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VISUALIZATION OF THE RIEMANN DARBOUX SUM AND ITS PROPERTIES WITH GEOGEBRA

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Abstract: Underachievement in mathematics is one major challenge for undergraduate - level students in Nepal. The results of Tribhuvan University (TU) show that students struggled with the subject Real analysis. Many teachers also struggle to teach this subject effectively due to the lack of conceptual knowledge, and technology. Therefore, conceptual understanding of fundamental terms of real analysis needs to be taught effectively at the undergraduate level to ensure smooth content progression into a higher level. Several studies advocated that the integration of information communication technology (ICT) with teaching and learning activities in mathematics enhances students' learning of the mathematics contents. So, this study visualizes the conceptual understanding of Riemann sums and proof of their properties using GeoGebra software of experimental teaching for the undergraduate level on TU. The study was based on constructivism-learning theory. The experiment showed that blended GeoGebra and usual practice processes promote conceptual understanding of students of the Riemann sum and proof of its properties.

Keywords and Phrases: Partition, bounded function, upper sum, lower sum, technology.

2020 Mathematics Subject Classification: 26E40.

1. Introduction

The Riemann Darboux sum is one of the fundamental key concepts of real analysis, with a broader application in social science, industrial, economic, and in many sciences and mathematics itself. In Nepal, the concept of a Riemann sum and its properties is first introduced in real analysis at the undergraduate level in the second year of Tribhuvan University (TU), which is a completely new idea for Nepalese students. Most of the students were realized that real analysis is a difficult subject at University Level. Very few students understand the conceptual understanding of the Riemann sum and its properties. The conceptual understanding has been the broader scope of many researchers. The conceptual understanding classified into operational and structural ways of understanding. While the operational understanding includes highly manipulative skills and uses them as principal means in their quest after meaning, the structural is more capable of direct-grasp understanding [11]. She defines as reification the transition from an operational to a structural way of thinking and states that this transition is a basic phenomenon in the formation of a mathematical object.

1.1. Theoretical Framework

This study was conducted according to the constructivist learning theory, which advocates that knowledge is built in the mind of an individual through active participation in certain experiences [13]. The fundamental belief of constructivist theory is that students actively construct knowledge, and contrary to the idea that knowledge is transmitted by the educator [15, 16]. Students are seen as active builders of knowledge rather than passive recipients [13]. Constructivists distinguish between cognitive constructivism and social constructivism. Social constructivists believe that knowledge is the result of collaborative construction in a socio-cultural context and that learning is favored by information sharing, negotiation, and discussions [5]. Thus, social constructivists emphasize the learning environment that should allow easy communication and collaboration with others [2, 5]. For cognitive constructivists, learning is first an individual matter; therefore, instructional design should support and meet the needs of individual students to create knowledge, and conceptual understanding [8]. So, the perspectives of constructivism theory the conceptual understanding of the mathematical knowledge have been constructed through the active participant, and emphasize a student-centered pedagogical approach [3, 12].

The use of mathematical software like Geogebra, Mathematica, Matlab, etc. gives students the chance to engage in high-level thinking such as analysis and reflection [10]. When using software, students are involved in learning knowledge, testing conjectures, and checking counterexamples, therefore, they have the opportunity to visualize the task and reflect on their thinking [14]. Thus, this study focused on integrating GeoGebra software in the teaching of Riemann Darboux sum, and its properties. In accordance with constructivist theories, it is believed that technology can help students build and conceptual understanding of mathematics content. The conceptual understanding of mathematical contents will eventually translate into improvement in carrying out mathematical exercises. The technology could stimulate the student to perform a series of actions and processes in order to objectively build their own patterns. The students keep going back and forth as they build their own knowledge, based on experience provided by technology. It can promote students to connect the visual, numerical, symbolic representations, explore and deepen the understanding of the concepts learned.

1.2. Objective

This study aimed to explore the conceptual understanding in learning Riemann Darboux sum and proof its properties with GeoGebra at undergraduate level students at Tribhuvan University.

