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INTUITIONISTIC FUZZY CONGRUENCES ON PRODUCT LATTICES

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Abstract: In this work, the concept of intuitionistic fuzzy congruences on lattice X was introduced and was defined direct product between them. Also some characterizations of them were established. Finally, isomorphism between factor lattices of similarity classes was investigated.

Keywords and Phrases: Fuzzy set theory, intuitionistic fuzzy set, lattices and related structures, congruence relations, direct product, isomorphisms.

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1. Introduction

Zadeh [36] introduced the concepts of a fuzzy set. Intuitionistic fuzzy set (in short IFS) introduced by Atanassov [1]. Intuitionistic fuzzy sets have been found to be very useful in diversely applied areas of science and technology. A lattice is an abstract structure studied in the mathematical subdisciplines of order theory and abstract algebra. It consists of a partially ordered set in which every two elements have a unique supremum (also called a least upper bound or join) and a unique infimum (also called a greatest lower bound or meet). An example is given by the natural numbers, partially ordered by divisibility, for which the unique supremum is the least common multiple and the unique infimum is the greatest common divisor. In the history of fuzzy mathematics, fuzzy relations were early considered to be useful in various applications, and have therefore been extensively investigated. For a contemporary general approach to fuzzy relations one should look in Belohlavek's book [2], and also to other general publications e.g., the books by Klir and Yuan [8] and Turunen [34]. Relational equations and applications are presented by Di Nola, Sessa, Pedrycz and Sanchezin [5], and some new approaches to fuzzy relations are given by Ignjatovic, Ciric and Bogdanovicin [7, 3]. Das [4] and Yijia [35] have introduced the concept of fuzzy congruences in the background of semigroups. The author investigated some properties of fuzzy algebraic structures [10-33]. In this paper, the concepts of intuitionistic fuzzy equivalence relation and intuitionistic fuzzy congruence on lattices are introduced and discussed. Let X and Y be lattices such that $A = (\mu_A, \nu_A) \in IFS(X \times X)$ and $B = (\mu_B, \nu_B) \in IFS(Y \times Y)$. We define intuitionistic fuzzy congruences on lattice X as $IFC(X \times X)$ and investigate some properties of them. We introduce direct product of A and B and we prove that if $A \in IFC(X \times X)$ and $B \in IFC(Y \times Y)$, then $A \times B \in IFC(X \times Y \times X \times Y)$ and under some conditions, we show the converse of about assertion. Finally, we prove isomorphism $\dfrac{X \times X}{A} \times \dfrac{Y \times Y}{B} \cong \dfrac{X \times Y \times X \times Y}{A \times B}$ as factor lattices of similarity classes.

2. Preliminaries

This section emphasizes on basic definitions, results and properties of lattices, fuzzy sets and intuitionistic fuzzy sets, which serve as a prerequisite for the research work done in the paper. For details we refer to [1, 6, 9].

Definition 2.1. *Let P be a nonempty set. A partial order P is a binary relation $\leq$ on P such that, for all $x, y, z \in P$, the following conditions are hold:*
(1) $x \leq x$ (reflexivity);
(2) $x \leq y$ and $y \leq x$ imply $x = y$ (antisymmetry);
(3) $x \leq y$ and $y \leq z$ imply $x = z$ (transivity).
A set P equipped with an order relation $\leq$ is said to be an ordered set (or partially ordered set or poset).

Definition 2.2. *A partially ordered set in which every pair of elements has a join (or least upper bound) and a meet (or greatest lower bound) is called a lattice.*

Definition 2.3. *Let L and K be lattices. Then map $\varphi : L \to K$ is an isomorphism if φ is one-to-one, onto and if $\varphi(a \wedge b) = \varphi(a) \wedge \varphi(b)$ and $\varphi(a \vee b) = \varphi(a) \vee \varphi(b)$ for all $a, b \in L$.*

Definition 2.4. *Let L and K be lattices. Define $\wedge : L \times K \to L \times K$ by $(l_1, k_1) \wedge (l_2, k_2) = (l_1 \wedge l_2, k_1 \wedge k_2)$ and $\vee : L \times K \to L \times K$ by $(l_1, k_1) \vee (l_2, k_2) = (l_1 \vee l_2, k_1 \vee k_2)$ for all $l_1, l_2 \in L$ and $k_1, k_2 \in K$. Then $L \times K$ will be a lattice called the direct product of L and K.*

Definition 2.5. *Let X be an arbitrary set. A fuzzy set of X, we mean a function from X into $[0,1]$. A fuzzy binary relation on X is a fuzzy set defined on $X \times X$.*

Definition 2.6. *For sets X, Y and Z, $f = (f_1, f_2) : X \to Y \times Z$ is called a complex mapping if $f_1 : X \to Y$ and $f_2 : X \to Z$ are mappings.*

Definition 2.7. *Let X be a nonempty set. A complex mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called an intuitionistic fuzzy set (in short, IFS) in X if $\mu_A + \nu_A \leq 1$ where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) for each $x \in X$ to A, respectively. In particular $\emptyset_X$ and U_X denote the intuitionistic fuzzy empty set and intuitionistic fuzzy whole set in X defined by $\emptyset_X(x) = (0,1) \sim 0$ and $U_X(x) = (1,0) \sim 1$, respectively. We will denote the set of all $IFSs$ in X as $IFS(X)$.*

Definition 2.8. *Let X be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be $IFSs$ in X. Then*
(1) Inclusion: $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.
(2) Equality:$A = B$ iff $A \subseteq B$ and $B \subseteq A$.

3. Intuitionistic Fuzzy Congruences on Product Lattices

Definition 3.1. *Let X be a non empty set and $A = (\mu_A, \nu_A) \in IFS(X \times X)$. We say that $A = (\mu_A, \nu_A)$ is an equivalence relation on $X \times X$ if the following conditions hold:*
(1) $A(x,x) = (1,0)$,
(2) $A(x,y) = A(y,x)$,
(3) $A(x,z) \supseteq (\sup_{y\in X}\{\mu_A(x,y) \wedge \mu_A(y,z)\}, \inf_{y\in X}\{\nu(x,y)_A \vee \nu_A(y,z)\})$,
for all $x, y, z \in X$.

