{t^k}\sum_{m=k}^{\infty}T_{2,\lambda}(m,k)\sum_{i=m}^{\infty}S_{1,\lambda}(i,m)\frac{t^i}{i!}\right)$$
$$=\left(\sum_{n=0}^{\infty}b^{(k)}_{n,\lambda}\frac{t^n}{n!}\right)\left(\frac{k!}{t^k}\sum_{i=k}^{\infty}\left(\sum_{m=k}^{i}T_{2,\lambda}(m,k)S_{1,\lambda}(i,m)\right)\frac{t^i}{i!}\right)$$
$$=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\sum_{m=k}^{i+k}\frac{\binom{n}{i}}{\binom{i+k}{i}}T_{2,\lambda}(m,k)S_{1,\lambda}(i+k,m)b^{(k)}_{n-i,\lambda}(1)\right)\frac{t^n}{n!}. \tag{2.21}$$

Comparing the coefficients of t on both sides, we get the result.

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