Definition 1.1. (Partition) *Let $[a, b]$ be a close interval in the real line $\mathbb{R}$. The partition P of $[a, b]$ is a finite set of points $x_0, x_1, x_2, \cdots, x_n$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$.*

Symbolically, we write $P = \{x_0, x_1, x_2, \cdots, x_n\}$, where $x_0 = a$ and $x_n = b$. The partition P has $(n + 1)$ points. The intervals $[x_0, x_1]$,$[x_1, x_2]$,$[x_2, x_3]$,$\cdots$,$[x_{n-1}, x_n]$ are called the subintervals of the partition P. The length of the r^{th} subinterval $[x_{r-1}, x_r]$ is denoted by $\delta_r = x_r - x_{r-1}$ i.e. δ_r, where $r = 1, 2, 3, \cdots, n$. The set of partition of [a,b] is denoted by $P[a, b]$. Thus, $P[a, b]$=$\{P : P$ is a partition of $[a, b]\}$. The length of largest subinterval is called the norm of the partition P and it is denoted by $||P||$ or $\mu(P)$ i.e. $||P|| = max\{\delta_r = x_r - r_{r-1}, 1 \leq r \leq n\}$. Let $P_1, P_2 \in P[a, b]$, then P_2 is said to be finer of P_1, If $P_1 \subset P_2$ i.e. if each point of P_1 is also a point of P_2. In this case, we say that P_2 is refinement of P_1 [1, 4, 7].

Definition 1.2. (Darbox Lower and Upper Sum) *Let $P = \{x_0, x_2, x_3, \cdots, x_n\}$ be a partitions of $[a, b]$. Let $f : [a, b] \to \mathbb{R}$ be a bounded real valued function and let*

$$
\begin{aligned}
m &= inf\{f(x) : x \in [a, b]\} \\
M &= sup\{f(x) : x \in [a, b]\} \\
m_r &= inf\{f(x) : x \in [x_{r-1}, x_r]\} \\
M_r &= sup\{f(x) : x \in [x_{r-1}, x_r]\}
\end{aligned}
$$

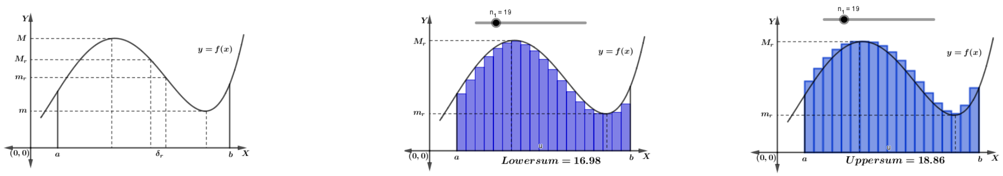
$Then\,the\,sums\,defined\,by$

$$L(P, f) = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + \cdots + m_n\delta_n$$

$$
\begin{aligned}
&= \sum_{r=1}^{n} m_r \delta_r \quad and \\
U\left(P, f\right) &= M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + \cdots + M_n\delta_n \\
&= \sum_{r=1}^{n} M_r \delta_r
\end{aligned}
$$

are respectively called the ***Darboux Lower Sum*** *and* ***Darboux Upper Sum*** *of f corresponding to the partition P of [a,b]. Also, $t_r \in [x_{r-1} - x_r]$, then the sum* $f(t_1)\delta_1 + f(t_2)\delta_2 + f(t_3)\delta_3 + \cdots + f(t_n)\delta_n = \sum_{r=1}^{n} f(t_r)\delta_r$ [1, 6, 7, 9].

The geometrical idea of Darboux sums is indicated in figures 1. The Lower sum is the area of the shaded rectangles whose width of r^{th} rectangle is δ_r and the height of the shaded rectangle is m_r which is indicated by figure 1(b). Similarly,the Upper sum is the area of the shaded rectangles,the width of r^{th} rectangle is δ_r and the height of the shaded rectangle is M_r also indicated by figure 1(c).