Remark 3.2. *Note that in Definition 3.1 we get the follwing statements for all $x, y, z \in X$.*
(1)

$$A(x,x) = (1,0) \Longleftrightarrow A(x,x) = (\mu_A(x,x), \nu_A(x,x)) = (1,0) \Longleftrightarrow \mu_A(x,x) = 1, \nu_A(x,x) = 0.$$

(2)

$$A(x,y) = A(y,x) \Longleftrightarrow A(x,y) = (\mu_A(x,y), \nu_A(x,y)) = A(y,x) = (\mu_A(y,x), \nu_A(y,x))$$
$$\Longleftrightarrow \mu_A(x,y) = \mu_A(y,x) \quad \text{and} \quad \nu_A(x,y) = \nu_A(y,x).$$

(3)

$$A(x,z) \supseteq (\sup_{y\in X}\{\mu(x,y) \wedge \mu(y,z)\}, \inf_{y\in X}\{\mu(x,y) \vee \mu(y,z)\})$$

$$\Longleftrightarrow A(x,z) = (\mu_A(x,z), \nu_A(x,z)) \supseteq (\sup_{y\in X}\{\mu_A(x,y) \wedge \mu_A(y,z)\}, \inf_{y\in X}\{\nu(x,y)_A \vee \nu_A(y,z)\})$$

$\Longleftrightarrow (\mu_A(x,z) \geq \sup_{y\in X}\{\mu_A(x,y) \wedge \mu_A(y,z)\}$ *and* $\nu_A(x,z) \leq \inf_{y\in X}\{\nu_A(x,y) \vee \nu_A(y,z)\}$.

Definition 3.3. *Let $A = (\mu_A, \nu_A) \in IFS(X \times X)$ be an equivalence relation on $X \times X$. The similarity class for each $x \in X$ is the intuitionistic fuzzy set $A_x : X \to [0,1] \times [0,1]$ such that $A_x(y) = A(x,y)$ for all $y \in X$.*

Lemma 3.4. *Let X be a non empty set and $A = (\mu_A, \nu_A) \in IFS(X \times X)$ be an equivalence relation on $X \times X$. Then $A_x = A_y$ if and only if $A(x,y) = (1,0)$ for all $x, y \in X$.*

Proof. Let $x, y \in X$. If $A_x = A_y$, then $A_x(y) = A_y(y) = A(y,y) = (1,0)$ and then $A(x,y) = (1,0)$.
Conversely, if $A(x,y) = (1,0)$, then $A_x(y) = (1,0) = A_y(y)$ and so $A_x = A_y$.

Definition 3.5. *Let X be a lattice and $A = (\mu_A, \nu_A) \in IFS(X \times X)$ be an equivalence relation on $X \times X$. Then $A = (\mu_A, \nu_A)$ is join compatible if*

$$A(x_1 \vee x_2, y_1 \vee y_2) = (\mu_A(x_1 \vee x_2, y_1 \vee y_2), \nu_A(x_1 \vee x_2, y_1 \vee y_2))$$

$$\supseteq (\mu_A(x_1, y_1) \wedge \mu_A(x_2, y_2), \nu_A(x_1, y_1) \vee \nu_A(x_2, y_2))$$

and $A = (\mu_A, \nu_A)$ is meet compatible if

$$A(x_1 \wedge x_2, y_1 \wedge y_2) = (\mu_A(x_1 \wedge x_2, y_1 \wedge y_2), \nu_A(x_1 \wedge x_2, y_1 \wedge y_2))$$

$$\supseteq (\mu_A(x_1, y_1) \wedge \mu_A(x_2, y_2), \nu_A(x_1, y_1) \vee \nu_A(x_2, y_2))$$

for all x_1, x_2, y_1, y_2 in X. If $A = (\mu_A, \nu_A)$ is both join compatible and meet compatible, then$A = (\mu_A, \nu_A)$ is an intuitionistic fuzzy congruence on $X \times X$.
Denote by $IFC(X \times X)$, the set of all intuitionistic fuzzy congruences on lattice X.

Example 3.6. Let X be a non empty lattice and $A = (\mu_A, \nu_A) \in IFS(X \times X)$. Define

$$\mu_A(x,y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_A(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Then $A = (\mu_A, \nu_A) \in IFC(X \times X)$.

Lemma 3.7. *Let X be a lattice and $A = (\mu_A, \nu_A) \in IFS(X \times X)$ be an equivalence relation on $X \times X$. Then*

(1) $A = (\mu_A, \nu_A)$ *is join compatible if and only if* $A(x_1 \vee t, y_1 \vee t) \supseteq A(x_1, y_1)$,
(2) $A = (\mu_A, \nu_A)$ *is meet compatible if and only if* $A(x_1 \wedge t, y_1 \wedge t) \supseteq A(x_1, y_1)$,
for all x_1, y_1, t *in* X.

Proof. Let x_1, x_2, y_1, y_2, t in X.
(1) If $A = (\mu_A, \nu_A)$ is join compatible, then

$$A(x_1 \vee t, y_1 \vee t) = (\mu_A(x_1 \vee t, y_1 \vee t), \nu_A(x_1 \vee t, y_1 \vee t))$$

$$\supseteq (\mu_A(x_1, y_1) \wedge \mu_A(t,t), \nu_A(x_1, y_1) \vee \nu_A(t,t))$$

which means that

$$\mu_A(x_1 \vee t, y_1 \vee t) \geq \mu_A(x_1, y_1) \wedge \mu_A(t,t) = \mu_A(x_1, y_1) \wedge 1 = \mu_A(x_1, y_1)$$

and then

$$\mu_A(x_1 \vee t, y_1 \vee t) \geq \mu_A(x_1, y_1). \quad \text{(a)}$$

Also

$$\nu_A(x_1 \vee t, y_1 \vee t) \leq \nu_A(x_1, y_1) \vee \nu_A(t,t) = \nu_A(x_1, y_1) \vee 0 = \nu_A(x_1, y_1)$$