(a) Sup and Inf of r^{th} interval (b) Lower Sum of f(x) (c) Upper Sum of f(x)

Figure 1

Example 1.3. Let $P = \{1, 2, 3, 4, 5\}$ be the partition of [1,5] and $f : [a, b] \to \mathbb{R}$ be a function defined by $f(x) = x^2$, then find (i) $U(P, f)$ and (ii) $L(p, f)$

Solution. Since, $f : [a, b] \to \mathbb{R}$ be a bounded function defined by $f(x) = x^2$, and $P = \{1, 2, 3, 4, 5\}$ be the partition of [5, 11], then we have to visualization of (i) $U(P, f)$ and (ii) $L(P, f)$ by using GeoGebra Software.

Now,

$$
\begin{aligned}
f(x) &= x^2 \\
f(2) &= 4 \\
f(3) &= 9 \\
f(4) &= 16 \\
f(5) &= 25
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
U(p,f) &= \sum_{r=1}^{n} M_r\delta_r \\
&= M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4 + M_5\delta_5 \\
&= 1\cdot 4 + 1\cdot 9 + 1\cdot 16 + 1\cdot 25 \\
&= 54
\end{aligned}
$$

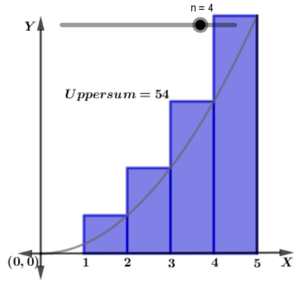


Figure 2: Upper sum of $f(x) = x^2$, $x \in [1, 5]$

Again,

$$
\begin{aligned}
L(p,f) &= \sum_{r=1}^{n} m_r\delta_r \\
&= m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4 + m_5\delta_5 \\
&= 1\cdot 1 + 1\cdot 4 + 1\cdot 9 + 1\cdot 16 \\
&= 30
\end{aligned}
$$

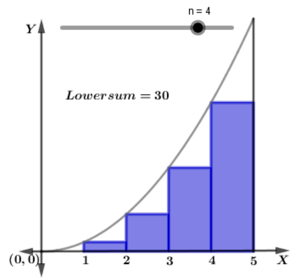


Figure 3: Lower sum of $f(x) = x^2$, $x \in [1, 5]$

Example 1.4. Let $P = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\}$ be partition of $[0,\pi]$ and $f : [0,\pi] \to \mathbb{R}$ be function defined by $f(x) = sinx$, then find $U(P,f)$ and $L(P,f)$.

Solution. Since, the given function $f(x) = sinx$ is bounded on $[0,\pi]$ and $P = \left\{0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi\right\}$ be partition of $[0,\pi]$, then we have to finds of (i) $U(P,f)$ and (ii) $L(P,f)$ and visualization its by using GeoGebra Software.

Since, $f(x) = sinx$, then $f(0) = sin0 = 0$, $f\left(\frac{\pi}{6}\right) = sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $f\left(\frac{\pi}{3}\right) = sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $f\left(\frac{\pi}{2}\right) = sin\left(\frac{\pi}{2}\right) = 1$, $f\left(\frac{2\pi}{3}\right) = sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$, $f\left(\frac{5\pi}{6}\right) = sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$, $f\left(\pi\right) = sin\left(\pi\right) = 0$ Now, we have

$$
\begin{aligned}
U(p,f) &= \sum_{r=1}^{n} M_r\delta_r \\
&= M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4 + M_5\delta_5 + M_6\delta_6 \\
&= sin\left(\frac{\pi}{6}\right)\frac{\pi}{6} + sin\left(\frac{\pi}{3}\right)\frac{\pi}{6} + sin\left(\frac{\pi}{2}\right)\frac{\pi}{6} + sin\left(\frac{\pi}{2}\right)\frac{\pi}{6} \\
&+ sin\left(\frac{2\pi}{3}\right)\frac{\pi}{6} + sin\left(\frac{5\pi}{6}\right)\frac{\pi}{6}
\end{aligned}
$$