thus

$$\nu_A(x_1 \vee t, y_1 \vee t) \leq \nu_A(x_1, y_1). \quad \text{(b)}$$

Now from (a) and (b) we will have that

$$A(x_1 \vee t, y_1 \vee t) = (\mu_A(x_1 \vee t, y_1 \vee t), \nu_A(x_1 \vee t, y_1 \vee t))$$

$$\supseteq (\mu_A(x_1, y_1), \nu_A(x_1, y_1)) = A(x_1, y_1).$$

Conversely, let $A(x_1, y_1) \subseteq A(x_1 \vee t, y_1 \vee t)$ and $A(x_2, y_2) \subseteq A(x_2 \vee t, y_2 \vee t)$. Then $\mu_A(x_1, y_1) \leq \mu_A(x_1 \vee t, y_1 \vee t)$ and $\mu_A(x_2, y_2) \leq \mu_A(x_2 \vee t, y_2 \vee t)$. Now

$$\mu_A(x_1 \vee x_2, y_1 \vee y_2) = \mu_A((x_1, y_1) \vee (x_2, y_2))$$

$$\geq \sup_{(t,t) \in X \times X} \{\mu_A((x_1, y_1) \vee (t,t)) \wedge \mu_A((t,t) \vee (x_2, y_2))\}$$

$$= \sup_{(t,t) \in X \times X} \{\mu_A((x_1, y_1) \vee (t,t)) \wedge \mu_A((x_2, y_2) \vee (t,t))\}$$

$$= \sup_{(t,t) \in X \times X} \{\mu_A(x_1 \vee t, y_1 \vee t) \wedge \mu_A(x_2 \vee t, y_2 \vee t)\}$$

$$\geq \mu_A(x_1, y_1) \wedge \mu_A(x_2, y_2)$$

and then

$$\mu_A(x_1 \vee x_2, y_1 \vee y_2) \geq \mu_A(x_1, y_1) \wedge \mu_A(x_2, y_2). \quad \text{(a)}$$

Also as $A(x_1, y_1) \subseteq A(x_1 \vee t, y_1 \vee t)$ and $A(x_2, y_2) \subseteq A(x_2 \vee t, y_2 \vee t)$ so $\nu_A(x_1, y_1) \geq \nu_A(x_1 \vee t, y_1 \vee t)$ and $\nu_A(x_2, y_2) \geq \nu_A(x_2 \vee t, y_2 \vee t)$. Thus

$$\nu_A(x_1 \vee x_2, y_1 \vee y_2) = \nu_A((x_1, y_1) \vee (x_2, y_2))$$

$$\leq \inf_{(t,t) \in X \times X} \{\nu_A((x_1, y_1) \vee (t, t)) \vee \nu_A((t, t) \vee (x_2, y_2))\}$$

$$= \inf_{(t,t) \in X \times X} \{\nu_A((x_1, y_1) \vee (t, t)) \vee \nu_A((x_2, y_2) \vee (t, t))\}$$

$$= \inf_{(t,t) \in X \times X} \{\nu_A(x_1 \vee t, y_1 \vee t) \vee \nu_A(x_2 \vee t, y_2 \vee t)\}$$

$$\leq \nu_A(x_1, y_1) \vee \nu_A(x_2, y_2)$$

and

$$\nu_A(x_1 \vee x_2, y_1 \vee y_2) \leq \nu_A(x_1, y_1) \vee \nu_A(x_2, y_2). \quad \text{(b)}$$

Therefore (a) and (b) give us that

$$A(x_1 \vee x_2, y_1 \vee y_2) = (\mu_A(x_1 \vee x_2, y_1 \vee y_2), \nu_A(x_1 \vee x_2, y_1 \vee y_2))$$

$$\supseteq (\mu_A(x_1, y_1) \wedge \mu_A(x_2, y_2), \nu_A(x_1, y_1) \vee \nu_A(x_2, y_2))$$

which means that $A = (\mu_A, \nu_A)$ is join compatible.
(2) The proof is similar as (1).

Definition 3.8. *Let X and Y be lattices such that $A = (\mu_A, \nu_A) \in IFS(X \times X)$ and $B = (\mu_B, \nu_B) \in IFS(Y \times Y)$. Define $A \times B \in IFS(X \times Y \times X \times Y)$ as*

$$A \times B = (\mu_A, \nu_A) \times (\mu_B, \nu_B) = (\mu_A \times \mu_B, \nu_A \times \nu_B) = (\mu_{A \times B}, \nu_{A \times B})$$

such that

$$\mu_{A \times B}((x_1, y_1), (x_2, y_2)) = \mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2)$$

and

$$\nu_{A \times B}((x_1, y_1), (x_2, y_2)) = \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2)$$

for all x_1, x_2 in X and y_1, y_2 in Y.
Thus

$$(A \times B)((x_1, y_1), (x_2, y_2)) = (\mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2), \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2))$$

for all x_1, x_2 in X and y_1, y_2 in Y.

Preposition 3.9. *Let X and Y be lattices and $A = (\mu_A, \nu_A) \in IFC(X \times X)$ and $B = (\mu_B, \nu_B) \in IFC(Y \times Y)$. Then $A \times B \in IFC(X \times Y \times X \times Y)$.*

Proof. (1) Let $x \in X$ and $y \in Y$. Then

$$(A \times B)((x, y), (x, y)) = (\mu_{A\times B}((x, y), (x, y)), \nu_{A\times B}((x, y), (x, y)))$$

$$= (\mu_A(x, x) \wedge \mu_B(y, y), \nu_A(x, x) \vee \nu_B(y, y)) = (1 \wedge 1, 0 \vee 0) = (1, 0).$$