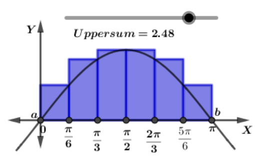


Figure 4: Uppersum of $f(x) = sinx$,$x \in [0,\pi]$

$$= \frac{1}{2}\cdot\frac{\pi}{6}+\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6}+1\cdot\frac{\pi}{6}+1\cdot\frac{\pi}{6}+\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6}+\frac{1}{2}\cdot\frac{\pi}{6}$$
$$= \frac{\pi}{12}\left(3+\sqrt{3}\right)$$
$$= 2.48$$

Again,

$$L(p,f) = \sum_{r=1}^{n} m_r\delta_r$$
$$= m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4 + m_5\delta_5 + m_6\delta_6$$
$$= 0 + sin\left(\frac{\pi}{6}\right)\frac{\pi}{6} + sin\left(\frac{\pi}{3}\right)\frac{\pi}{6} + sin\left(\frac{2\pi}{3}\right)\frac{\pi}{6}$$
$$+ sin\left(\frac{5\pi}{6}\right)\frac{\pi}{6} + 0$$
$$= \frac{1}{2}\cdot\frac{\pi}{6}+\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6}++\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6}+\frac{1}{2}\cdot\frac{\pi}{6}$$
$$= \frac{\pi}{6}\left(1+\sqrt{3}\right)$$
$$= 1.43$$

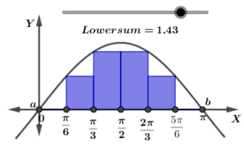


Figure 5: Lowersum of $f(x) = sinx, x \in [0,\pi]$

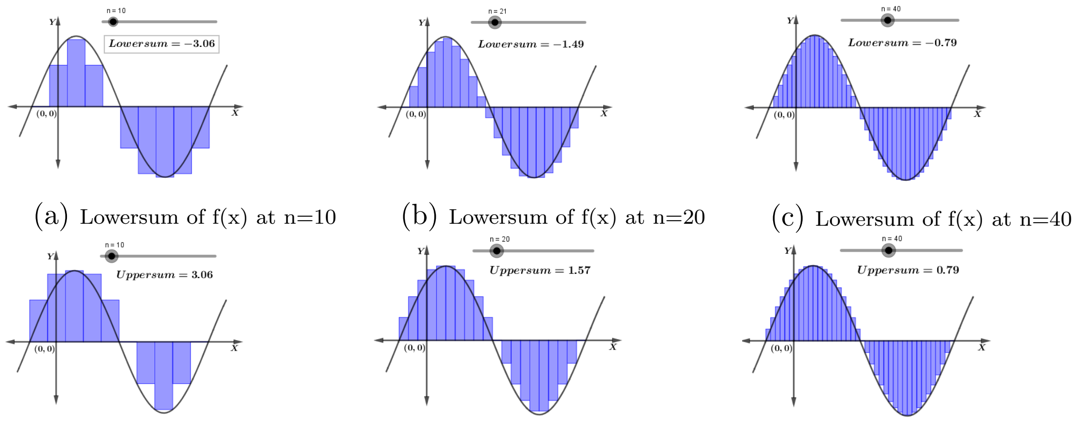
2. Properties of Darboux Sums

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function and P be a partition of [a,b], then $U(P,f)$ and $L(P,f)$ are bounded and $U(P,f) \le S(P,f) \le L(P,f)$* [1, 7].

Proof. Let m and M be lower and upper bounds of f on [a,b] respectively, then

$$m \le m_r \le f(t_r) \le M_r \le M$$
$$= m\delta_r \le m_r\delta_r \le f(t_r)\delta_r \le M_r\delta_r \le M\delta_r$$
$$= \sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} f(t_r)\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$$
$$= m\sum_{r=1}^{n}\delta_r \le L(P,f) \le S(P,f) \le U(P,f) \le M\sum_{r=1}^{n}\delta_r$$
$$= m(b-a) \le L(P,f) \le S(P,f) \le U(P,f) \le M(b-a)$$

Thus, $L(P,f) \le S(P,f) \le U(P,f)$ and these sum are bound. Hence, the set of lower and upper sums are bounded sets.



(a) Lowersum of f(x) at n=10 (b) Lowersum of f(x) at n=20 (c) Lowersum of f(x) at n=40

(d) Uppersum of f(x) at n=10 (e) Uppersum of f(x) at n=20 (f) Uppersum of f(x) at n=40

Figure 6: Uppersum and Lowersum of $f(x) = 1.5sinx + 2cosx$,at$(1.36 \leq x \leq 5.98)$

.

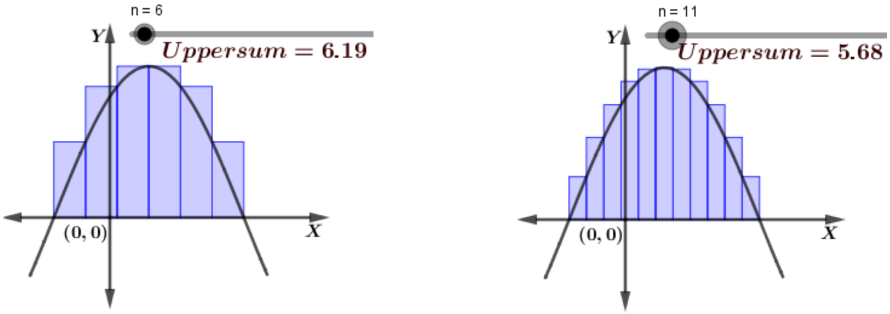
Theorem 2.2. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, if $P_1, P_2 \in P[a, b]$ and $P_1 \subset P_2$, then (i) $U(P_1, f) \geq U(P_2, f)$ and (ii) $L(P_1, f) \leq L(P_2, f)$* [1, 6].
Proof. Let $P_1 = \{x_0, x_1, x_2, \cdots, x_n\}$ be the partition of [a,b]. First consider the case when P_2 contains one point x' then P_1.
Suppose, this additional point x' belong to the r^{th} subintervals $[x_{r-1} - x_r]$.
Let $M_{r_1} = sup\{f(x) : x \in [x_{r-1}, x']\}$; $\quad M_{r_2} = sup\{f(x) : x \in [x', x_r]\}$
Then, $M_{r_1} \leq M_r$,and $M_{r_2} \leq M_r \Rightarrow M_r - M_{r_1} \geq 0$,and $M_r - M_{r_2} \geq 0$
Now,

$$\begin{aligned}
U(P_1, f) - U(P_2, f) &= [M_1\delta_1 + M_2\delta_2 + \cdots + M_r\delta_r + \cdots + M_n\delta_n] \\
&\quad - [M_1\delta_1 + M_2\delta_2 + \cdots + \cdots + M_{r_1}\delta_{r_1} + M_{r_2}\delta_{r_2} + \cdots + M_n\delta_n] \\
&= M_r\delta_r + M_{r_1}\delta_{r_1} + M_{r_2}\delta_{r_2} \\
&= M_r(x_r - x_{r-1}) + M_{r_1}(x' - x_{r-1}) + M_{r_2}(x_r - x') \\
&= M_r(x_r - x_{r-1}) + M_{r_1}(x' - x_{r-1}) + M_{r_2}(x_r - x') \\
&= M_r(x_r - x') + M_r(x' - x_{r-1}) - M_{r_1}(x' - x_{r-1}) - M_{r_2}(x_r - x') \\
&= (M_r - M_{r_2})(x_r - x') + (M_r - M_{r_1})(x' - x_{r-1}) \geq 0 \qquad (1)
\end{aligned}$$

$\therefore U(P_1, f) \geq U(P_2, f)$. If P_2 contains p more additional points than as equation (1), we can show $U(P_1, f) \geq U(P_2, f)$. Similarly, we can show that $L(P_1, f) \leq L(P_2, f)$

Theorem 2.3. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and $P_1, P_2 \in P[a, b]$, then $U(P_1, f) \geq L(P_2, f)$* [1, 6, 7].
Proof. Let $P = P_1 \cup P_2$, with $P_1 \subset P$ and $P_2 \subset P$, then by the theorem 2.2 we have

$U(P_1, f) \geq U(P, f)$ and $L(P_2, f) \leq L(P, f)$. Also, we have $L(P, f) \leq U(P, f)$.
Thus, $U(P_1, f) \geq U(P, f) \geq L(P, f) \geq L(P_2, f)$
$\Rightarrow U(P_1, f) \geq L(P_2, f)$