(2) Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then

$$(A \times B)((x_1, y_1), (x_2, y_2))$$

$$= (\mu_{A\times B}((x_1, y_1), (x_2, y_2)), \nu_{A\times B}((x_1, y_1), (x_2, y_2)))$$

$$= (\mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2), \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2))$$

$$= (\mu_A(x_2, x_1) \wedge \mu_B(y_2, y_1), \nu_A(x_2, x_1) \vee \nu_B(y_2, y_1))$$

$$= (\mu_{A\times B}((x_2, y_2), (x_1, y_1)), \nu_{A\times B}((x_2, y_2), (x_1, y_1)))$$

$$= (A \times B)((x_2, y_2), (x_1, y_1))$$

(3) Let $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Then

$$\mu_{A\times B}((x_1, y_1), (x_3, y_3)) = \mu_A(x_1, x_3) \wedge \mu_B(y_1, y_3)$$

$$\geq \sup_{x_2\in X}\{\mu_A(x_1, x_2) \wedge \mu_A(x_2, x_3)\} \wedge \sup_{y_2\in Y}\{\mu_B(y_1, y_2) \wedge \mu_B(y_2, y_3)\}$$

$$= \sup_{(x_2,y_2)\in X\times Y}\{\mu_A(x_1, x_2) \wedge \mu_A(x_2, x_3) \wedge \mu_B(y_1, y_2) \wedge \mu_B(y_2, y_3)\}$$

$$= \sup_{(x_2,y_2)\in X\times Y}\{\mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2) \wedge \mu_A(x_2, x_3) \wedge \mu_B(y_2, y_3)\}$$

$$= \sup_{(x_2,y_2)\in X\times Y}\{\mu_{A\times B}((x_1, y_1), (x_2, y_2)) \wedge \mu_{A\times B}((x_2, y_2), (x_3, y_3))\}.$$

Also

$$\nu_{A\times B}((x_1, y_1), (x_3, y_3)) = \nu_A(x_1, x_3) \vee \nu_B(y_1, y_3)$$

$$\leq \inf_{x_2\in X}\{\nu_A(x_1, x_2) \vee \nu_A(x_2, x_3)\} \vee \inf_{y_2\in Y}\{\nu_B(y_1, y_2) \vee \nu_B(y_2, y_3)\}$$

$$= \inf_{(x_2,y_2)\in X\times Y}\{\nu_A(x_1, x_2) \vee \nu_A(x_2, x_3) \vee \nu_B(y_1, y_2) \vee \nu_B(y_2, y_3)\}$$

$$= \inf_{(x_2,y_2)\in X\times Y}\{\nu_A(x_1, x_2) \vee \nu_B(y_1, y_2) \vee \nu_A(x_2, x_3) \vee \nu_B(y_2, y_3)\}$$

$$= \inf_{(x_2,y_2)\in X\times Y}\{\nu_{A\times B}((x_1, y_1), (x_2, y_2)) \vee \nu_{A\times B}((x_2, y_2), (x_3, y_3))\}.$$

Thus

$$(A\times B)((x_1,y_1),(x_3,y_3)) \supseteq (\sup_{(x_2,y_2)\in X\times Y}\{\mu_{A\times B}((x_1,y_1),(x_2,y_2))\wedge\mu_{A\times B}((x_2,y_2),(x_3,y_3))\}$$

$$,\inf_{(x_2,y_2)\in X\times Y}\{\nu_{A\times B}((x_1,y_1),(x_2,y_2))\vee\nu_{A\times B}((x_2,y_2),(x_3,y_3))\}).$$

Therefore from (1)-(3) we get that $A\times B$ is an equivalence relation on $X\times Y\times X\times Y$. Now we show that $A\times B$ is join and meet compatible. Let $(t_1,t_2)\in X\times Y$ then

$$\mu_{A\times B}((x_1,y_1)\vee(t_1,t_2),(x_2,y_2)\vee(t_1,t_2))$$

$$= \mu_{A\times B}((x_1\vee t_1,y_1\vee t_2),(x_2\vee t_1,y_2\vee t_2))$$

$$= \mu_A(x_1\vee t_1,y_1\vee t_2)\wedge\mu_B(x_2\vee t_1,y_2\vee t_2)$$

$$\geq \mu_A(x_1,y_1)\wedge\mu_B(x_2,y_2) = \mu_{A\times B}((x_1,y_1),(x_2,y_2))$$

and so

$$\mu_{A\times B}((x_1,y_1)\vee(t_1,t_2),(x_2,y_2)\vee(t_1,t_2)) \geq \mu_{A\times B}((x_1,y_1),(x_2,y_2)). \quad \text{(a)}$$

Also

$$\nu_{A\times B}((x_1,y_1)\vee(t_1,t_2),(x_2,y_2)\vee(t_1,t_2))$$

$$= \nu_{A\times B}((x_1\vee t_1,y_1\vee t_2),(x_2\vee t_1,y_2\vee t_2))$$

$$= \nu_A(x_1\vee t_1,y_1\vee t_2)\vee\nu_B(x_2\vee t_1,y_2\vee t_2)$$

$$\leq \nu_A(x_1,y_1)\vee\nu_B(x_2,y_2) = \nu_{A\times B}((x_1,y_1),(x_2,y_2))$$

and so

$$\nu_{A\times B}((x_1,y_1)\vee(t_1,t_2),(x_2,y_2)\vee(t_1,t_2)) \leq \nu_{A\times B}((x_1,y_1),(x_2,y_2)). \quad \text{(b)}$$

Now from (a) and (b) we get that

$$(A\times B)((x_1,y_1)\vee(t_1,t_2),(x_2,y_2)\vee(t_1,t_2)) \supseteq (A\times B)((x_1,y_1),(x_2,y_2))$$

and so by Lemma 3.7 (part(1)) we obtain that $A\times B$ is join compatible. Also

$$\mu_{A\times B}((x_1,y_1)\wedge(t_1,t_2),(x_2,y_2)\wedge(t_1,t_2))$$

$$= \mu_{A\times B}((x_1\wedge t_1,y_1\wedge t_2),(x_2\wedge t_1,y_2\wedge t_2))$$

$$= \mu_A(x_1\wedge t_1,y_1\wedge t_2)\wedge\mu_B(x_2\wedge t_1,y_2\wedge t_2)$$

$$\geq \mu_A(x_1, y_1) \wedge \mu_B(x_2, y_2) = \mu_{A\times B}((x_1, y_1), (x_2, y_2))$$

and so

$$\mu_{A\times B}((x_1, y_1) \wedge (t_1, t_2), (x_2, y_2) \wedge (t_1, t_2)) \geq \mu_{A\times B}((x_1, y_1), (x_2, y_2)). \quad \text{(a)}$$

Also

$$\begin{aligned} &\nu_{A\times B}((x_1, y_1) \wedge (t_1, t_2), (x_2, y_2) \wedge (t_1, t_2)) \\ &= \nu_{A\times B}((x_1 \wedge t_1, y_1 \wedge t_2), (x_2 \wedge t_1, y_2 \wedge t_2)) \\ &= \nu_A(x_1 \wedge t_1, y_1 \wedge t_2) \vee \nu_B(x_2 \wedge t_1, y_2 \wedge t_2) \\ &\leq \nu_A(x_1, y_1) \vee \nu_B(x_2, y_2) = \nu_{A\times B}((x_1, y_1), (x_2, y_2)) \end{aligned}$$

and

$$\nu_{A\times B}((x_1, y_1) \wedge (t_1, t_2), (x_2, y_2) \wedge (t_1, t_2)) \leq \nu_{A\times B}((x_1, y_1), (x_2, y_2)). \quad \text{(b)}$$

Thus from (a) and (b) we can say

$$(A \times B)((x_1, y_1) \wedge (t_1, t_2), (x_2, y_2) \wedge (t_1, t_2)) \supseteq (A \times B)((x_1, y_1), (x_2, y_2))$$

and as Lemma 3.7 (part(2)) so $A \times B$ is meet compatible.
Therefore $A \times B \in IFC(X \times Y \times X \times Y)$.

Example 3.10. Let $A = (\mu_A, \nu_A) \in IFS(X \times X)$ such that

$$\mu_A(x, y) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_A(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

Define $A \times A = (\mu_{A\times A}, \nu_{A\times A}) \in IFS(X \times Y \times X \times Y)$ as:

$$\mu_{A\times A}((x, y), (z, t) = \mu_A(x, z) \wedge \mu_A(y, t) = \begin{cases} 1 & \text{if } (x, y) = (z, t) \\ 0 & \text{otherwise} \end{cases}$$

and

$$\nu_{A\times A}((x, y), (z, t) = \nu_A(x, z) \vee \nu_A(y, t) = \begin{cases} 0 & \text{if } (x, y) = (z, t) \\ 1 & \text{otherwise} \end{cases}$$

for all $x, y, z, t \in X$.
Then $A = (\mu_A, \nu_A) \in IFC(X \times X)$ and $A \times A = (\mu_{A\times A}, \nu_{A\times A}) \in IFC(X \times Y \times$

$X \times Y$).

Proposition 3.11. *Let $C = (\mu_C, \nu_C) \in IFC(X \times Y \times X \times Y)$. Then for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have the following statements.*
(1) $C((x_1, y_1), (x_2, y_1)) = C((x_1, y_2), (x_2, y_2))$.
(2) $C((x_1, y_1), (x_1, y_2)) = C((x_2, y_1), (x_2, y_2))$.
Proof. Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then
(1)

$$C((x_1, y_1), (x_2, y_1)) \leq C((x_1, y_1) \vee (x_1 \wedge x_2, y_2), (x_2, y_1) \vee (x_1 \wedge x_2, y_2))$$

(Lemma 3.7 part(1))

$$= \rho((x_1 \vee x_1 \wedge x_2, y_1 \vee y_2), (x_2 \vee x_1 \wedge x_2, y_1 \vee y_2)) = C((x_1, y_1 \vee y_2), (x_2, y_1 \vee y_2))$$

$$\leq C((x_1, y_1 \vee y_2) \wedge (x_1 \vee x_2, y_2), (x_2, y_1 \vee y_2) \wedge (x_1 \vee x_2, y_2)) \text{ (Lemma 3.7 part(2))}$$

$$= C((x_1 \wedge x_1 \vee x_2, y_1 \vee y_2 \wedge y_2), (x_2 \wedge x_1 \vee x_2, y_1 \vee y_2 \wedge y_2)) = C((x_1, y_2), (x_2, y_2)).$$

Similarly we can prove that $C((x_1, y_1), (x_2, y_1)) \geq C((x_1, y_2), (x_2, y_2))$ and thus $C((x_1, y_1), (x_2, y_1)) = C((x_1, y_2), (x_2, y_2))$.
(2) The proof is similar as (1).

Not that we can prove the converse of Proposition 3.9 such that if $C = (\mu_C, \nu_C) \in FCT(X \times Y \times X \times Y)$, then $C = A \times B$ where $A = (\mu_A, \nu_A) \in IFC(X \times X)$ and $B = (\mu_B, \nu_B) \in IFC(Y \times Y)$.

Proposition 3.12. *Let $C = (\mu_C, \nu_C) \in IFC(X \times Y \times X \times Y)$. Define $A = (\mu_A, \nu_A) \in IFS(X \times X)$ and $B = (\mu_B, \nu_B) \in IFS(Y \times Y)$ by:*

$$A(x_1, x_2) = (\mu_A(x_1, x_2), \nu_A(x_1, x_2))$$

$$= C((x_1, y_1), (x_2, y_1)) = (\mu_C((x_1, y_1), (x_2, y_1)), \nu_C((x_1, y_1), (x_2, y_1))$$

and

$$B(y_1, y_2) = (\mu_B(y_1, y_2), \nu_B(y_1, y_2))$$

$$= C((x_1, y_1), (x_1, y_2)) = (\mu_C((x_1, y_1), (x_1, y_2)), \nu_C((x_1, y_1), (x_1, y_2))$$

for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then $C = A \times B$.
Proof. Using Proposition 3.11 we obtain that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ are well defined. Firstly, we must prove that $A = (\mu_A, \nu_A) \in IFS(X \times X)$ be an equivalence relation on $X \times X$ and $B = (\mu_B, \nu_B) \in IFS(Y \times Y)$ be an equivalence

relation on $Y \times Y$.
(1) Let $x_1 \in X$ and $y_1 \in Y$. Then

$$A(x_1, x_1) = (\mu_A(x_1, x_1), \nu_A(x_1, x_1)) = C((x_1, y_1), (x_1, y_1))$$

$$= (\mu_C((x_1, x_1), (x_1, x_1)), \nu_C((x_1, x_1), (x_1, x_1)) = ((1, 0) \sim 1, (0, 1) \sim 0) = (1, 0).$$