(a) Uppersum of f(x) at n=6 (b) Lowersum of f(x) at n=11

Figure 7: $U(P_1, f) \geq U(P_2, f)$,$f(x) = 1.5sinx + 2cosx$,at$(-1.32 \leq x \leq 2.6)$

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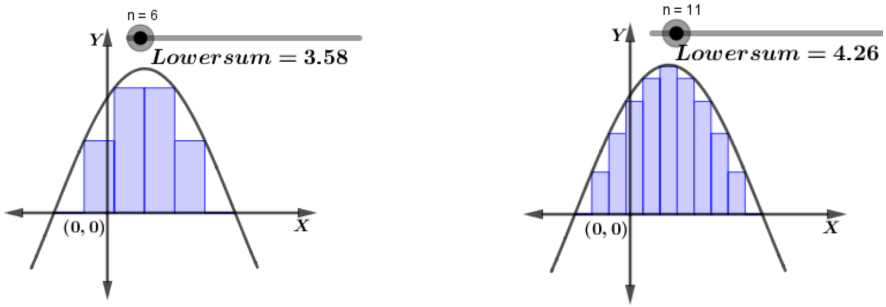
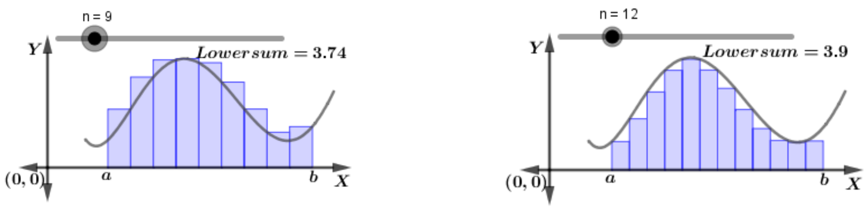


(a) Lowersum of f(x) at n=6 (b) Lowersum of f(x) at n=11

Figure 8: $L(P_1, f) \leq L(P_2, f)$,$f(x) = 1.5sinx + 2cosx$, at$(-1.32 \leq x \leq 2.6)$



(a) $U(P_1, f)$, at n=9 (b) $L(P_2, f)$, at n=12

Figure 9: $U(P_1, f) \geq L(P_2, f)$, where
$f(x) = -0.04x^5 + 0.68x^4 - 4.17x^3 + 10.9x^2 - 11.16x + 4.22$, at$(0.68 \leq x \leq 5.12)$

.

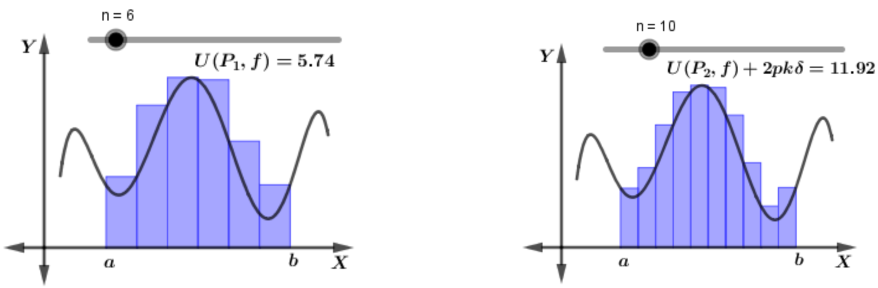
Theorem 2.4. *Let $f : [a, b] \to \mathbb{R}$ be a bounded function, $P_1, P_2 \in P[a, b]$, and $P_2 \supset P_1$ with p additional points. If $||P|| \leq \delta$ and $|f(x)| \leq k$, then (i) $U(P_1, f) \leq U(P_2, f) + 2pk\delta$ and (ii) $L(P_2, f) \leq L(P_1, f) + 2pk\delta$* [1, 6, 7].