(2) Let $x_1, x_2 \in X$ and $y_1, y_2 \in Y$. Then

$$A(x_1, x_2) = C((x_1, y_1), (x_2, y_1)) = C((x_2, y_1), (x_1, y_1)) = A(x_2, x_1).$$

(3) Let $x_1, x_3 \in X$ and $y_1, y_3 \in Y$. Then

$$A(x_1, x_3) = (\mu_A(x_1, x_3), \nu_A(x_1, x_3))$$

$$= C((x_1, y_1), (x_3, y_1)) = (\mu_C((x_1, y_1), (x_3, y_1)), \nu_C((x_1, y_1), (x_3, y_1)).$$

As

$$\mu_A(x_1, x_3) = \mu_C((x_1, y_1), (x_3, y_1))$$

$$\geq \sup_{(x_2, y_2) \in (X \times Y)} \{\mu_C((x_1, y_1), (x_2, y_2)) \wedge \mu_C((x_2, y_2), (x_3, y_1))\}$$

$$\geq \sup_{x_2 \in X} \{\mu_C((x_1, y_1), (x_2, y_1)) \wedge \mu_C((x_2, y_1), (x_3, y_1))\} = \sup_{x_2 \in X} \{\mu_A(x_1, x_2) \wedge \mu_A(x_2, x_3)\}$$

and so

$$\mu_A(x_1, x_3) \geq \sup_{x_2 \in X} \{\mu_A(x_1, x_2) \wedge \mu_A(x_2, x_3)\}. \quad \text{(a)}$$

Also

$$\nu_A(x_1, x_3) = \nu_C((x_1, y_1), (x_3, y_1))$$

$$\leq \inf_{(x_2, y_2) \in (X \times Y)} \{\nu_C((x_1, y_1), (x_2, y_2)) \vee \nu_C((x_2, y_2), (x_3, y_1))\}$$

$$\leq \inf_{x_2 \in X} \{\nu_C((x_1, y_1), (x_2, y_1)) \vee \nu_C((x_2, y_1), (x_3, y_1))\} = \inf_{x_2 \in X} \{\nu_A(x_1, x_2) \vee \nu_A(x_2, x_3)\}$$

and thus

$$\nu_A(x_1, x_3) \leq \inf_{x_2 \in X} \{\nu_A(x_1, x_2) \vee \nu_A(x_2, x_3)\}. \quad \text{(b)}$$

Therefore

$$A(x_1, x_3) = (\mu_A(x_1, x_3), \nu_A(x_1, x_3)) \supseteq$$

$$(\sup_{x_2 \in X} \{\mu_A(x_1, x_2) \wedge \mu_A(x_2, x_3)\}, \inf_{x_2 \in X} \{\nu_A(x_1, x_2) \vee \nu_A(x_2, x_3)\})$$

which means that $A = (\mu_A, \nu_A) \in IFS(X \times X)$ be an equivalence relation on $X \times X$.
Now

$$A(x_1 \vee x_3, x_2 \vee x_3) = C((x_1 \vee x_3, y_1), (x_2 \vee x_3, y_1)) = C((x_1, y_1) \vee (x_3, y_1), (x_2, y_1) \vee (x_3, y_1))$$

$$\geq C((x_1, y_1), (x_2, y_1)) = A(x_1, x_2)$$

and

$$A(x_1 \wedge x_3, x_2 \wedge x_3) = C((x_1 \wedge x_3, y_1), (x_2 \wedge x_3, y_1)) = C((x_1, y_1) \wedge (x_3, y_1), (x_2, y_1) \wedge (x_3, y_1))$$

$$\geq C((x_1, y_1), (x_2, y_1)) = A(x_1, x_2)$$

mean that $A = (\mu_A, \nu_A) \in IFC(X \times X)$.
In a similar way it can be proved that $B = (\mu_B, \nu_B) \in IFC(Y \times Y)$.
Next we must show that $C = A \times B$ as

$$C((x_1, y_1), (x_2, y_2)) = (\mu_C((x_1, y_1), (x_2, y_2)), \nu_C((x_1, y_1), (x_2, y_2)))$$

$$= (\mu_{A \times B}((x_1, y_1), (x_2, y_2))), \nu_{A \times B}((x_1, y_1), (x_2, y_2))))$$

$$= (\mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2), \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2))$$

for all $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Firstly, we must prove that

$$\mu_C((x_1, y_1), (x_2, y_2)) = \mu_{A \times B}((x_1, y_1), (x_2, y_2))).$$

As

$$\mu_C((x_1, y_1), (x_2, y_2)) \leq \mu_C((x_1, y_1) \wedge (x_1 \vee x_2, y_1 \wedge y_2)), (x_2, y_2) \wedge (x_1 \vee x_2, y_1 \wedge y_2))$$

(Lemma 3.7(1))

$$= \mu_C((x_1 \wedge x_1 \vee x_2, y_1 \wedge y_1 \wedge y_2), (x_2 \wedge x_1 \vee x_2, y_2 \wedge y_1 \wedge y_2)) = \mu_C((x_1, y_1 \wedge y_2), (x_2, y_1 \wedge y_2))$$

$$= \mu_A(x_1, x_2).$$

Thus

$$\mu_C((x_1, y_1), (x_2, y_2)) \leq \mu_A(x_1, x_2). \tag{a}$$

Also

$$\mu_C((x_1, y_1), (x_2, y_2)) \leq \mu_C((x_1, y_1) \wedge (x_1 \wedge x_2, y_1 \vee y_2)), (x_2, y_2) \wedge (x_1 \wedge x_2, y_1 \vee y_2))$$

(Lemma 3.7(1))

$$= \mu_C((x_1 \wedge x_1 \wedge x_2, y_1 \wedge y_1 \vee y_2), (x_2 \wedge x_1 \wedge x_2, y_2 \wedge y_1 \vee y_2)) = \mu_C((x_1 \wedge x_2, y_1), (x_1 \wedge x_2, y_2)$$

$$= \mu_B(y_1, y_2).$$

Then

$$\mu_C((x_1, y_1), (x_2, y_2)) \leq \mu_B(y_1, y_2). \qquad (b)$$

Now from (a) and (b) we get that

$$\mu_C((x_1, y_1), (x_2, y_2)) = \mu_C((x_1, y_1), (x_2, y_2)) \wedge \mu_C((x_1, y_1), (x_2, y_2)))$$

$$\leq \mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2)) = \mu_{A\times B}((x_1, y_1), (x_2, y_2))$$

and then

$$\mu_C((x_1, y_1), (x_2, y_2)) \leq \mu_{A\times B}((x_1, y_1), (x_2, y_2)). \qquad (c)$$

Now

$$\mu_{A\times B}((x_1, y_1), (x_2, y_2)) = \mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2)$$

$$= \mu_C((x_1, y_3), (x_2, y_3)) \wedge \mu_C((x_3, y_1), (x_3, y_2))$$

$$= \mu_C((x_1, y_1 \wedge y_2), (x_2, y_1 \wedge y_2)) \wedge \mu_C((x_1 \wedge x_2, y_1), (x_1 \wedge x_2, y_2)) \quad \text{(Proposition 3.11)}$$

$$\leq \mu_C((x_1, y_1 \wedge y_2) \vee (x_1 \wedge x_2, y_1), (x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_2, y_2))) \qquad \text{(Definition 3.5)}$$

$$= \mu_C((x_1 \vee x_1 \wedge x_2, y_1 \wedge y_2 \vee y_1), (x_2 \vee x_1 \wedge x_2, y_1 \wedge y_2 \vee y_2)) = \mu_C((x_1, y_1), (x_2, y_2)).$$

Then

$$\mu_{A\times B}((x_1, y_1), (x_2, y_2))) \leq \mu_C((x_1, y_1), (x_2, y_2)). \qquad (d)$$

Then by (c) and (d) we obtain that

$$\mu_C((x_1, y_1), (x_2, y_2)) = \mu_{A\times B}((x_1, y_1), (x_2, y_2))). \qquad (e)$$

Now we prove that

$$\nu_C((x_1, y_1), (x_2, y_2)) = \nu_{A\times B}((x_1, y_1), (x_2, y_2))).$$

Then

$$\nu_C((x_1, y_1), (x_2, y_2)) \geq \nu_C((x_1, y_1) \wedge (x_1 \vee x_2, y_1 \wedge y_2)), (x_2, y_2) \wedge (x_1 \vee x_2, y_1 \wedge y_2))$$

(Lemma 3.7(1))

$$= \nu_C((x_1 \wedge x_1 \vee x_2, y_1 \wedge y_1 \wedge y_2), (x_2 \wedge x_1 \vee x_2, y_2 \wedge y_1 \wedge y_2)) = \nu_C((x_1, y_1 \wedge y_2), (x_2, y_1 \wedge y_2))$$

$$= \mu_A(x_1, x_2).$$

Thus

$$\nu_C((x_1, y_1), (x_2, y_2)) \geq \nu_A(x_1, x_2). \tag{f}$$

Also

$$\nu_C((x_1, y_1), (x_2, y_2)) \geq \nu_C((x_1, y_1) \wedge (x_1 \wedge x_2, y_1 \vee y_2)), (x_2, y_2) \wedge (x_1 \wedge x_2, y_1 \vee y_2)$$

(Lemma 3.7(1))

$$= \nu_C((x_1 \wedge x_1 \wedge x_2, y_1 \wedge y_1 \vee y_2), (x_2 \wedge x_1 \wedge x_2, y_2 \wedge y_1 \vee y_2)) = \nu_C((x_1 \wedge x_2, y_1), (x_1 \wedge x_2, y_2)$$

$$= \nu_B(y_1, y_2).$$

Then

$$\nu_C((x_1, y_1), (x_2, y_2)) \geq \nu_B(y_1, y_2). \tag{g}$$

Now from (f) and (g) we get that

$$\nu_C((x_1, y_1), (x_2, y_2)) = \nu_C((x_1, y_1), (x_2, y_2)) \vee \nu_C((x_1, y_1), (x_2, y_2)))$$
$$\geq \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2)) = \nu_{A\times B}((x_1, y_1), (x_2, y_2))$$

and then

$$\nu_C((x_1, y_1), (x_2, y_2)) \geq \nu_{A\times B}((x_1, y_1), (x_2, y_2)). \tag{h}$$

Now

$$\nu_{A\times B}((x_1, y_1), (x_2, y_2)) = \nu_A(x_1, x_2) \vee \mu_B(y_1, y_2)$$
$$= \nu_C((x_1, y_3), (x_2, y_3)) \vee \nu_C((x_3, y_1), (x_3, y_2))$$

$= \nu_C((x_1, y_1 \wedge y_2), (x_2, y_1 \wedge y_2)) \vee \nu_C((x_1 \wedge x_2, y_1), (x_1 \wedge x_2, y_2))$ (Proposition 3.11)

$\geq \nu_C((x_1, y_1 \wedge y_2) \vee (x_1 \wedge x_2, y_1), (x_2, y_1 \wedge y_2) \vee (x_1 \wedge x_2, y_2)))$ (Difinition 3.5)

$= \nu_C((x_1 \vee x_1 \wedge x_2, y_1 \wedge y_2 \vee y_1), (x_2 \vee x_1 \wedge x_2, y_1 \wedge y_2 \vee y_2)) = \nu_C((x_1, y_1), (x_2, y_2)).$

Then

$$\nu_{A\times B}((x_1, y_1), (x_2, y_2))) \geq \nu_C((x_1, y_1), (x_2, y_2)). \tag{i}$$

Then by (c) and (d) we obtain that

$$\nu_C((x_1, y_1), (x_2, y_2)) = \nu_{A\times B}((x_1, y_1), (x_2, y_2))). \tag{j}$$

Therefore (e) and (j) give us that

$$C((x_1, y_1), (x_2, y_2)) = (\mu_C((x_1, y_1), (x_2, y_2)), \nu_C((x_1, y_1), (x_2, y_2)))$$

$= (\mu_{A\times B}((x_1,y_1),(x_2,y_2))), \nu_{A\times B}((x_1,y_1),(x_2,y_2)))) = (A\times B)((x_1,y_1),(x_2,y_2))))$ which yields $C = A \times B$.