Proof. Let $P_1 = \{x_0, x_1, x_2, \cdots, x_n\}$. Also, consider the case when P_2 contains one more point say x' than P_1.Suppose $x' \in (x_{r-1}, x_r)$, then by the theorem 2.2 we have

$U(P_1, f) - U(P_2, f) = (M_r - M_{r_2})(x_r - x') + (M_r - M_{r_1})(x' - x_{r-1})$

Since, $-k \leq M_{r_1} \leq M_r \leq k$ and $-k \leq M_{r_2} \leq M_r \leq k$.

Thus, $0 \leq M_r - M_{r_1} \leq 2k$ and $0 \leq M_r - M_{r_2} \leq 2k$.



(a) $U(P_1, f) = 5.74$ (b) $U(P_2, f) + 2pk\delta = 11.92$

Figure 10: $f(x) = -0.19x^6 + 2.83x^5 - 15.75x^4 + 40.34x^3 - 49.74x^2 + 26.29x - 2.95, \text{at}(a = 0.26 \leq x \leq 4.58 = b)$

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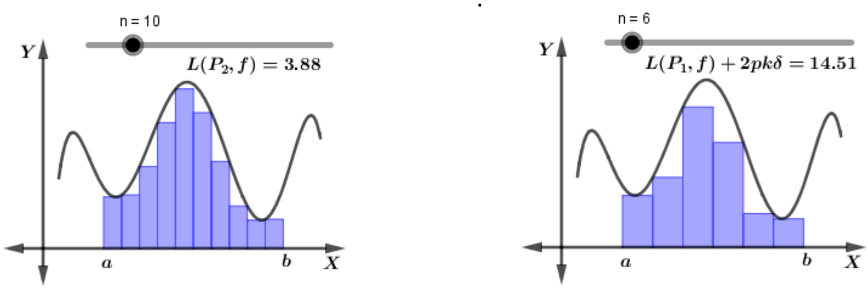


(a) $L(P_2, f) = 3.88$ (b) $L(P_1, f) + 2pk\delta = 14.51$

Figure 11: $f(x) = -0.19x^6 + 2.83x^5 - 15.75x^4 + 40.34x^3 - 49.74x^2 + 26.29x - 2.95, \text{at}(a = 0.26 \leq x \leq 4.58 = b)$

.

$$\begin{aligned}\therefore U(P_1, f) - U(P_2, f) &\leq 2k(x_r - x') + 2k(x' - x_{r-1}) \\ &= 2k(x_r - x' + x' - x_{r-1}) \\ &= 2k(x_r - x_{r-1}) \\ &= 2k\delta_r \\ &\leq 2k\delta\end{aligned}$$

If P_2 contains p more additional points than P_1 into same manner we can show that $U(P_1, f) - U(P_2, f) \leq 2kp\delta$
$\Rightarrow U(P_1, f) \leq U(P_2, f) + 2kp\delta$
Similarly, we can show that $L(P_2, f) \leq L(P_1, f) + 2kp\delta$

Theorem 2.5. *If $P \in P[a, b]$ also, f ,and g are two bounded functions on [a,b], then (i) $U(P, f + g) \leq U(P, f) + U(P, g)$ and (ii) $L(P, f + g) \geq L(P, f) + L(P, g)$* [12, 15].
Proof. Let $P_1 = \{x_0, x_1, x_2, \cdots, x_n\}$ be a partition of [a,b] and δ_r be the length of r^{th} subinterval $[x_{r-1}, x_r]$.

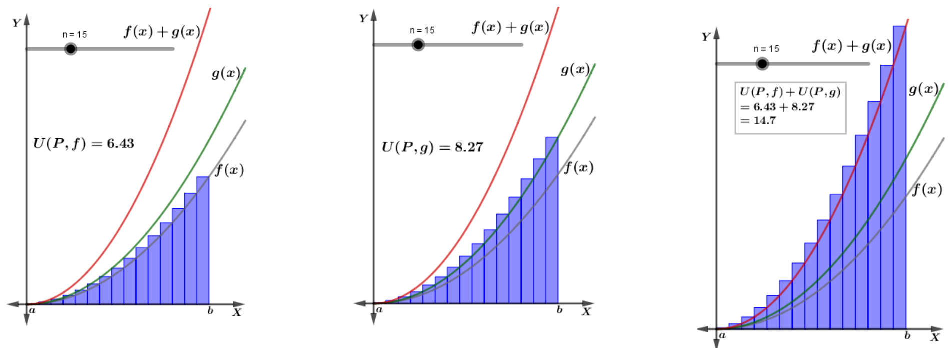
$$\begin{aligned}Let \quad M_r &= sup\{(f + g)(x) : x \in [x_{r-1}, x_r]\} \\ M_{r'} &= sup\{f(x) : x \in [x_{r-1}, x_r]\} \\ M_{r''} &= sup\{g(x) : x \in [x_{r-1}, x_r]\}\end{aligned}$$