Remark 3.13. *In the Propositions 3.9 and 3.12 it is also proved that if $C \in IFC(X\times Y\times X\times Y)$, then we can define $A \in IFC(X\times X)$ and $B \in IFC(Y\times Y)$ such that $C = A\times B$ where $A(x_1,x_2) = C((x_1,y),(x_2,y))$ and $B(y_1,y_2) = C((x,y_1),(x,y_2))$ for all $x,x_1,x_2 \in X$ and $y,y_1,y_2 \in Y$.*

Definition 3.14. *If $A \in IFC(X\times X)$, then $A = (\mu_A,\nu_A)$ is an equivalence relation on $X\times X$ and so it determines similarity classes. Let $\dfrac{X\times X}{A}$ denote the set of all similarity classes of $X\times X$ determined by A. Suppose $\dfrac{X\times X}{A} = \{A_x \mid x\in X\}$ where $A_x : X \to [0,1]$ such that $A_x(y) = \mu(x,y)$ for all $y \in X$. Define two binary operations $\coprod$ and $\prod$ on $\dfrac{X\times X}{A}$ by $A_x \coprod A_y = A_{x\vee y}$ and $A_x \prod A_y = A_{x\wedge y}$ for all $x,y\in X$. Then $\dfrac{X\times X}{A}$ together with the binary operations $\coprod$ and $\prod$ is a lattice, which is called the factor lattice of $X\times X$ corresponding to A.*

Proposition 3.15. *Let $X, Y, A = (\mu_A,\nu_A), B = (\mu_B,\nu_B)$ and $A\times B$ be as in Proposition 3.9. Then the factor lattice $\dfrac{X\times Y\times X\times Y}{A\times B}$ corresponding to $A\times B$ is isomorphic to the product of the corresponding factor lattices $\dfrac{X\times X}{A}$ and $\dfrac{Y\times Y}{B}$.*

Proof. Let $\dfrac{X\times X}{A} = \{A_x \mid x\in X\}$ and $\dfrac{Y\times Y}{B} = \{B_y \mid y\in Y\}$ such that

$$\frac{X\times Y\times X\times Y}{A\times B} = \{(A\times B)_{(x,y)} \mid (x,y)\in X\times Y\}.$$

Define the map

$$\varphi : \frac{X\times X}{A}\times\frac{Y\times Y}{B} \to \frac{X\times Y\times X\times Y}{A\times B}$$

as

$$\varphi(A_x,B_y) = (A\times B)_{(x,y)}.$$

Let $x_1,x_2\in X$ and $y_1,y_2\in Y$. Firstly, we show that φ is well defined. If $(A_{x_1},B_{y_1}) = (A_{x_2},B_{y_2})$, then $A_{x_1} = A_{x_2}$ and $B_{y_1} = B_{y_2}$ so $A(x_1,x_2) = (1,0)$ and $B(y_1,y_2) = (1,0)$ which mean that $(A\times B)((x_1,y_1),(x_2,y_2)) = (1,0)$ and then $(A\times B)_{(x_1,y_1)}(x_2,y_2) = (1,0)$ and Lemma 3.4 give us that $(A\times B)_{(x_1,y_1)} = (A\times B)_{(x_2,y_2)}$ and thus φ is well defined.
Secondly, we prove that φ is one to one. Let

$$(A\times B)_{(x_1,y_1)} = (A\times B)_{(x_2,y_2)}$$

then

$$(A \times B)((x_1, y_1), (x_2, y_2)) = (1, 0)$$

which means that

$$(\mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2), \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2)) = (1, 0)$$

and so

$$\mu_A(x_1, x_2) \wedge \mu_B(y_1, y_2) = 1 \qquad \text{and} \qquad \nu_A(x_1, x_2) \vee \nu_B(y_1, y_2)) = 0$$

and then

$$\mu_A(x_1, x_2) = 1 = \mu_B(y_1, y_2) \qquad \text{and} \qquad \nu_A(x_1, x_2) = 0 = \nu_B(y_1, y_2)).$$

Now we will have that

$A(x_1, x_2) = (\mu_A(x_1, x_2), \nu_A(x_1, x_2)) = 1$ and $B(y_1, y_2) = (\mu_B(y_1, y_2), \nu_B(y_1, y_2)) = 0.$

By Lemma 3.4 we get $A_{x_1} = A_{x_2}$ and $B_{y_1} = B_{y_2}$ and then $(A_{x_1}, B_{y_1}) = (A_{x_2}, B_{y_2})$ which yields φ is one to one.
Thirdly, It is clearly that φ is onto.
Finally, we prove that φ is a lattice homomorphism. Let

$$(A_{x_1}, B_{y_1}), (A_{x_2}, B_{y_2}) \in \frac{X \times X}{A} \times \frac{Y \times Y}{B}$$

and $\coprod(\prod)$ be the join(meet) in factor lattice. Then

$$\varphi((A_{x_1}, B_{y_1}) \coprod (A_{x_2}, B_{y_2})) = \varphi(A_{x_1} \coprod A_{x_2}, B_{y_1} \coprod B_{y_2}) = \varphi(A_{x_1 \vee x_2}, B_{y_1 \vee y_2})$$

(Definition 3.14)

$$= (A \times B)_{(x_1 \vee x_2, y_1 \vee y_2)} = (A \times B)_{(x_1, y_1) \vee (x_2 \vee y_2)} = (A \times B)_{(x_1, y_1)} \coprod (A \times B)_{(x_2, y_2)}$$

$$= \varphi(A_{x_1}, B_{y_1}) \coprod \varphi(A_{x_2}, B_{y_2}).$$

Similarly

$$\varphi((A_{x_1}, B_{y_1}) \prod (A_{x_2}, B_{y_2})) = \varphi(A_{x_1}, B_{y_1}) \prod \varphi(A_{x_2}, B_{y_2}).$$

Therefore φ is a lattice homomorphism and proof is complete.

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