Since, $M_{r'} \geq f(x)$ and $M_{r''} \geq g(x)$, for all $x \in [x_{r-1}, x_r]$.

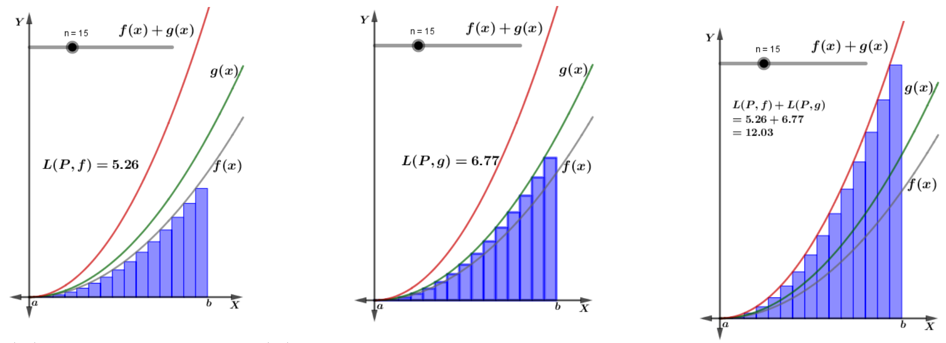
$$\begin{aligned}&\Rightarrow M_{r'} + M_{r''} \geq f(x) + g(x) \\ &\Rightarrow M_{r'} + M_{r''} \geq (f + g)(x), \quad \forall \in [x_{r-1}, x_r] \\ &\therefore M_{r'} + M_{r''} \geq M_r \\ &\Rightarrow M_{r'}\delta_r + M_{r''}\delta_r \geq M_r\delta_r \\ &\Rightarrow \sum M_{r'}\delta_r + \sum M_{r''}\delta_r \geq \sum M_r\delta_r \\ &\Rightarrow U(P, f) + U(P, g) \geq U(P, f + g)\end{aligned}$$

Also, $f(x) \geq m_{r'}$ and $g(x) \geq m_{r''}$ for all $x \in [x_{r-1}, x_r]$

$$\begin{aligned}&\Rightarrow f(x) + g(x) \geq m_{r'} + m_{r''}, \forall x \in [x_{r-1}, x_r] \\ &\Rightarrow m_r \geq m_{r'} + m_{r''} \\ &\Rightarrow m_r\delta_r \geq m_{r'}\delta_r + m_{r''}\delta_r \\ &\Rightarrow \sum m_r\delta_r \geq \sum m_{r'}\delta_r + \sum m_{r''}\delta_r \\ &\Rightarrow L(P, f + g) \geq L(P, f) + L(P, g)\end{aligned}$$



(a) Uppersum of f(x) at n=15 (b) Uppersum of g(x) at n=15 (c) Uppersum of f(x)+g(x)



(d) Lowersum of f(x) at n=15 (e) Lowersum of g(x) at n=15 (f) Lowersum of f(x)+g(x)

Figure 12

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3. Conclusion. GeoGebra software's ability is to transfer abstractions into concrete mathematical concepts and have a positive effect on learning directly contributed to the development of students' understanding of concepts. A conceptual understanding of mathematical concepts plays an important role in the teaching and learning process of mathematics and is provide using GeoGebra software. Therefore, it is recommended for use by students of all levels to enrich their conceptual understanding of mathematical content in teaching-learning activities.

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